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A CLASSIFICATION OF THE CUBIC SEMISYMMETRIC GRAPHS OF ORDER $34p^2$

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Abstract. A simple undirected graph is called semisymmetric if it is regular and edge transitive but not vertex-transitive. In this paper, we classify all connected cubic semisymmetric graph of order $34p^2$, where p be a prime.

Key words: Symmetric graph, semisymmetric graph, regular coverings.

Abstrak. Suatu graf tak berarah sederhana disebut semisimetris jika graf tersebut regular dan transitif sisi tapi tidak transitif titik. Dalam paper ini, kami mengklasifikasikan semua graf semisimetris kubik terhubung berorde $34p^2$ dengan p adalah bilangan prima.

Key words: Graf simetris, graf semisimetris, selimut regular.

1. **Introduction**

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, the edge set, the arc set and the full automorphism group of X, respectively. For $u \in V(X)$, $N_X(u)$ is the set of vertices adjacent to u in X. For a graph X and a subgroup G of $\mathrm{Aut}(X)$, X is said to be *G-vertex-transitive* or *G-edge-transitive* if G is transitive on the sets of vertices or edges of X respectively. A graph is *G-semisymmetric* if it is *G-vertex-transitive* but not *G-edge-transitive.* Furthermore, a graph X is said to be vertex-transitive or edge-transitive if in the above definition, $G = \mathrm{Aut}(X)$. It can be shown that a G-edge-transitive but not G-vertex-transitive graph is necessarily bipartite, where the two partite parts of

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the graph are orbits of G. Moreover, if X is regular these two partite sets have equal cardinality. A regular edge- but not vertex-transitive graph will be referred to as a *semisymmetric* graph.

The class of semisymmetric graphs was first introduced by Folkman [4], and Malnič et al. [7] classified cubic semisymmetric graphs of order $2p^3$ for a prime p, while Folkman [4] proved that there is no cubic semisymmetric graphs of order $2p$ or $2p^2$. In this paper, we show that there are no connected cubic semisymmetric graphs of order $34p^2$.

Let X be a graph and N a subgroup of $\mathrm{Aut}(X)$. Denote by X_N *the quotient graph* corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in X_N whenever there is an edge between those orbits in X. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $\wp : \widetilde{X} \to X$ if there is a surjection $\wp : V(\widetilde{X}) \to V(X)$ such that $\wp|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection $\wp$ is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}_K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}_K$ is the composition $\wp h$ of $\wp$ and h.

Proposition 2.1. (1) [9, Theorem 8.5.3] *Let p and q be primes and let a and b be non-negative integers. Then every group of order $p^a p^b$ is solvable.*
(2) [3, Feit-Thompson Theorem] *Every finite group of odd order is solvable.*

Proposition 2.2. [10, p. 236] *Let G be a finite group. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \bigcap Z(G)|$.*

Proposition 2.3. [5] *The vertex stabilizers of a G-semisymmetric cubic graph has order 2^r. 3, where $0 \le r \le 7$.*

The next preposition is special case of [8, Lemma 3.2].

Proposition 2.4. *Let X be a connected G-semisymmetric cubic with bipartition sets then $U(X)$ and $W(X)$, and let N be a normal subgroup of G. If N is intransitive on bipartition sets then N acts semiregularly on both $U(X)$ and $W(X)$, and X is an N-covering of a G/N-semirsymmetric graph.*

2. **Main Results**

The following is the main result of this paper.

Theorem 2.1. *Let p be a prime. Then there is no connected cubic semisymmetric graph of order $34p^2$.*

Proof. For $p \le 3$, by [1], there is no connected cubic semisymmetric graph of order $34p^2$. Thus we may assume that $p \ge 5$. To prove the theorem, we only need to show that no connected cubic semisymmetric graph of order $34p^2$ exits for $p \geq 5$. Suppose to the contrary that X is such a graph. Since semisymmetric graphs are bipartite, one may denote by $U(X)$ and $W(X)$ the bipartite sets of X. Clearly, $|U(X)| = |W(X)| = 17p^2$. Set $A :=$Aut(X). By proposition 2.3, $|A| = 2^r.\ 3.\ 17.\ p^2$ for some integer $0 \leq r \leq 7$. Let N be a minimal normal subgroup of A.

Assume that N is unsolvable. Then N is a product of isomorphic non-abelian simple groups and hence a non-abelian simple group because $|A| = 2^s.\ 3.\ 17.\ p^2$. Suppose that N is not transitive on bipartition sets $U(X)$ and $W(X)$. By Proposition 2.4, N acts semiregularly on bipartition sets $U(X)$ and $W(X)$. Thus, $|N| \mid 34p^2$. This forces that N is solvable, a contradiction. It follows that N is transitive on at least one of the bipartition sets $U(X)$ and $W(X)$, implying $17p^2 \mid |N|$. By Proposition 2.1, $|N| = 2^t.\ 17.\ p^2$ or $2^t.\ 3.\ 17.\ p^2$. Let q be a prime. Then by [6, pp. 12-14] and [2], a non-abelian simple $\{2,\ q,\ p\}$-group is one of the following groups

$$A_5,\ A_6,\ PSL_2(7),\ PSL_2(8),\ PSL_2(17),\ PSL_3(3),\ PSU_3(3) \text{ and } PSU_4(2), \qquad (1)$$

With orders $2^2.\ 3.\ 5,\ 2^3.\ 3^2.\ 5,\ 2^3.\ 3.\ 7,\ 2^3.\ 3^2.\ 7,\ 2^4.\ 3^2.\ 17,\ 2^4.\ 3^3.\ 13,\ 2^5.\ 3^3.\ 7$ and $2^6.\ 3^4.\ 5$, respectively. This implies that for $p \geq 5$, there is no simple group of order $2^t.\ 17.\ p^2$. Hence $|N| = 2^t.\ 3.\ 17.\ p^2$.

Assume that L is a proper subgroup of N. If L is unsolvable, then L has a non-abelian simple composite factor L_1/L_2. Since $p \geq 5$ and $|L_1/L_2| \mid 2^t.\ 3.\ 17.\ p^2$, by simple group listed in (1), L_1/L_2 cannot be a $\{2,\ 3,\ 17\}$- or $\{2,\ 17,\ p\}$-group. If L_1/L_2 is $\{2,\ 3,\ p\}$-group, then $p = 5$ or 7 and hence N is A_5 or $PSL_2(7)$, which is impossible, because by [2, pp. 239], there is no non-abelian simple group of order $|N| = 2^t.\ 3.\ 17.\ p^2$ for $p = 5,\ 7$. Thus, L_1/L_2 is a $\{2,\ 3,\ 17,\ p\}$-group. One may assume that $|L| = 2^s.\ 3.\ 17.\ p^2$ or $2^s.\ 3.\ 17.\ p$, where $s \geq 2$. Let $|L| = 2^s.\ 3.\ 17.\ p^2$. Since $N = 2^t.\ 3.\ 17.\ p^2$ for some $0 \leq t \leq 7$, one has $|N : L| \leq 32$. Consider the action of N on the right cosets of L in N by right multiplication. The simplicity of N implies that this action is faithful. It follows $N \leq S_{32}$ and hence $p = 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29$ or 31. But by [2, pp. 239], there is no non-abelian simple group of order $|N| = 2^t.\ 3.\ 17.\ p^2$ for $p = 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31$, a contradiction. Thus, L is solvable and hence N is a minimal non-abelian simple group, that is, N is a non-abelian simple group and every proper subgroup of N is solvable. By [11, Corollary 1], N is one of the groups in Table I. It can be easily verified that the order of groups in Table I are not of the form $2^t.\ 3.\ 17.\ p^2$.

Thus $|L| = 2^s.\ 3.\ 17.\ p$. By the same argument as in the preceding paragraph (replacing N by L) L is one of the groups in Table I. Since $|L| = 2^s.\ 3.\ 17.\ p$, the possible candidates for L is $PSL_2(m)$. Clearly, $m = p$. We show that $|L| \leq 2,500,224$. If $17 \nmid (p-1)/2$, then $(p-1)/2 \mid 2^7.\ 3$, which implies that $p \leq 769$. If $p = 769$, then $2^8 \mid |L|$, a contradiction. Thus $p < 769$ and hence $p \leq 193$ because $(p-1)/2 \mid 2^7.\ 3$. It follows that $|L| \leq 2^7.\ 3.\ 17.\ 193 = 1,259,904 < 2,500,224$. If $17 \mid (p-1)/2$, Then $p+1 \mid 2^7.\ 3$. Consequently $p \leq 383$, implying $|L| \leq 2^7.\ 3.\ 17.\ 383 \leq 2,500,224$. Thus, $|L| \leq 2,500,224$. Then by [2, pp. 239], is isomorphic to $PSL_2(16)$. It follows that $p = 5$ and hence $|N| = 2^t.\ 3.\ 17.\ 5^2$, which is impossible by [2, pp. 239].

Table I. The possible for non-abelian simple group N.

N	$\|N\|$
$PSL(2,m),\ m>3$ a prime and $m^2 \neq 3 \pmod 5$	$\frac{1}{2}m(m-1)(m+1)$
$PSL(2,2^n)$, n a prime	$2^n(2^{2n}-1)$
$PSL(2,3^n)$, n an odd prime	$\frac{1}{2}3^n(3^{2n}-1)$
$PSL(3,3)$, n a prime	$13.3^3.2^4$
Suzuki group $Sz(2^n)$, n an odd prime	$2^{2n}(2^{2n}+1)(2^n-1)$

Thus N is solvable and hence elementary abelian. Therefore N is intransitive on both $U(X)$ and $W(X)$ and by Proposition 2.4, N is semiregular on $U(X)$ and $W(X)$. Set $Q := O_p(A)$. If $|Q| = p^2$, then by Proposition 2.4, the quotient graph X_Q of X corresponding to the orbits of Q is a A/Q-semisymmetric graph of order 34, which is impossible by [1]. Thus $|Q| = 1$ or p. Suppose first that $Q = 1$. Now we consider the quotient graph X_N of X corresponding to the orbits of N. The semiregularity of N implies that $N \cong \mathbb{Z}_{17}$, because $Q = 1$. By Proposition 2.4 X_N is A/N-semisymmetric. We denote by $U(X_N)$ and $W(X_N)$ the bipartition sets of X_N. Clearly, $|U(X_N)| = |W(X_N)| = p^2$. Let L/N be a minimal normal subgroup of A/N. By the same argument as above we may prove that L/N is solvable and hence elementary abelian, which implies that L/N is intransitive on bipartite sets of X_N. Then by Proposition 2.4, L/N is semiregular on $U(X_N)$ and $W(X_N)$, implying $|L/N| \mid p^2$. Hence $L/N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. Thus $|L| = 17p$ or $17p^2$. Since $p \geq 5$, by Sylow's theorem L has a normal subgroup of order p or p^2, which is characteristic in L and hence is normal in A, because $L \lhd A$. This contradicts our assumption that $Q = 1$.

Suppose now that $Q \cong \mathbb{Z}_p$. Let $C := C_A(Q)$ be the centralizer of Q in A. By Proposition 2.2, $p \nmid |C' \bigcap Z(C)|$ and hence $C' \bigcap Q = 1$, where C' is the derived subgroup of C. This forces $p^2 \nmid |C'|$, because C' is normal in A. It follows that C' is intransitive on $U(X)$ and $W(X)$. As C' is normal in A, by Proposition 2.4, it is semiregular on $U(X)$ and $W(X)$. Moreover, the quotient graph $X_{C'}$ is A/C'-semisymmetric and since $p^2 \nmid |C'|$, the semiregularity of C' implies that $|C'| \mid 17p$. Since the Sylow p-subgroups of A are abelian, one has $p^2 \mid |C|$ and so $p \mid |C/C'|$. Now let K/C' be a Sylow p-subgroup of the abelian group C/C'. As K/C' is characteristic in C/C' and $C/C' \lhd A/C'$, we have that $K/C' \lhd A/C'$. Hence K is normal in A. Clearly $|K| = 17p^2$ because $|Q| = p$. Since $p \geq 5$, K has a normal subgroup of order p^2, which is characteristic in K and hence is normal in A, contradicting to $Q \cong \mathbb{Z}_p$. □

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