

FIRST EIGENVALUE OF p -LAPLACIAN ALONG THE NORMALIZED RICCI FLOW ON BIANCHI CLASSES

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Abstract. Consider M as a 3-homogeneous manifold. In this paper, we are going to study the behavior of the first eigenvalue of p -Laplace operator in a case of Bianchi classes along the normalized Ricci flow. Also we will give some upper and lower bounds for a such eigenvalue.

Key words and Phrases: Ricci flow, p -Laplace operator, Eigenvalue.

1. INTRODUCTION

Over the last few years, studying the geometric flows, specially the Ricci flow have become a topic of active research in both mathematics and physics. Generally, a geometric flow is an evolution of a geometric structure under a differential equation related to a functional usually associated with a curvature in a manifold. Although, the Ricci flow was introduced first by Hamilton [10] in mathematics and in the work of Friedan [9] in the context of string theory, Perelman has made a current wide interests by the proof of Poincare' conjecture using the Ricci flow in [16].

Consider M as a manifold with Riemannian metric g_0 , the family $g(t)$ of Riemannian metrics on M have been called as an un-normalized Ricci flow when it satisfies the equation

$$\frac{d}{dt}g(t) = -2Ric(g(t)) \quad g(0) = g_0, \quad (1)$$

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where Ric is known Ricci tensor of $g(t)$.

And also one can consider the normalized Ricci flow as follow

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + \frac{2r}{n}g \quad g(0) = g_0, \quad (2)$$

where $r = \frac{\int_M R d\mu}{\int_M d\mu}$ is the average of scalar curvature.

Hamilton in [10], has shown that there is a unique solution for the Ricci flow (1), on the interval $[0, T)$ for a sufficient T . Now consider $g(t)$ as a solution of the Ricci flow (1), the customary normalization on 3-manifolds is setting

$$\bar{g}(\bar{t}) = \psi(t)g(t), \quad \bar{t} = \int_0^t \psi(\nu) d\nu,$$

with $\frac{1}{\psi} \frac{\partial \psi}{\partial t} = \frac{2r}{n}$ where $n = 3$ and r is as same as what mentioned above is average of scalar curvature. In this case $\bar{g}(\bar{t})$ will be the solution of normalized Ricci flow (2).

In [16], Perelman has shown that the first eigenvalue of $-\Delta + \frac{R}{4}$ is nondecreasing under the Ricci flow. Later Cao [4] has shown the similar result for the eigenvalues of $-\Delta + \frac{R}{2}$ on a manifolds with non-negative curvature operator. Also similar results hold for the first eigenvalue of $-\Delta + aR$ ($a \geq \frac{1}{4}$) along the Ricci flow, for more details see [5, 14].

There are some other published work in monotonicity of eigenvalues of geometric operators under some geometric flows. For example second author in [2], has studied the evolution for the first eigenvalue of p -Laplacian along the Yamabe flow and also in [3] shown the monotonicity of eigenvalues of Witten-Laplace operator along the Ricci-Bourguignon flow. Also for more details in a case of p -Laplacian operator, Wang in [17], has shown the eigenvalue estimate for the weighted p -Laplacian and later in [18] shown the gradient estimate on the weighted p -Laplace heat equation. Beside what mentioned before A. Abolarinwa in [1], has studied the evolution and monotonicity of the first eigenvalue of p -Laplacian under the Ricci-harmonic flow and also you can find some useful results in eigenvalue monotonicity of the p -Laplace operator under the Ricci flow in [20], also Cao and Songbo Hou have worked on monotonicity of the first eigenvalue under Ricci flow and you can see their results in [8, 12]. Finally we will use some results of [19] in this work.

In this paper we will investigate the evolution of the first eigenvalue of p -Laplacian operator and then we will find some bounds in a case of Bianchi classes.

2. PRELIMINARIES AND EVOLUTION EQUATION

Let (M, g) be a locally homogeneous closed 3-manifold, there are nine classes of such manifolds. They are divided into two groups, the first consists of $H(3)$, $H(2) \times \mathbb{R}^1$ and $So(3) \times \mathbb{R}^1$ and the other one includes \mathbb{R}^3 , $SU(2)$, $SL(2, \mathbb{R})$, *Heisenberg*, $E(1, 1)$ and $E(2)$ which are called Bianchi classes. Milnor [15], has provided a frame $\{X_i\}_{i=1}^3$ where both the metric and Ricci tensors are diagonalized and this property is preserved by the Ricci flow (1). Now let $\{\theta\}_{i=1}^3$ be a dual to Milnor's frame, we

consider the metric $g(t)$ as

$$g(t) = A(t) (\theta_1)^2 + B(t) (\theta_2)^2 + C(t) (\theta_3)^2,$$

then the Ricci flow becomes a system of ODE with three variables $\{A(t), B(t), C(t)\}$. Consider M as a compact Riemannian manifold and $u : M \rightarrow \mathbb{R}$ be a smooth function on M or we can consider $u \in W^{1,p}(M)$ the Sobolev space. The p -Laplacian of u for $1 < p < \infty$ is defined as

$$\begin{aligned} \Delta_p u &= \operatorname{div} (|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-2} \Delta u + (p-2) |\nabla u|^{p-4} (\operatorname{Hess} u) (\nabla u, \nabla u), \end{aligned}$$

where

$$(\operatorname{Hess} u) (X, Y) = \nabla (\nabla u) (X, Y) = X. (Y.u) - (\nabla_X Y).u \quad X, Y \in \chi(M).$$

In this case we say that λ is an eigenvalue of p -Laplace operator whenever there exist a function u on M such that

$$\Delta_p u = -\lambda |u|^{p-1} u, \tag{3}$$

$$\lambda(t) = \operatorname{inf} \left\{ \int_M |\nabla u|^p d\mu \mid u \in W^{1,p}(M) \setminus \{0\}, \int_M |u|^p d\mu = 1 \right\}.$$

The theorem below from [19] gives us the continuity of the first eigenvalue of p -Laplace operator.

Theorem 2.1. *If g_1 and g_2 are two metrics which satisfy*

$$(1 + \varepsilon)^{-1} g_1 \leq g_2 \leq (1 + \varepsilon) g_1,$$

then for any $p > 1$, we have

$$(1 + \varepsilon)^{-(n+\frac{p}{2})} \lambda(g_2) \leq \lambda(g_1) \leq (1 + \varepsilon)^{(n+\frac{p}{2})} \lambda(g_2).$$

In particular $\lambda(g(t))$ is a continuous function in a t -variable.

Discussing about the monotonicity of the first eigenvalue of the p -Laplace operator powerfully is dependent to the differentiability of the eigenvalue function. In this section we are following the X. Cao's argument [4], where we introduce the smooth eigenvalue function $\lambda(u, t)$ which is smooth then we can write the monotonicity of $\lambda(t)$. We assume at time t_0 , $u_0 = u(t_0)$ is eigenfunction for the first eigenvalue $\lambda(t_0)$ of p -Laplacian. Then we have

$$\int_M |u(t_0)|^p d\mu_{g(t_0)} = 1.$$

We consider the following smooth function

$$\varphi(t) := u_0 \left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))} \right]^{\frac{1}{2(p-1)}},$$

and normalize this smooth function as

$$u(t) = \frac{\varphi(t)}{\left(\int_M |\varphi(t)|^p d\mu \right)^{\frac{1}{p}}},$$

under the Ricci flow. Now we define a general smooth function as

$$\begin{aligned} \lambda(u, t) &:= - \int_M \Delta_p u(t) \cdot u(t) d\mu \\ &= \int_M |\nabla u|^p d\mu, \end{aligned}$$

where u is any smooth function satisfying

$$\int_M |u|^p d\mu = 1 \quad \text{and} \quad \int_M |u|^{p-2} u d\mu = 0. \tag{4}$$

In general $\lambda(u, t)$ is not equal to $\lambda(t)$. But at time t_0 we conclude that

$$\lambda(u(t_0), t_0) = \lambda(t_0).$$

Now we are ready to give an evolution formula for $\lambda(u, t)|_{t=t_0}$ along the normalized Ricci flow in a case of 3-homogeneous manifold. In this case it is not hard to see $R = r$ where R is scalar curvature and r is as same as what explained in the definition of the normalized Ricci flow (2).

Proposition 2.2. *Let $(M, g(t))$ be a solution of the normalized Ricci flow (2), on a locally homogeneous 3-manifold. If $\lambda_{1,p}(t)$ denotes the first eigenvalue of the p -Laplacian (3), then*

$$\frac{d}{dt} \lambda(u, t)|_{t=t_0} = p \int |\nabla u|^{p-2} R_{ij} u^i u^j d\mu - \frac{pR}{3} \lambda(t_0),$$

where $u^i = \nabla^i u$ and $u^j = \nabla^j u$.

Proof. By the direct computation it will be easy to see that under the normalized Ricci flow (2) we have

$$\frac{\partial}{\partial t} |\nabla u|^p = p |\nabla u|^{p-2} \left(\left(R^{ij} - \frac{r}{3} g^{ij} \right) u_i u_j + u_i \frac{\partial u_i}{\partial t} \right), \tag{5}$$

and also

$$\frac{\partial}{\partial t} (d\mu) = -(R - r) d\mu. \tag{6}$$

The function $\lambda(u, t)$ is smooth so it concludes that

$$\begin{aligned} \frac{d}{dt} \lambda(u, t)|_{t=t_0} &= \int_M \frac{\partial}{\partial t} |\nabla u|^p d\mu - \int_M |\nabla u|^p (R - r) d\mu \\ &= p \int_M |\nabla u|^{p-2} \left(\left(R^{ij} - \frac{r}{3} g^{ij} \right) u_i u_j \right) d\mu + p \int_M |\nabla u|^{p-2} u_i \frac{\partial u_i}{\partial t} d\mu \\ &\quad - \int_M |\nabla u|^p (R - r) d\mu. \end{aligned}$$

As we mentioned before in homogeneous manifold $R = r$ also the condition $\int_M |u|^p d\mu = 1$ results that

$$\int_M |\nabla u|^{p-2} u_i \frac{\partial u_i}{\partial t} d\mu = 0.$$

Hence

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &= p \int_M |\nabla u|^{p-2} R^{ij} u_i u_j d\mu - \frac{pR}{3}\lambda(t_0) \\ &= p \int_M |\nabla u|^{p-2} R_{ij} u^i u^j d\mu - \frac{pR}{3}\lambda(t_0), \end{aligned}$$

which implies what we looking for. □

3. ESTIMATE OF $\lambda(t)$ ON BIANCHI CLASSES

In this section we are going to give some useful bounds for $\lambda(t)$ separately in Bianchi classes. Before Hou in [11] has given bounds for the first eigenvalue of Δ in a case of $u > 0$ under the backward Ricci flow and also he proved the eigenvalue evolves toward zero in a case that the backward Ricci flow converges to a sub-Riemannian geometry by a proper rescaling. Later Razavi and Korouki in [13] have done similar work for the first eigenvalue of $(-\Delta - R)$ under the Ricci flow.

Remark 3.1. *In homogeneous condition and in a case of the un-normalized Ricci flow we get that*

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} = p \int_M |\nabla u|^{p-2} R_{ij} u_i u_j d\mu.$$

Now we study the behavior of the first eigenvalue of p -Laplacian in each classes separately.

Case 1: \mathbb{R}^3

In this case all metrics are flat, so for all $t \geq 0$ we have $g(t) = g_0$ where g_0 is initial metric, therefore $\lambda(t)$ is constant.

Case 2: Heisenberg

This class is isomorphic to the set of upper-triangular 3×3 matrices endowed with the usual matrix multiplication. Under the metric g_0 we choose a frame $\{X_i\}_{i=1}^3$ in which

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

also under the normalization $A_0 B_0 C_0 = 1$ we have

$$\begin{aligned} R_{11} &= \frac{1}{2}A^3, & R_{22} &= -\frac{1}{2}A^2B, & R_{33} &= -\frac{1}{2}A^2C, \\ R &= -\frac{1}{2}A^2. \end{aligned}$$

Theorem 3.2. *Let $\lambda(t)$ be the first eigenvalue of p -Laplace operator on Heisenberg Riemannian manifold (\mathcal{H}^3, g_0) and also assume that $B_0 \geq C_0$. Then in a sufficient neighborhood as $[0, t]$, the quantities $\lambda(t)e^{\int_0^t (-\frac{2}{3}pA^2)d\tau}$ is nondecreasing and $\lambda(t)e^{\int_0^t (\frac{1}{3}pA^2)d\tau}$ is nonincreasing along the normalized Ricci flow (2), where*

$$-\frac{1}{3}pA^2\lambda(t_0) \leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{2}{3}pA^2\lambda(t_0).$$

Proof. Under the proposition 2.2 and Ricci coordinates in *Hiesenberg* case, we get

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &= \int_{\mathcal{H}^3} |\nabla u|^{p-2} [g_{11}\nabla^1 u \nabla^1 u - g_{22}\nabla^2 u \nabla^2 u \\ &\quad - g_{33}\nabla^3 u \nabla^3 u] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\leq \frac{1}{2}pA^2 \int_{\mathcal{H}^3} |\nabla u|^{p-2} [g_{11}\nabla^1 u \nabla^1 u + g_{22}\nabla^2 u \nabla^2 u \\ &\quad + g_{33}\nabla^3 u \nabla^3 u] d\mu - \frac{p}{3}R\lambda(t_0) \\ &= \frac{1}{2}pA^2\lambda(t_0) - \frac{p}{3}R\lambda(t_0). \end{aligned} \tag{7}$$

By substituting R into formula (7) we obtain

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{2}{3}pA^2\lambda(t_0).$$

Since $\lambda(f, t)$ is smooth function with respect to time t , hence in any sufficiently small neighborhood of t_0 , we have

$$\frac{d}{dt}\lambda(f(t), t) \leq \frac{2}{3}pA^2\lambda(f(t), t). \tag{8}$$

Since t_0 is arbitrary then for any $t \in [0, T)$ the inequality (8) holds and it implies

$$\frac{d}{dt} \left(\lambda(t)e^{\int_0^t (-\frac{2}{3}pA^2)d\tau} \right) \geq 0. \tag{9}$$

Therefore the quantity $\lambda(t)e^{\int_0^t (-\frac{2}{3}pA^2)d\tau}$ is nondecreasing. Also in a similar way we have

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\geq \frac{1}{2}pA^2 \int_{\mathcal{H}^3} |\nabla u|^{p-2} [-g_{11}\nabla^1 u \nabla^1 u - g_{22}\nabla^2 u \nabla^2 u \\ &\quad - g_{33}\nabla^3 u \nabla^3 u] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\geq -\frac{1}{2}pA^2\lambda(t_0) - \frac{p}{3}R\lambda(t_0) \\ &= -\frac{1}{3}pA^2\lambda(t_0). \end{aligned}$$

Which implies what we are looking for. □

Remark 3.3. *In this case by [11], for the tensors A, B and C we have*

$$\begin{aligned} A &= A_0 \left(1 + \frac{16}{3}R_0 t \right)^{-\frac{1}{2}}, \quad B = B_0 \left(1 + \frac{16}{3}R_0 t \right)^{\frac{1}{4}}, \\ C &= C_0 \left(1 + \frac{16}{3}R_0 t \right)^{\frac{1}{4}}, \end{aligned}$$

where $R_0 = -\frac{1}{2}A_0^2$. Now by substituting these formulas into the formula (7) which is hold for arbitrary t_0 , and integrating from both sides in $[t_0, t]$, we get

$$\ln\lambda(t) \leq \frac{1}{2} \cdot \frac{pA_0^2}{1 + \frac{16}{3}R_0} \ln\left(1 + \frac{16}{3}R_0t\right),$$

and similarly

$$\ln\lambda(t) \geq -\frac{1}{2} \cdot \frac{pA_0^2}{1 + \frac{16}{3}R_0} \ln\left(1 + \frac{16}{3}R_0t\right).$$

Case 3: E(2)

Manifold E(2) is the group of isometries of Euclidian plane. In this case we have an Einstein metric and Ricci flow converges exponentially to flat metrics. Dependent to the metric g_0 we choose the frame $\{X_i\}_{i=0}^3$ such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = 0,$$

In this case under the normalization $A_0B_0C_0 = 1$ we have

$$\begin{aligned} R_{11} &= \frac{1}{2}A(A^2 - B^2), & R_{22} &= \frac{1}{2}B(B^2 - A^2), \\ R_{33} &= -\frac{1}{2}C(A - B)^2, & R &= -\frac{1}{2}(A - B)^2. \end{aligned}$$

X. Cao in [7] has proved that for initial tensors A_0 and B_0

- If $A_0 = B_0$ then $A = B$, in this case $g(t) = g_0$ where g_0 is constant.
- If $A_0 > B_0$ then $A > B$ in this case we have

Theorem 3.4. Consider $\lambda(t)$ as a first eigenvalue of p -Laplace operator on 3-homogeneous Riemannian manifold $(E(2), g_0)$ and also let $A_0 > B_0$ then in a sufficient neighborhood as $[0, t]$, the quantities $\lambda(t)e^{\int_0^t -\frac{1}{2}p((A^2 - B^2) - \frac{1}{3}(A - B)^2)d\tau}$ and $\lambda(t)e^{\int_0^t \frac{1}{2}p((A^2 - B^2) - \frac{1}{3}(A - B)^2)d\tau}$ are non-decreasing and non-increasing along the normalized Ricci flow (2), respectively, where

$$\begin{aligned} -\frac{1}{2}p\left((A^2 - B^2) - \frac{1}{3}(A - B)^2\right)\lambda(t_0) &\leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \\ &\leq \frac{1}{2}p\left((A^2 - B^2) - \frac{1}{3}(A - B)^2\right)\lambda(t_0). \end{aligned}$$

Proof. Since by [6] we have $A > B$ and also under the proposition 2.2, we get

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &= p \int_{E(2)} |\nabla u|^{p-2} \left[\frac{1}{2}A(A^2 - B^2) \nabla^1 u \nabla^1 u - \frac{1}{2}B(A^2 - B^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. - \frac{1}{2}C(A - B)^2 \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R, \end{aligned}$$

where by the assumption $A_0 > B_0$ we have

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\geq p \int_{E(2)} |\nabla u|^{p-2} \left[-\frac{1}{2}A(A^2 - B^2) \nabla^1 u \nabla^1 u - \frac{1}{2}B(A^2 - B^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. - \frac{1}{2}C(A^2 - B^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\geq -\frac{1}{2}p(A^2 - B^2) \lambda(t_0) - \frac{p}{3}R\lambda(t_0). \end{aligned}$$

In a similar way

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p(A^2 - B^2) \lambda(t_0) - \frac{p}{3}R\lambda(t_0),$$

now apply R from above, since t_0 is arbitrary it implies what mentioned before in the theorem. \square

Case 4: E(1,1)

Manifold E(1,1) is the group of isometries of the plane with flat Lorentz metric, there is no Einstein metric here and Ricci flow fails to converge, they all are asymptotically cigar degeneracies. For a given metric g_0 similarly by a frame $\{X_i\}_{i=0}^3$ we have

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2.$$

Also under the normalization $A_0 B_0 C_0 = 1$ we obtain

$$\begin{aligned} R_{11} &= \frac{1}{2}A(A^2 - C^2), & R_{22} &= -\frac{1}{2}B(A + C)^2, \\ R_{33} &= \frac{1}{2}C(C^2 - A^2), & R &= -\frac{1}{2}(A + C)^2. \end{aligned}$$

Theorem 3.5. *Let $\lambda(t)$ denotes the first eigenvalue of p -Laplace operator on 3-homogeneous Riemannian manifold $(E(1,1), g_0)$ then in a sufficient neighborhood $[0, t]$ we get*

- *If $A_0 = C_0$ then $\lambda(t)e^{\int_0^t (-\frac{1}{3}p(A+C)^2)d\tau}$ and $\lambda(t)e^{\int_0^t (\frac{1}{3}p(A+C)^2)d\tau}$ are non-decreasing and non-increasing along the normalized Ricci flow (2), respectively, where*

$$\left(-\frac{1}{3}p(A + C)^2\right) \lambda(t_0) \leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \left(\frac{1}{3}p(A + C)^2\right) \lambda(t_0).$$

- *If $A_0 > C_0$ then $\lambda(t)e^{\int_0^t (-\frac{1}{2}p((A^2-C^2)-\frac{1}{3}(A+C)^2))dt}$ and $\lambda(t)e^{\int_0^t (\frac{2}{3}p(A+C)^2)dt}$ are non-decreasing and non-increasing respectively, where*

$$-\frac{1}{2}p\left((A^2 - C^2) - \frac{1}{3}(A + C)^2\right) \lambda(t_0) \leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \left(\frac{2}{3}p(A + C)^2\right) \lambda(t_0).$$

Proof. In the case of $A_0 = C_0$ by [6] we get that $A = C$, it is easy to see

$$R_{11} = R_{33} = 0,$$

which means

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p(A + C)^2 \lambda(t_0) - \frac{p}{3}R\lambda(t_0).$$

Similarly in a case of $A_0 > C_0$ it is easy to get

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} \geq p \int_{E(1,1)} |\nabla u|^{p-2} & \left[-\frac{1}{2}A(A^2 - C^2) \nabla^1 u \nabla^1 u - \frac{1}{2}B(A^2 - C^2) \nabla^2 u \nabla^2 u \right. \\ & \left. - \frac{1}{2}C(A^2 - C^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0), \end{aligned}$$

and in a similar way

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p(A + C)^2 \lambda(t_0) - \frac{p}{3}R\lambda(t_0),$$

now similar to the above theorems, the proof is completed. □

Case 5: SU(2)

Similarly in this class we have Einstein metrics and Ricci flow converges exponentially in to these metrics, also by the frame $\{X_i\}_{i=0}^3$ we have

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

In this case under the normalization $A_0 B_0 C_0 = 1$, we have

$$\begin{aligned} R_{11} &= \frac{1}{2}A[A^2 - (B - C)^2], & R_{22} &= \frac{1}{2}B[B^2 - (A - C)^2], \\ R_{33} &= \frac{1}{2}C[C^2 - (A - B)^2], \end{aligned}$$

where

$$R = \frac{1}{2}[A^2 - (B - C)^2] + \frac{1}{2}[B^2 - (A - C)^2] + \frac{1}{2}[C^2 - (A - B)^2].$$

Theorem 3.6. *Consider $\lambda(t)$ as a first eigenvalue of p -Laplace operator on 3-homogeneous Riemannian manifold $(SU(2), g_0)$ then there is a time \tilde{t} and the interval $[\tilde{t}, t]$ in which we have*

- If $A_0 = B_0 = C_0$ then $\lambda(t) = \lambda(0)$.
- If $A_0 = B_0 > C_0$ then

$$\lambda(t)e^{\int_{\tilde{t}}^t (p(B-C)^2 - \frac{1}{2}p(A^2 + C^2))d\tau} \quad \text{and} \quad \lambda(t)e^{\int_{\tilde{t}}^t (\frac{1}{2}pA^2((B-C)^2 + 1))d\tau}$$

are non-decreasing and non-increasing respectively, where

$$\left(p(B - C)^2 - \frac{1}{2}p(A^2 + C^2) \right) \lambda(t_0) \leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \left(\frac{1}{2}pA^2((B - C)^2 + 1) \right) \lambda(t_0).$$

- If $A_0 > B_0 \geq C_0$ then

$$\lambda(t)e^{\int_{\tilde{t}}^t \frac{1}{2}p((C^2 - (A - C)^2) - (A^2 - (B - C)^2))d\tau} \quad \text{and} \quad \lambda(t)e^{\int_{\tilde{t}}^t \frac{1}{2}p((A^2 - (B - C)^2) - (C^2 - (A - C)^2))d\tau}$$

are non-decreasing and non-increasing along the normalized Ricci flow (2) respectively, where

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \geq \frac{1}{2}p \left((C^2 - (A - C)^2) - (A^2 - (B - C)^2) \right) \lambda(t_0),$$

and

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p \left((A^2 - (B - C)^2) - (C^2 - (A - C)^2) \right) \lambda(t_0).$$

Proof. By X. Cao [7], the proof of first and second section will be easy and similarly for the third section we have

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\leq p \int_{SU(2)} |\nabla u|^{p-2} \left[\frac{1}{2}A (A^2 - (B - C)^2) \nabla^1 u \nabla^1 u + \frac{1}{2}B (A^2 - (A - C)^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. + \frac{1}{2}C (A^2 - (A - B)^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\leq p \int_{SU(2)} |\nabla u|^{p-2} \left[\frac{1}{2}A (A^2 - (B - C)^2) \nabla^1 u \nabla^1 u + B (A^2 - (B - C)^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. + C (A^2 - (B - C)^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\leq \frac{1}{2}p (A^2 - (B - C)^2) \lambda(t_0) - \frac{p}{3}R\lambda(t_0). \end{aligned}$$

Also in a similar way

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\geq p \int_{SU(2)} |\nabla u|^{p-2} \left[\frac{1}{2}A (C^2 - (B - C)^2) \nabla^1 u \nabla^1 u + \frac{1}{2}B (C^2 - (A - C)^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. + \frac{1}{2}C (C^2 - (A - B)^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\geq p \int_{SU(2)} |\nabla u|^{p-2} \left[\frac{1}{2}A (C^2 - (A - C)^2) \nabla^1 u \nabla^1 u + \frac{1}{2}B (C^2 - (A - C)^2) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. + \frac{1}{2}C (C^2 - (A - C)^2) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\geq \frac{1}{2}p (C^2 - (A - C)^2) \lambda(t_0) - \frac{p}{3}R\lambda(t_0), \end{aligned}$$

which if you substitute R , it is completed the proof. □

Case 6: $SL(2, \mathbb{R})$

On $SL(2, \mathbb{R})$ there is no Einstein metric and the Ricci flow doesn't converge and develops a pancake degeneracy, also by the frame $\{X_i\}_{i=0}^3$, we get

$$[X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3,$$

in this case we also have

$$\begin{aligned} R_{11} &= \frac{1}{2}A[A^2 - (B - C)^2], & R_{22} &= \frac{1}{2}B[B^2 - (A + C)^2], \\ R_{33} &= \frac{1}{2}C[C^2 - (A + B)^2]. \end{aligned}$$

In which

$$R = \frac{1}{2}[A^2 - (B - C)^2] + \frac{1}{2}[B^2 - (A + C)^2] + \frac{1}{2}[C^2 - (A + B)^2].$$

Theorem 3.7. *Let $\lambda(t)$ be the first eigenvalue of p -Laplace operator on 3-homogeneous Riemannian manifold $(SL(2, \mathbb{R}), g_0)$ and also let there is a time \tilde{t} and interval $[\tilde{t}, t]$ we get*

- If $A > B = C$ then

$$\lambda(t)e^{\int_{\tilde{t}}^t \frac{1}{2}p((C^2 - (A+B)^2) - A^2)d\tau} \quad \text{and} \quad \lambda(t)e^{\int_{\tilde{t}}^t \frac{1}{2}p(A^2 - (C^2 - (A+B)^2))d\tau},$$

are non-decreasing and non-increasing along the normalized Ricci flow (2), where

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \geq \frac{1}{2}p \left((C^2 - (A + B)^2) - A^2 \right) \lambda(t_0),$$

and

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p \left(A^2 - (C^2 - (A + B)^2) \right) \lambda(t_0).$$

- If $A \leq B - C$ then

$$\lambda(t)e^{\int_{\tilde{t}}^t -\frac{1}{2}p((B-C)^2 - A^2 + B^2)d\tau} \quad \text{and} \quad \lambda(t)e^{\int_{\tilde{t}}^t -\frac{1}{2}pC^2d\tau},$$

are non-decreasing and non-increasing along the normalized Ricci flow (2), where

$$-\frac{1}{2}p \left((B - C)^2 - A^2 + B^2 \right) \lambda(t_0) \leq \frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq -\frac{1}{2}pC^2.$$

Proof. By X. Cao [6, 7] we can easily calculate that

- for the first section we have

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\geq p \int_{SL(2, \mathbb{R})} |\nabla u|^{p-2} \left[\frac{1}{2}A \left(C^2 - (A + C)^2 \right) \nabla^1 u \nabla^1 u + \frac{1}{2}B \left(C^2 - (A + C)^2 \right) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. + \frac{1}{2}C \left(C^2 - (A + C)^2 \right) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}r\lambda(t_0) \\ &\geq \frac{1}{2}p \left(C^2 - (A + C)^2 \right) \lambda(t_0) - \frac{p}{3}r\lambda(t_0), \end{aligned}$$

also similarly we get

$$\frac{d}{dt}\lambda(u, t)|_{t=t_0} \leq \frac{1}{2}p \left(A^2 - (B - C)^2 \right) \lambda(t_0) - \frac{p}{3}R\lambda(t_0).$$

- For the second section also we get

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &= p \int_{SL(2, \mathbb{R})} |\nabla u|^{p-2} \left[-\frac{1}{2}A \left((B - C)^2 - A^2 \right) \nabla^1 u \nabla^1 u + \frac{1}{2}B \left(B^2 - (A + c)^2 \right) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. - \frac{1}{2}C \left((A + B)^2 - C^2 \right) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\leq \frac{1}{2}pB^2\lambda(t_0), \end{aligned}$$

also under consideration $A \leq B - C$ we have

$$(A + B)^2 > (B - C)^2,$$

now we can get

$$\begin{aligned} \frac{d}{dt}\lambda(u, t)|_{t=t_0} &\geq p \int_{SL(2, \mathbb{R})} |\nabla u|^{p-2} \left[-\frac{1}{2}A \left((B-C)^2 - A^2 \right) \nabla^1 u \nabla^1 u - \frac{1}{2}B \left((B-C)^2 - A^2 \right) \nabla^2 u \nabla^2 u \right. \\ &\quad \left. - \frac{1}{2}C \left((B-C)^2 - A^2 \right) \nabla^3 u \nabla^3 u \right] d\mu - \frac{p}{3}R\lambda(t_0) \\ &\geq -\frac{1}{2}p \left((B-C)^2 - A^2 \right) \lambda(t_0) - \frac{p}{3}R\lambda(t_0), \end{aligned}$$

now we should substitute R , this is making the proof complete. \square

REFERENCES

- [1] A. Abolarinwa, *Evolution and monotonicity of the first eigenvalue of p -Laplacian under the Ricci harmonic flow*, J. Appl. Anal., 21(2) (2015), 147-160.
- [2] S. Azami, *Eigenvalue variation of the p -Laplacian under the Yamabe flow*, Cogent mathematics, 3 (2016), 1236566.
- [3] S. Azami, *Monotonicity of eigenvalues of Witten-Laplace operator along the Ricci-Bourguignon flow*, AIMS mathematics, 2(2)(2017), 230-243.
- [4] X. Cao, *Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator*, Math. Ann., 337(2) (2007), 435-442.
- [5] X. Cao, *First eigenvalues of geometric operators under the Ricci flow*, Proc. Amer. Math. Soc, 136(11) (2008), 4075-4078.
- [6] X. Cao, J. Guckenheimer and L. Saloff-Coste, *The backward behavior of the Ricci and cross curvature flows on $SL(2, \mathbb{R})$* , Comm. Anal. Geom., 17(4) (2009), 777-796.
- [7] X. Cao and L. Saloff-Coste, *Backward Ricci flow on locally homogeneous three-manifolds*, Comm. Anal. Geom., 12(2), (2009) 305-325.
- [8] X. Cao, S. Hou and J. Ling, *Estimates and monotonicity of the first eigenvalue under the Ricci flow*, Math. Ann. 345 (2012), no. 2, 451-463.
- [9] D. Friedan, *Nonlinear models in $2+\epsilon$ dimensions*, Annals of physics 163 (31) (1985), 318-419.
- [10] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), 255-306.
- [11] S. Hou, *Eigenvalues under the backward Ricci flow on locally homogeneous closed 3-manifolds*, Acta Mathematica Sinica, English series, 136(11) (2018), 1179-1194.
- [12] S. Hou, *Eigenvalues under the Ricci flow of model geometries*, (Chinese) Acta Math. Sinica (Chin. ser.) 60 (2017), no. 4, 583-594.
- [13] F. Korouki and A. Razavi, *Bounds for the first eigenvalue of $(-\Delta - R)$ under the Ricci flow on Bianchi classes*, Bull. Braz. Math. Soc, (2019).
- [14] J. Li, *Eigenvalues and energy functionals with monotonicity formula under Ricci flow*, Math. Ann., 338 (2007), 927-946.
- [15] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math., 21(3) (1976), 293-329.
- [16] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Arxiv (2002).
- [17] L. Wang, *Eigenvalue estimate for the weighted p -Laplace*, Annali di Matematica, 191 (2012), 539-550.
- [18] L. Wang, *Gradient estimates on the weighted p -Laplace heat equation*, J. Diff. Equ., 264 (2018), 506-524.
- [19] J. Wu, E. Wang and Y. Zheng, *First eigenvalue of the p -Laplace operator along the Ricci flow*, Ann. Glob. Anal. Geom., 38(1) (2009), 27-55.

- [20] J. Wu, *First eigenvalue monotonicity for the p -Laplace operator under the Ricci flow*, Acta Mathematica Sinica, English series, (2011), 1591-1598.