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CONVERGENCE RESULTS FOR PROXIMAL POINT ALGORITHM IN COMPLETE CAT(0) SPACE FOR MULTIVALUED MAPPINGS

Samir Dashputre[1], Padmavati[2], Kavita Sakure[3]

[1] **Department of Mathematics, Govt. College, Arjunda, Balod, Chhattisgarh, India, email: samir231973@gmail.com,**
[2] **Department of Mathematics, Govt. V.Y.T. College, Durg, Chhattisgarh, India, email: padmavati.sudha62@gmail.com**
[3] **Department of Mathematics, Govt. Digvijay Auto. P.G. College, Rajnandgaon, Chhattisgarh, India email: kavitaaage@gmail.com**

Abstract. In this paper, we propose the modified proximal point algorithm with the process for three nearly Lipschitzian asymptotically nonexpansive mappings and multivalued mappings in CAT(0) space under certain conditions. We prove some convergence theorems for the algorithm which was introduced by Shamshad Hussain *et al.* [22]. A numerical example is given to illustrate the efficiency of proximal point algorithm for supporting our result.

Key words and Phrases: CAT(0) spaces, Nearly Lipschitzian mappings, Proximal point algorithm, $\Delta-$convergence.

1. Introduction

The proximal point algorithm (PPA) is a method for finding a minimizers of convex lower semicontinuous (lsc) function defined on Hilbert spaces was initiated by Martinet [29] in 1970. The PPA has since become extremely popular among the various researchers inclination in the theory of optimization and also exposed many challenging mathematical problems. The rich literature on this subject has become too extensive (see e.g. [7–9, 12, 13, 17, 18, 21, 24, 25, 36–38]). In particular, the PPA was studied in the framework of Riemannian manifold [10, 20], in Hadamard manifold [4–6, 11, 27, 41] and in CAT(0) space [8, 14–16, 34, 40].

On the otherhand, Markin [28] and Nadler [31] introduced the study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff

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metric. Shimizu *et al.* [39] proved the existence of fixed points for multivalued nonexpansive mappings in convex metric space was established by Shimizu *et al.* [39], i.e. he proved that *every multivalued mapping* $\mathcal{T} : Y \to \mathcal{C}(Y)$ *has a fixed point in a bounded, complete and uniformly convex metric space* (Y, d)*, where* $\mathcal{C}(Y)$ *is family of all compact subsets of* Y. In this direction to generalize the nonlinear multivalued mappings, Kim *et al.* [26] introduced the nearly Lipschitzian multivalued mapping.

In 2019, Hussain *et al.* [22] has been introduced modified proximal point algorithm in complete CAT(0) space (Y, d) as follows : suppose that h is a convex, proper and lower semi-continuous function on Y. The modified proximal point algorithm is given by for $s_1 \in Y$ and $\pi_m > 0$

$$\begin{cases} p_m = \operatorname{argmin}_{r\in Y}\left\{h(r) + \frac{1}{2\pi_m}d^2(r, s_m)\right\} \\ q_m = (1-\gamma_m)p_m \oplus \gamma_m A^m z_m, \\ r_m = (1-\beta_m)q_m \oplus \beta_m B^m y_m, \\ s_{m+1} = (1-\alpha_m)r_m \oplus \alpha_m C^m x_m, \qquad \forall m \in N\ . \end{cases} \tag{1}$$

where $z_m \in P_{\mathcal{T}}(p_m)$, $y_m \in P_{\mathcal{T}}(q_m)$ and $x_m \in P_{\mathcal{T}}(r_m)$ for each $m \in N$. Let $\{a_m\}, \{b_m\}$ and $\{c_m\}$ be a sequence in $[0, 1]$ for all $m \in N$ and $\{\pi_m\}$ be a sequence with $\pi_m > 0$ for all $m \in N$ and established some Δ-convergence theorems of the proposed algorithm to common fixed points of nonexpansive mappings including a total asymptotically nonexpansive mapping, multivalued mapping and minimizer of a convex function.

In the view of above literature, we propose the modified proximal point algorithm with the process for three nearly Lipschitzian asymptotically nonexpansive mappings and multivalued mappings in CAT(0) space under certain conditions. We prove $\Delta-$ convergence, strong and weak convergence results for the algorithm which was defined in (1) by Shamshad Hussain *et al.* [22]. A numerical example is given to illustrate the efficiency of proximal point algorithm for supporting our result.

2. PRELIMINARIES

Throughout in this paper, we assume that

$$G(T) = \{x \in \mathcal{W} : Tx = x\}$$

denote the set of fixed point where $\mathcal{W}$ is subset of CAT(0) space (Y, d) and $\mathcal{T} : \mathcal{W} \to \mathcal{W}$ is a mapping. A metric space (Y, d) is called a CAT(0) space if it is geodesically connected and every geodesic triangle in Y is atleast as **thin** as its comparison triangle in the Euclidean plane.

A subset $\mathcal{W}$ of a CAT(0) space Y is said to be convex, if for any $s, r \in \mathcal{W}$, we have $[s, r] \subset \mathcal{W}$, where

$$[s, r] := \{ts \oplus (1-t)r : 0 \le t \le 1\}$$

is unique geodesic joining s and r. In this paper, we can write $ts \oplus (1-t)r$ for the unique point q in the geodesic segment joining s to r such that

$$d(s,q) = td(s,r), \quad d(r,q) = (1-t)d(s,r)$$

where $t \in [0,1]$.

Definition 2.1. *Let $\{s_m\}$ be bounded sequence in a $\mathrm{CAT}(0)$ space (Y,d). For any $s \in Y$, we put*

$$\hat{r}(s,\{s_m\}) = \lim_{m\to\infty} \sup d(s,s_m).$$

Then,
1. The asymptotic radius of $\hat{r}(\{s_m\})$ of $\{s_m\}$ is given by

$$\hat{r}(\{s_m\}) = \inf\{\hat{r}(s,\{s_m\}) : s \in Y\}.$$

2. The asymptotic center $\mathcal{A}(\{s_m\})$ of $\{s_m\}$ is the set

$$\mathcal{A}(\{s_m\}) = \{s \in Y : \hat{r}(\{s, s_m\}) = \hat{r}(\{s_m\})\}.$$

In complete $\mathrm{CAT}(0)$ space, $\mathcal{A}(\{s_m\})$ consists of exactly one point [15].

Definition 2.2. *A sequence $\{s_m\}$ in a $\mathrm{CAT}(0)$ space (Y,d) is said to be $\Delta-$convergence to a point $s \in Y$, if s is a unique asymptotic center of $\{u_m\}$ for every subsequence $\{u_m\}$ of $\{s_m\}$. In this case, we write $\Delta\lim_{m\to\infty} s_m = s$ of $\{s_m\}$ and denote $\mathcal{W}_{\triangle}(s_m) := \cup\mathcal{A}(\{u_m\})$, where the union is sum over all subsequences $\{u_m\}$ of $\{s_m\}$.*

Lemma 2.3. *Let Y be a geodesic space in $\mathrm{CAT}(0)$ space. For all $s,r,q \in Y$ and $t \in [0,1]$, we have*
(i) $d^2((1-t)s \oplus tr, q) \le (1-t)d^2(s,q) + td^2(r,q) - t(1-t)d^2(s,r)$;
(ii) $d((1-t)s \oplus tr, q) \le (1-t)d(s,q) + td(r,q)$.

Lemma 2.4. *([15]) If $\{s_m\}$ is a bounded sequence in a complete $\mathrm{CAT}(0)$ space with $\mathcal{A}(\{s_m\}) = \{s\}$, $\{u_m\}$ is subspace of $\{s_m\}$ with $\mathcal{A}(\{u_m\}) = \{u\}$, and the sequence $\{d(s_m,u)\}$ converges, then $s = u$.*

Lemma 2.5. *([2]) Assume that a subset of a complete $\mathrm{CAT}(0)$ space (Y,d) is closed, convex and $\mathcal{T} : \mathcal{W} \to \mathcal{W}$ is nearly Lipschitzian mapping. Let $\{s_m\}$ be a bounded sequence in $\mathcal{W}$ such that $\Delta\lim_{m\to\infty} s_m = t$ and $\lim_{m\to\infty} d(s_m, \mathcal{T}s_m) = 0$. Then $\mathcal{T}t = t$.*

Let $\mathcal{CB}(\mathcal{W})$ be a collection of all nonempty and closed bounded subsets and $\mathcal{P}(\mathcal{W})$ be a collection of all nonempty proximal bounded and closed subsets of $\mathcal{W}$, respectively. Let $\mathcal{H}(.,.)$ be the Hausdorff distance on $\mathcal{CB}(\mathcal{W})$ defined by

$$\mathcal{H}(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{s\in\mathcal{A}} \mathrm{dist}(s,\mathcal{B}), \sup_{r\in\mathcal{B}} \mathrm{dist}(r,\mathcal{A})\right\} \qquad \forall \mathcal{A},\mathcal{B} \in \mathcal{CB}(\mathcal{W}).$$

A subset $\mathcal{W} \subset Y \neq \phi$ is said to be proximal if for each $s \in Y$, there exists an element $r \in \mathcal{W}$ such that

$$d(s,r) = \mathrm{dist}(s,\mathcal{W}) = \inf\{d(s,q) : q \in \mathcal{W}\}.$$

It is well known that each weakly compact convex subset of a Banach space is proximal as well as each closed convex subset of a uniformly convex Banach space is also proximal. Many authors have been discussed fixed point in CAT(0) space (see [1, 19, 32]).

Let $\mathcal{T} : Y \to 2^Y$ be a multivalued mapping. An element $s \in Y$ is said to fixed point of $\mathcal{T}$ if $s \in \mathcal{T}s$.

Definition 2.6. *A multivalued mapping $\mathcal{T} : Y \to \mathcal{CB}(Y)$ is called nonexpansive, if for $x, y \in Y$ and for $m \in N$, we have*

$$\mathcal{H}(\mathcal{T}^m s, \mathcal{T}^m r) \leq d(\mathcal{T}s, \mathcal{T}r), \qquad \forall s, r \in Y. \tag{2}$$

Definition 2.7. *[26] A multivalued mapping $\mathcal{T} : Y \to \mathcal{CB}(Y)$ is called nearly Lipschitzian with respect to sequence $\{v_m\}$, if for $x, y \in Y$ and for $m \in N$, there exists a constant $k_m \geq 0$, such that*

$$\mathcal{H}(\mathcal{T}^m s, \mathcal{T}^m r) \leq k_m\big(d(\mathcal{T}s, \mathcal{T}r) + v_m\big), \qquad \forall s, r \in Y. \tag{3}$$

where the sequence $\{v_m\}$ in $[0, \infty)$ such that $\lim_{m\to\infty} v_m = 0$. The infimum of constants k_m in (3) is called the nearly Lipschitzian constant of $\mathcal{T}^m$, denoted by $\eta(\mathcal{T}^m)$.

A multivalued nearly Lipschitzian mapping $\mathcal{T}$ with sequence $(v_m, \eta(\mathcal{T}^m))$ is said to be

1. multivalued nearly nonexpansive, if $\eta(\mathcal{T}^m) = 1$ for all $m \in N$,
2. multivalued nearly asymptotically nonexpansive, if $\eta(\mathcal{T}^m) \geq 1$ for all $m \in N$ and $\lim_{m\to\infty} \eta(\mathcal{T}^m) = 1$,
3. multivalued nearly uniformly k-Lipschitzian, if $\eta(\mathcal{T}^m) \leq k$ for all $m \in N$,
4. multivalued nearly uniformly k-contractive, if $\eta(\mathcal{T}^m) \leq k < 1$ for all $m \in N$.

The following example of nearly Lipschitzian mapping given by Abbas *et al.* [2] as follows.

Example 2.8. *Assume that $A : (0, \infty) \to (0, \infty)$ is defined by*

$$A(s) = \begin{cases} 1 + (s)^{\frac{1}{2}} & s \in (0, 1] \\ 2 & s \in (1, \infty); \end{cases}$$

Similarly, we define here two nearly Lipschitzian mappings in our next two examples as follows.

Example 2.9. *Assume that $B : (0, \infty) \to (0, \infty)$ is defined by*

$$B(s) = \begin{cases} 1 + (s)^{\frac{1}{3}} & s \in (0, 1] \\ 2 & s \in (1, \infty); \end{cases}$$

Here

$$B^m x = B^m y = 2 \quad \forall m \geq 2.$$

Thus

$$d(B^m x, B^m y) \leq k_m(d(x, y) + v_m) \quad \forall m \geq 2$$

is true for $k_m \geq 0$ and for any sequence $\{v_m\}$ in $[0, \infty)$ with $v_m \to 0$.

Example 2.10. *Assume that* $C : (0, \infty) \to (0, \infty)$ *is defined by*

$$C(s) = \begin{cases} 1 + (s)^{\frac{1}{4}} & s \in (0, 1] \\ 2 & s \in (1, \infty); \end{cases}$$

Here

$$C^m x = C^m y = 2 \quad \forall m \geq 2.$$

Thus

$$d(C^m x, C^m y) \leq k_m(d(x, y) + v_m) \quad \forall m \geq 2$$

is true for $k_m \geq 0$ *and for any sequence* $\{v_m\}$ *in* $[0, \infty)$ *with* $v_m \to 0$.

Recall that a function $h : \mathcal{W} \to (-\infty, \infty]$ is convex, if for any geodesic $[s, r] = \{c_{s,r}(a) : 0 \leq a \leq 1\} = \{as \oplus (1 - a)r : 0 \leq a \leq 1\}$ joining $s, r \in \mathcal{W}$, the function $h \circ c$ is convex, i.e.,

$$h(c_{s,r}(a)) = h(as \oplus (1 - a)r) \leq ah(s) + (1 - a)h(r).$$

The Moreau-Yoshida resolvent of function h in the CAT(0) space is given by

$$\mathcal{J}_\pi(x) = \text{argmin}_{r \in Y} \Big[h(r) + \frac{1}{2\pi} d^2(r, s) \Big]$$

for any $\pi > 0$ and for all $s \in Y$.

Remark 2.11. *(1) The resolvent* $\mathcal{J}_\pi$ *of function* h *is nonexpansive for all* $\pi > 0$ *(see [23]).*

(2) If h *is convex, proper and lower semi-continuous function, then the set of fixed point of the resolvent associated with* h *coincides with the set of minimizers of* h *(see [7]).*

Lemma 2.12. *(see [23, 30]) Suppose that* (Y, d) *is a* CAT(0) *space. Then for each* $r, s \in Y$ *and* $\pi > 0$

$$\frac{1}{2\pi} d^2(\mathcal{J}_\pi s, r) - \frac{1}{2\pi} d^2(s, r) + \frac{1}{2\pi} d^2(\mathcal{J}_\pi s, s) \leq h(r) - h(\mathcal{J}_\pi s).$$

Lemma 2.13. *([3]) Suppose that* (Y, d) *is a* CAT(0) *space. Then for each* $s \in Y$ *and* $\pi > \mu > 0$

$$\mathcal{J}_\pi s = \mathcal{J}_\mu \Big(\frac{\pi - \mu}{\pi} \mathcal{J}_\pi s \oplus \frac{\mu}{\pi} s \Big)$$

Lemma 2.14. *([39]) Let* $\{a_m\}$, $\{b_m\}$ *and* $\{c_m\}$ *be sequences of nonnegative real numbers such that*

$$a_{m+1} \leq (1 + b_m)a_m + c_m \quad \forall m \in N.$$

If $\sum_{m=1}^{\infty} b_m < \infty$ *and* $\sum_{m=1}^{\infty} c_m < \infty$, *then* $\lim_{m \to \infty} a_m$ *exists.*

3. Main Results

Theorem 3.1. *Let (Y, d) be a complete* CAT(0) *space and $\mathcal{W}$ be a nonempty closed convex subset of Y. Let $\mathcal{T} : \mathcal{W} \to P(\mathcal{W})$ be multivalued mapping and $P_{\mathcal{T}}$ be a nonexpansive mapping. Let $h : Y \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $A, B, C : \mathcal{W} \to \mathcal{W}$ be three nearly Lipschitzian mappings with $\{k_m\}, \{v_m\}$ being nonnegative real sequences such that $\sum_{m=1}^{\infty} k_m < \infty$ and $\sum_{m=1}^{\infty} \nu_m < \infty$. Let $\mathcal{E} := G(A) \cap G(B) \cap G(C) \cap G(\mathcal{T}) \cap \operatorname{argmin}_{r \in W} h(r) \neq \phi$. Let $\{s_m\}$ be defined by*

$$\begin{cases} p_m = \operatorname{argmin}_{r\in Y}\{h(r) + \frac{1}{2\pi_m} d^2(r, x_m)\} \\ q_m = (1 - c_m)p_m \oplus c_m A^m z_m \\ r_m = (1 - b_m)B^m y_m \oplus b_m q_m \\ s_{m+1} = (1 - a_m)r_m + a_m C^m x_m \qquad m \in N. \end{cases} \tag{4}$$

where $z_m \in P_{\mathcal{T}}(p_m)$, $y_m \in P_{\mathcal{T}}(q_m)$ and $x_m \in P_{\mathcal{T}}(r_m)$ for each $m \in N$. Let $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ be sequences in $[0, 1]$ for all $m \in N$ and $\{\pi_m\}$ be a sequence with $\pi_m > 0$ for all $m \in N$. Then $\lim_{m\to\infty} d(s_m, t)$ exists for all $t \in \mathcal{E}$.

Proof. Since $\mathcal{E} \neq \phi$. So we can assume that $t \in \mathcal{E}$ which implies that $t = At = Bt = Ct$ and $h(t) \leq h(r)$ for any $r \in \mathcal{W}$. Thus, we have

$$h(t) + \frac{1}{2\pi_m} d^2(t, t) \leq h(r) + \frac{1}{2\pi_m} d^2(r, t)$$

for each $r \in W$, and we have $t = \mathcal{J}_{\pi_m} t$ for each $m \in N$. Since $p_m = \mathcal{J}_{\pi_m} s_m$ and $\mathcal{J}_{\pi_m}$ is nonexpansive, so we have

$$d(p_m, t) = d(\mathcal{J}_{\pi_m} s_m, \mathcal{J}_{\pi_m} t) \leq d(s_m, t) \tag{5}$$

Now using (4), (5) and Lemma 2.3, we have

$$\begin{aligned}
d(q_m, t) &= d((1 - c_m)p_m \oplus c_m A^m z_m, t) \\
&\leq (1 - c_m)d(p_m, t) + c_m d(A^m z_m, t) \\
&\leq (1 - c_m)d(p_m, t) + c_m(k_m(d(z_m, t) + v_m)) \\
&= (1 - c_m)d(p_m, t) + c_m k_m d(z_m, t) + c_m k_m v_m \\
&\leq (1 - c_m) \operatorname{dist}(p_m, P_{\mathcal{T}}(t)) + c_m k_m \operatorname{dist}(z_m, P_{\mathcal{T}}(t)) + c_m k_m v_m \\
&\leq (1 - c_m)\mathcal{H}(P_{\mathcal{T}}(p_m), P_{\mathcal{T}}(t)) + c_m k_m \mathcal{H}(P_{\mathcal{T}}(z_m), P_{\mathcal{T}}(t)) + c_m k_m v_m \\
&\leq (1 - c_m)d(p_m, t) + c_m k_m d(p_m, t) + c_m k_m v_m \\
&= (1 - c_m(1 - k_m))d(p_m, t) + c_m k_m v_m \\
&\leq (1 - c_m(1 - k_m))d(s_m, t) + c_m k_m v_m \\
d(q_m, t) &\leq \mathcal{C} d(s_m, t) + \mathcal{G} \qquad (6)
\end{aligned}$$

where $\mathcal{C} = 1 - c_m(1 - k_m)$ and $\mathcal{G} = c_m k_m v_m$.
Now using (4), (6) and Lemma 2.3, we have

$$\begin{aligned}
d(r_m, t) &= d((1 - b_m)B^m y_m \oplus b_m q_m, t) \\
&\leq (1 - b_m)d(B^m y_m, t) + b_m d(q_m, t) \\
&\leq (1 - b_m)(k_m(d(y_m, t) + v_m)) + b_m d(q_m, t) \\
&= (1 - b_m)k_m d(y_m, t) + (1 - b_m)k_m v_m + b_m d(q_m, t) \\
&\leq (1 - b_m)k_m \operatorname{dist}(y_m, P_{\mathcal{T}}(t)) + (1 - b_m)k_m v_m + b_m \operatorname{dist}(q_m, P_{\mathcal{T}}(t)) \\
&\leq (1 - b_m)k_m \mathcal{H}(P_{\mathcal{T}}(y_m), P_{\mathcal{T}}(t)) + (1 - b_m)k_m v_m + b_m \mathcal{H}(\mathcal{T}(q_m), P_{\mathcal{T}}(t)) \\
&\leq (1 - b_m)k_m d(q_m, t) + (1 - b_m)k_m v_m + b_m d(q_m, t) \\
&\leq (1 - b_m(1 - k_m))d(q_m, t) + (1 - b_m)k_m v_m \\
&\leq (1 - b_m(1 - k_m))\big(\mathcal{C}d(s_m, t) + \mathcal{G}\big) + (1 - b_m)k_m v_m \\
d(r_m, t) &\leq \mathcal{B}\mathcal{C}d(s_m, t) + \mathcal{B}\mathcal{G} + \mathcal{F} \qquad (7)
\end{aligned}$$

where $\mathcal{B} = 1 - b_m(1 - k_m)$ and $\mathcal{F} = (1 - b_m)k_m v_m$.
Now using (4), (7) and Lemma 2.3, we have

$$\begin{aligned}
d(s_{m+1}, t) &= d((1 - a_m)r_m \oplus a_m C^m x_m, t) \\
&\leq (1 - a_m)d(r_m, t) + a_m d(C^m x_m, t) \\
&\leq (1 - a_m)d(r_m, t) + a_m(k_m(d(x_m, t) + v_m)) \\
&= (1 - a_m)d(r_m, t) + a_m k_m d(x_m, t) + a_m k_m v_m \\
&\leq (1 - a_m) \operatorname{dist}(r_m, P_{\mathcal{T}}(t)) + a_m k_m \operatorname{dist}(x_m, P_{\mathcal{T}}(t)) + a_m k_m v_m \\
&\leq (1 - a_m)\mathcal{H}(P_{\mathcal{T}}(r_m), P_{\mathcal{T}}(t)) + a_m k_m \mathcal{H}(P_{\mathcal{T}}(x_m), P_{\mathcal{T}}(t)) + a_m k_m v_m \\
&\leq (1 - a_m)d(r_m, t) + a_m k_m d(r_m, t) + a_m k_m v_m \\
&= (1 - a_m + a_m k_m)d(r_m, t) + a_m k_m v_m \\
&= (1 - a_m(1 - k_m))d(r_m, t) + a_m k_m v_m \\
&\leq (1 - a_m(1 - k_m))\Big(\mathcal{B}\mathcal{C}d(s_m, t) + \mathcal{B}\mathcal{G} + \mathcal{F}\Big) + a_m k_m v_m \\
d(s_{m+1}, t) &\leq \mathcal{A}\mathcal{B}\mathcal{C}d(s_m, t) + \mathcal{A}\mathcal{B}\mathcal{G} + \mathcal{A}\mathcal{F} + \mathcal{D} \qquad (8)
\end{aligned}$$

where $\mathcal{A} = 1 - a_m(1 - k_m)$ and $\mathcal{D} = a_m k_m v_m$. Since $\sum_{m=1}^{\infty} k_m < \infty$ and $\sum_{m=1}^{\infty} v_m < \infty$. Therefore $\sum_{m=1}^{\infty} \mathcal{A}\mathcal{B}\mathcal{C} < \infty$ and $\sum_{m=1}^{\infty} \mathcal{A}\mathcal{B}\mathcal{G} + \mathcal{A}\mathcal{F} + \mathcal{D} < \infty$. Thus, from Lemma 2.14 and inequality (8), $\lim_{m\to\infty} d(s_m, t)$ exists and we may assume that

$$\lim_{m\to\infty} d(s_m, t) = k \geq 0. \qquad (9)$$

By (9), $\{s_m\}$ is bounded and therefore $\{p_m\}$, $\{q_m\}$, $\{r_m\}$, $\{A^m s_m\}$, $\{B^m s_m\}$ and $\{C^m s_m\}$ are bounded. □

Theorem 3.2. *Let (Y, d) be a complete* CAT(0) *space and $\mathcal{W}$ be a nonempty closed convex subset of Y. Let $\mathcal{T} : \mathcal{W} \to P(\mathcal{W})$ be multivalued mapping and $P_{\mathcal{T}}$ be a nonexpansive mapping. Let $h : Y \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $A, B, C : \mathcal{W} \to \mathcal{W}$ be three nearly Lipschitzian mappings with $\{k_m\}, \{v_m\}$ being nonnegative real sequences such that $\sum_{m=1}^{\infty} k_m < \infty$,*

$\sum_{m=1}^{\infty} v_m < \infty$ *and* $k_m \leq 1\ \forall m \in N$. *Let* $\mathcal{E} := G(A) \cap G(B) \cap G(C) \cap G(\mathcal{T}) \cap \operatorname{argmin}_{r \in \mathcal{W}} h(r) \neq \phi$. *Let* $\{s_m\}$ *be defined by (4). Then*
(1) $\lim_{m\to\infty} d(s_m, p_m) = 0$;
(2) $\lim_{m\to\infty} d(s_m, As_m) = \lim_{m\to\infty} d(s_m, Bs_m) = \lim_{m\to\infty} d(s_m, Cs_m) = 0$.

Proof. (1) By Lemma 2.12, we get

$$\frac{1}{2\pi_m}\{d^2(p_m,t) - d^2(s_m,t) + d^2(s_m,p_m)\} \leq h(t) - h(p_m).$$

Since $h(t) \leq h(p_m)$ for each $m \in N$, we have

$$d^2(s_m,p_m) \leq d^2(s_m,t) - d^2(p_m,t). \tag{10}$$

From (8) and (9), we have

$$\begin{aligned} \liminf_{m\to\infty} d(s_{m+1},t) &\leq \liminf_{m\to\infty}(1-a_m(1-k_m))d(r_m,t) + \liminf_{m\to\infty} a_m k_m v_m \\ k &\leq \liminf_{m\to\infty} d(r_m,t) \end{aligned} \tag{11}$$

From (7), we have

$$\begin{aligned} \limsup_{m\to\infty} d(r_m,t) &\leq \limsup_{m\to\infty}(1-b_m(1-k_m))\big((1-c_m(1-k_m))d(s_m,t) + c_m k_m v_m\big) \\ &\quad + \limsup_{m\to\infty}(1-b_m)k_m v_m \\ \limsup_{m\to\infty} d(r_m,t) &\leq k. \end{aligned} \tag{12}$$

From (11) and (12), we have

$$\lim_{m\to\infty} d(r_m,t) = k. \tag{13}$$

From (7) and (13), we have

$$\begin{aligned} \liminf_{m\to\infty} d(r_m,t) &\leq \liminf_{m\to\infty}(1-b_m(1-k_m))d(q_m,t) + \liminf_{m\to\infty}(1-b_m)k_m v_m \\ k &\leq \liminf_{m\to\infty} d(q_m,t) \end{aligned} \tag{14}$$

From (6) and (9), we have

$$\begin{aligned} \limsup_{m\to\infty} d(q_m,t) &\leq \limsup_{m\to\infty}(1-c_m(1-k_m))d(s_m,t) + \limsup_{m\to\infty} c_m k_m v_m \\ \limsup_{m\to\infty} d(q_m,t) &\leq k. \end{aligned} \tag{15}$$

From (14) and (15), we have

$$\lim_{m\to\infty} d(q_m,t) = k. \tag{16}$$

From (6) and (16), we have

$$\begin{aligned} \liminf_{m\to\infty} d(q_m,t) &\leq \liminf_{m\to\infty}(1-c_m(1-k_m))d(p_m,t) + \liminf_{m\to\infty} c_m k_m v_m \\ k &\leq \liminf_{m\to\infty} d(p_m,t). \end{aligned} \tag{17}$$

From (6), we have

$$\limsup_{m\to\infty} d(p_m,t) \leq \limsup_{m\to\infty} d(s_m,t) \leq k. \tag{18}$$

From (17) and (18), we have

$$\lim_{m\to\infty} d(p_m, t) = k. \tag{19}$$

So from (10), we have

$$\lim_{m\to\infty} d(s_m, p_m) = 0. \tag{20}$$

(2) Suppose that $\mathcal{E}$ is nonempty, and let $t \in \mathcal{E}$. From (9), $\lim_{m\to\infty} d(s_m, t)$ exists and $\{s_m\}$ is bounded. From (4) and Lemma 2.3, we have

$$\begin{aligned}
d^2(q_m, t) &= d^2((1-c_m)p_m \oplus c_m A^m z_m, t)\\
&\le (1-c_m)d^2(p_m, t) + c_m d^2(A^m z_m, t) - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&\le (1-c_m)d^2(p_m, t) + c_m(k_m d(z_m, t) + v_m)^2 - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m(k_m d(z_m, t) + v_m)^2 + (1-c_m)d^2(p_m, t) - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m(k_m^2 d^2(z_m, t) + v_m^2 + 2k_m v_m d(z_m, t)) + (1-c_m)d^2(p_m, t)\\
&\quad -c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m k_m^2 d^2(z_m, t) + (1-c_m)d^2(p_m, t) + c_m(v_m^2 + 2k_m v_m d(z_m, t))\\
&\quad -c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m k_m^2 \operatorname{dist}(z_m, P_{\mathcal{T}}(t))^2 + (1-c_m)\operatorname{dist}(p_m, P_{\mathcal{T}}(t))^2\\
&\quad +c_m(v_m^2 + 2k_m v_m \operatorname{dist}(z_m, P_{\mathcal{T}}(t))) - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m k_m^2 \mathcal{H}(P_{\mathcal{T}}(p_m), P_{\mathcal{T}}(t))^2 + (1-c_m)\mathcal{H}(P_{\mathcal{T}}(p_m), P_{\mathcal{T}}(t))^2\\
&\quad +c_m(v_m^2 + 2k_m v_m \mathcal{H}(P_{\mathcal{T}}(p_m), P_{\mathcal{T}}(t))) - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m k_m^2 d^2(p_m, t) + (1-c_m)d^2(p_m, t) + c_m(v_m^2 + 2k_m v_m)d(p_m, t)\\
&\quad -c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= c_m d^2(p_m, t) + (1-c_m)d^2(p_m, t) + c_m(v_m^2 + 2k_m v_m)d(p_m, t)\\
&\quad -c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= d^2(p_m, t) + c_m(v_m^2 + 2k_m v_m)d(p_m, t) - c_m(1-c_m)d^2(p_m, A^m z_m)\\
&= d^2(p_m, t) + p v_m - c_m(1-c_m)d^2(p_m, A^m z_m)
\end{aligned}$$

where $p = c_m(v_m + 2k_m)d(p_m, t) > 0$. Therefore,

$$c_m(1-c_m)d^2(A^m z_m, p_m) \le d^2(s_m, t) - d^2(q_m, t) + p v_m.$$

Since $\lim_{m\to\infty} v_m = 0$, we have $c_m(1-c_m)d^2(A^m z_m, p_m) = 0$.
From $\liminf_{m\to\infty} c_m(1-c_m) > 0$, we have

$$\lim_{m\to\infty} d(A^m z_m, p_m) = 0. \tag{21}$$

From (4) and Lemma 2.3, we have

$$
\begin{aligned}
& d^2(r_m,t) = d^2((1-b_m)B^m y_m \oplus b_m q_m, t) \\
\leq \; & (1-b_m)d^2(B^m y_m,t) + b_m d^2(q_m,t) - b_m(1-b_m)d^2(B^m y_m, q_m) \\
\leq \; & (1-b_m)(k_m d(y_m,t)+v_m)^2 + b_m d^(q_m,t) - b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)(k_m d(y_m,t)+v_m)^2 + b_m d^2(q_m,t) - b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)(k_m^2 d^2(y_m,t) + v_m^2 + 2k_m v_m d(y_m,t)) + b_m d^2(q_m,t) \\
& -b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)k_m^2 d^2(y_m,t) + b_m d^2(q_m,t) + (1-b_m)(v_m^2 + 2k_m v_m d(y_m,t)) \\
& -b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)k_m^2 \operatorname{dist}(y_m, P_{\mathcal{T}}(t))^2 + b_m \operatorname{dist}(q_m, P_{\mathcal{T}}(t))^2 \\
& +(1-b_m)(v_m^2 + 2k_m v_m \operatorname{dist}(y_m, P_{\mathcal{T}}(t))) - b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)k_m^2 \mathcal{H}(P_{\mathcal{T}}(y_m), P_{\mathcal{T}}(t))^2 + b_m \mathcal{H}(P_{\mathcal{T}}(q_m), P_{\mathcal{T}}(t))^2 \\
& +(1-b_m)(v_m^2 + 2k_m v_m \mathcal{H}(P_{\mathcal{T}}(q_m), P_{\mathcal{T}}(t))) - b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)k_m^2 d^2(q_m,t) + b_m d^2(q_m,t) + (1-b_m)(v_m^2 + 2k_m v_m)d(q_m,t) \\
& -b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & (1-b_m)d^2(q_m,t) + b_m d^2(q_m,t) + (1-b_m)(v_m^2 + 2k_m v_m)d(q_m,t) \\
& -b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & d^2(q_m,t) + (1-b_m)(v_m^2 + 2k_m v_m)d(q_m,t) \\
& -b_m(1-b_m)d^2(B^m y_m, q_m) \\
= \; & d^2(q_m,t) + q v_m - b_m(1-b_m)d^2(B^m y_m, q_m)
\end{aligned}
$$

where $q = (1-b_m)(v_m + 2k_m)d(q_m,t) > 0$. Therefore

$$b_m(1-b_m)d^2(B^m y_m, q_m) \leq d^2(q_m,t) - d^2(r_m,t) + q v_m.$$

Since $\lim_{m\to\infty} v_m = 0$, we have

$$b_m(1-b_m)d^2(B^m y_m, q_m) = 0.$$

From $\liminf_{m\to\infty} b_m(1-b_m) > 0$, we have

$$\lim_{m\to\infty} d(B^m y_m, q_m) = 0. \tag{22}$$

Similarly from (4) and Lemma 2.3, we have

$$\begin{aligned}
& d^2(s_{m+1},t) = d^2((1-a_m)r_m \oplus a_m C^m x_m, t) \\
\leq\ & (1-a_m)d^2(r_m,t) + a_m d^2(C^m x_m, t) - a_m(1-a_m)d^2(r_m, C^m x_m) \\
\leq\ & (1-a_m)d^2(r_m,t) + a_m(k_m d(x_m,t)+v_m)^2 - a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m(k_m d(x_m,t)+v_m)^2 + (1-a_m)d^2(r_m,t) - a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m(k_m^2 d^2(x_m,t) + v_m^2 + 2k_m v_m d(x_m,t)) + (1-a_m)d^2(r_m,t) \\
& -a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m k_m^2 d^2(x_m,t) + (1-a_m)d^2(r_m,t) + a_m(v_m^2 + 2k_m v_m d(x_m,t)) \\
& -a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m k_m^2 \operatorname{dist}(x_m, P_{\mathcal{T}}(t))^2 + (1-a_m)\operatorname{dist}(r_m, P_{\mathcal{T}}(t))^2 + a_m(v_m^2 \\
& +2k_m v_m \operatorname{dist}(x_m, P_{\mathcal{T}}(t))) - a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m k_m^2 \mathcal{H}(P_{\mathcal{T}}(r_m), P_{\mathcal{T}}(t))^2 + (1-a_m)\mathcal{H}(P_{\mathcal{T}}(r_m), P_{\mathcal{T}}(t))^2 + a_m(v_m^2 \\
& +2k_m v_m \mathcal{H}(P_{\mathcal{T}}(r_m), P_{\mathcal{T}}(t))) - a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m k_m^2 d^2(r_m,t) + (1-a_m)d^2(r_m,t) + a_m(v_m^2 + 2k_m v_m)d(r_m,t) \\
& -a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & a_m d^2(r_m,t) + (1-a_m)d^2(r_m,t) + a_m(v_m^2 + 2k_m v_m)d(r_m,t) \\
& -a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & d^2(r_m,t) + a_m(v_m^2 + 2k_m v_m)d(r_m,t) \\
& -a_m(1-a_m)d^2(r_m, C^m x_m) \\
=\ & d^2(r_m,t) + r v_m - a_m(1-a_m)d^2(r_m, C^m x_m)
\end{aligned}$$

where $r = a_m(v_m + 2k_m)d(r_m,t) > 0$. Therefore

$$a_m(1-a_m)d^2(C^m x_m, r_m) \leq d^2(r_m,t) - d^2(s_{m+1},t) + r v_m.$$

Since $\lim_{m\to\infty} v_m = 0$, we have

$$a_m(1-a_m)d^2(C^m x_m, r_m) = 0.$$

From $\liminf_{m\to\infty} a_m(1-a_m) > 0$, we have

$$\lim_{m\to\infty} d(C^m x_m, r_m) = 0. \tag{23}$$

From (20) and (21), we have

$$\begin{aligned}
d(q_m, s_m) &= d((1-c_m)p_m \oplus c_m A^m z_m, s_m) \\
&\leq (1-c_m)d(p_m,s_m) \oplus c_m d(A^m z_m, s_m) \\
&\leq (1-c_m)d(p_m,s_m) \oplus c_m(d(A^m z_m, p_m) + d(p_m,s_m)) \\
&= (1-c_m)d(p_m,s_m) \oplus c_m d(A^m z_m, p_m) + c_m d(p_m,s_m)) \\
&= d(p_m,s_m) \oplus c_m d(A^m z_m, p_m) \\
&\to 0 \quad as\ m\to\infty.
\end{aligned} \tag{24}$$

From (22) and (24), we have

$$\begin{aligned} d(r_m, s_m) &= d((1-b_m)q_m \oplus b_m B^m y_m, s_m) \\ &\leq (1-b_m)d(q_m, s_m) \oplus b_m d(B^m y_m, s_m) \\ &\leq (1-b_m)d(q_m, s_m) \oplus b_m(d(B^m y_m, q_m) + d(q_m, s_m)) \\ &= (1-b_m)d(q_m, s_m) \oplus b_m d(B^m y_m, q_m) + b_m d(q_m, s_m)) \\ &= d(q_m, s_m) \oplus b_m d(B^m y_m, q_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{25}$$

From (23) and (25), we have

$$\begin{aligned} d(s_{m+1}, s_m) &= d((1-a_m)r_m \oplus a_m C^m x_m, s_m) \\ &\leq (1-a_m)d(r_m, s_m) \oplus a_m d(C^m x_m, s_m) \\ &\leq (1-a_m)d(r_m, s_m) \oplus a_m(d(C^m x_m, r_m) + d(r_m, s_m)) \\ &= (1-a_m)d(r_m, s_m) \oplus a_m d(C^m x_m, r_m) + a_m d(r_m, s_m)) \\ &= d(r_m, s_m) \oplus a_m d(C^m x_m, r_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{26}$$

By the triangular inequality, (20) and (21),

$$\begin{aligned} d(A^m s_m, s_m) &\leq d(A^m s_m, A^m z_m) + d(A^m z_m, p_m) + d(p_m, s_m) \\ &\leq k_m(d(s_m, z_m) + v_m) + d(A^m z_m, p_m) + d(p_m, s_m) \\ &\leq k_m(\mathcal{H}(P_{\mathcal{T}} s_m, P_{\mathcal{T}} z_m) + v_m) + d(A^m z_m, p_m) + d(p_m, s_m) \\ &\leq k_m(d(s_m, p_m) + v_m) + d(A^m z_m, p_m) + d(p_m, s_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{27}$$

By the triangular inequality, (22) and (24),

$$\begin{aligned} d(B^m s_m, s_m) &\leq d(B^m s_m, B^m y_m) + d(B^m y_m, q_m) + d(q_m, s_m) \\ &\leq k_m(d(s_m, y_m) + v_m) + d(B^m y_m, q_m) + d(q_m, s_m) \\ &\leq k_m(\mathcal{H}(P_{\mathcal{T}} s_m, P_{\mathcal{T}} y_m) + v_m) + d(B^m y_m, q_m) + d(q_m, s_m) \\ &\leq k_m(d(s_m, q_m) + v_m) + d(B^m y_m, q_m) + d(q_m, s_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{28}$$

By the triangular inequality, (23) and (25),

$$\begin{aligned} d(C^m s_m, s_m) &\leq d(C^m s_m, C^m x_m) + d(C^m x_m, r_m) + d(r_m, s_m) \\ &\leq k_m(d(s_m, x_m) + v_m) + d(C^m y_m, r_m) + d(r_m, s_m) \\ &\leq k_m(\mathcal{H}(P_{\mathcal{T}} s_m, P_{\mathcal{T}} x_m) + v_m) + d(C^m x_m, r_m) + d(r_m, s_m) \\ &\leq k_m(d(s_m, r_m) + v_m) + d(C^m x_m, r_m) + d(r_m, s_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{29}$$

From (26) and (27), we have

$$\begin{aligned} d(s_m, As_m) &\leq d(s_m, s_{m+1}) + d(s_{m+1}, A^{m+1}s_{m+1}) + d(A^{m+1}s_{m+1}, A^{m+1}s_m) \\ &\quad +d(A^{m+1}s_m, As_m) \\ &\leq d(s_m, s_{m+1}) + d(s_{m+1}, A^{m+1}s_{m+1}) + k_m(d(s_{m+1}, s_m) + v_m) \\ &\quad +k_m(d(A^m s_m, s_m) + v_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{30}$$

From (26) and (28), we have

$$\begin{aligned} d(s_m, Bs_m) &\leq d(s_m, s_{m+1}) + d(s_{m+1}, B^{m+1}s_{m+1}) + d(B^{m+1}s_{m+1}, B^{m+1}s_m) \\ &\quad +d(B^{m+1}s_m, Bs_m) \\ &\leq d(s_m, s_{m+1}) + d(s_{m+1}, B^{m+1}s_{m+1}) + k_m(d(s_{m+1}, s_m) + v_m) \\ &\quad +k_m(d(B^m s_m, s_m) + v_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{31}$$

From (26) and (29), we have

$$\begin{aligned} d(s_m, Cs_m) &\leq d(s_m, s_{m+1}) + d(s_{m+1}, C^{m+1}s_{m+1}) + d(C^{m+1}s_{m+1}, C^{m+1}s_m) \\ &\quad +d(C^{m+1}s_m, Cs_m) \\ &\leq d(s_m, s_{m+1}) + d(s_{m+1}, C^{m+1}s_{m+1}) + k_m(d(s_{m+1}, s_m) + v_m) \\ &\quad +k_m(d(B^m s_m, s_m) + v_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{32}$$

□

Theorem 3.3. *Let* (Y, d) *be a complete* CAT(0) *space and* $\mathcal{W}$ *be a nonempty closed convex subset of* Y*. Let* $\mathcal{T} : \mathcal{W} \to P(\mathcal{W})$ *be multivalued mapping and* $P_{\mathcal{T}}$ *be a nonexpansive mapping. Let* $h : Y \to (-\infty, \infty]$ *be a proper convex and lower semi-continuous function,* $A, B, C : \mathcal{W} \to \mathcal{W}$ *be three nearly Lipschitzian mappings. Then* $\{s_m\}$ *defined in* (4) *is* $\Delta-$*convergent to a common fixed point of* $\mathcal{E}$.

Proof. From Lemma 2.13 and (20), we have

$$\begin{aligned} d(\mathcal{J}s_m, s_m) &\leq d(\mathcal{J}s_m, p_m) + d(p_m, s_m) \\ &= d(\mathcal{J}s_m, \mathcal{J}_{\pi_m}p_m) + d(p_m, s_m) \\ &= d\big(\mathcal{J}s_m, \mathcal{J}_\pi\big(\frac{\pi_m - \pi}{\pi_m}\mathcal{J}_{\pi_m}s_m \oplus \frac{\pi}{\pi_m}\big)\big) + d(p_m, s_m) \\ &\leq d\big(s_m, \big(1 - \frac{\pi}{\pi_m}\big)\mathcal{J}_{\pi_m}s_m \oplus \frac{\pi}{\pi_m}s_m\big) + d(p_m, s_m) \\ &\leq \big(1 - \frac{\pi}{\pi_m}\big)d(s_m, \mathcal{J}_{\pi_m}s_m) + \frac{\pi}{\pi_m}d(s_m, s_m) + d(p_m, s_m) \\ &\leq \big(1 - \frac{\pi}{\pi_m}\big)d(s_m, p_m) + d(p_m, s_m) \\ &\to 0 \quad as\ m \to \infty. \end{aligned} \tag{33}$$

By Theorem 3.1, we have $\lim_{m\to\infty} d(s_m, t)$ exists for all $t \in \mathcal{E}$ and

$$\lim_{m\to\infty} d(s_m, As_m) = \lim_{m\to\infty} d(s_m, Bs_m) = \lim_{m\to\infty} d(s_m, Cs_m) = 0.$$

Now we have to show that

$$\mathcal{W}_{\triangle}(s_m) = \cup_{\{u_m\}\subset\{s_m\}}\mathcal{A}(\{u_m\}) \subset \mathcal{E}.$$

Let $u \in \mathcal{W}_{\triangle}(s_m)$. Then there exists a subsequence $\{u_m\}$ of $\{s_m\}$ such that $\mathcal{A}(u_m\}) = \{u\}$. From Definition 2.2, there exists a subsequence $\{v_m\}$ of $\{u_m\}$ such that $\Delta - \lim_{m\to\infty} v_m = v$ for some $v \in \mathcal{W}$. By Lemma 2.5, $v \in \mathcal{E}$. By Lemma 2.4, $u = v$. This shows that $\mathcal{W}_{\triangle}(\{s_m\}) \subset \mathcal{E}$.

Now we have to prove that the sequence $\{s_m\}$ $\Delta-$converges to a point in $\mathcal{E}$, which will prove that $\mathcal{W}_{\triangle}(\{s_m\})$ consists of exactly one point. Let $\{u_m\}$ be a subsequence of $\{s_m\}$ with $\mathcal{A}(u_m\}) = \{u\}$, and let $\mathcal{A}(s_m\}) = \{s\}$. Since $u \in \mathcal{W}_{\triangle}(\{s_m\}) \subset \mathcal{E}$ and $\{d(s_m, u)\}$ converges by Lemma 2.4, we have $s = u$. Therefore $\mathcal{W}_{\triangle}(\{s_m\}) = \{s\}$. □

Corollary 3.4. *Let (Y, d) be a complete* CAT(0) *space and $\mathcal{W}$ be a nonempty closed convex subset of Y. Let $\mathcal{T} : \mathcal{W} \to P(\mathcal{W})$ be multivalued mapping and $P_{\mathcal{W}}$ be a nonexpansive mapping. Let $h : Y \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $A, B, C : \mathcal{W} \to \mathcal{W}$ be three nearly Lipschitzian mappings and $z_m \in P_{\mathcal{T}}(p_m)$, $y_m \in P_{\mathcal{T}}(q_m)$, $x_m \in P_{\mathcal{T}}(r_m)$, $\{a_m\}$, $\{b_m\}$, $\{c_m\}$ and π_m satisfy all the conditions of Theorem 3.1. Let $\{s_m\}$ be sequence defined by (4). Then the sequence $\{s_m\}$ converges weakly to a common point in $\mathcal{E}$.*

Now we construct and prove strong convergence theorems.

Let $\mathcal{W}$ be a nonempty closed convex subset of CAT(0) space (Y, d). A family $\{A, B, C, \mathcal{T}\}$ of mappings is said to satisfy Condition $(\mathcal{E})$, if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(w) > 0$ for all $w \in (0, \infty)$ such that

$$d(s, As) \geq h(d(s, \mathcal{G})),$$

or

$$d(s, Bs) \geq h(d(s, \mathcal{G})),$$

or

$$d(s, Cs) \geq h(d(s, \mathcal{G})),$$

or

$$d(s, Ts) \geq h(d(s, \mathcal{G})),$$

for all $s \in Y$, where $\mathcal{G} = G(A) \cap G(B) \cap G(C) \cap G(\mathcal{T})$.

Theorem 3.5. *Let (Y, d) be a complete* CAT(0) *space and $\mathcal{W}$ be a nonempty closed convex subset of Y. Let $\mathcal{T} : \mathcal{W} \to P(\mathcal{W})$ be multivalued mapping and $P_{\mathcal{T}}$ be a nonexpansive mapping. Let $h : Y \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function and $A, B, C : \mathcal{W} \to \mathcal{W}$ be three nearly Lipschitzian mappings and $z_m \in P_{\mathcal{T}}(p_m)$, $y_m \in P_{\mathcal{T}}(q_m)$, $x_m \in P_{\mathcal{T}}(r_m)$, $\{a_m\}$, $\{b_m\}$, $\{c_m\}$ and $\{\pi_m\}$ satisfy all the conditions of Theorem 3.1 and $\{A, B, C, \mathcal{J}_\pi\}$ satisfy the Condition $(\mathcal{E})$. Then the sequence $\{s_m\}$ defined in (4) strongly converges to an element of $\mathcal{E}$.*

Proof. From Theorem 3.1, we have $\lim_{m\to\infty} d(s_m, t)$ exists for all $t \in \mathcal{E}$. Also it follows that $\lim_{m\to\infty} d(s_m, \mathcal{E})$ exists. On the otherhand, by Condition ($\mathcal{E}$), we have

$$\lim_{m\to\infty} d(d(s_m, \mathcal{E})) \geq \lim_{m\to\infty} d(s_m, As_m) = 0.$$

or

$$\lim_{m\to\infty} d(d(s_m, \mathcal{E})) \geq \lim_{m\to\infty} d(s_m, Bs_m) = 0.$$

or

$$\lim_{m\to\infty} d(d(s_m, \mathcal{E})) \geq \lim_{m\to\infty} d(s_m, Cs_m) = 0.$$

or

$$\lim_{m\to\infty} d(d(s_m, \mathcal{E})) \geq \lim_{m\to\infty} d(s_m, \mathcal{J}_\pi s_m) = 0.$$

Thus, we have $\lim_{m\to\infty} d(s_m, \mathcal{E}) = 0$. Using the property of h, we have

$$\lim_{m\to\infty} d(s_m, \mathcal{E}) = 0.$$

Therefore, $\{s_m\}$ is a Cauchy sequence in Y, and so $\{s_m\}$ converges to a point $t \in Y$ and hence $d(t, \mathcal{E}) = 0$. Since $\mathcal{E}$ is closed, so we have $t \in \mathcal{E}$. □

Remark 3.6. *(1) Our results extends the results of Hussain et al. [22] in the framework of CAT(0) spaces. They established convergence theorems for different classes of generalized nonexpansive mappings including a total asymptotically nonexpansive mapping, a multivalued mapping, and a minimizer of a convex function for solving the convex minimization problem and the common fixed point problem.*

(2) Our results is generalization of the results of Pakkaranang et al. [33] in the framework of CAT(0) *spaces. They established convergence theorems for three asymptotically quasinonexpansive mappings involving the convex and lower semi-continuous function for solving the convex minimization problem and the common fixed point problem.*

4. Numerical Examples

In this section, we discuss a numerical result to illustrate the convergence of the iterative algorithm (4) to support our example.

Example 4.1. *Consider $Y = R$ with its usual metric, then Y is complete* CAT(0) *space (see [35, Example 3]). Assume that $C = [0, 5000]$. Here C is closed and bounded subset of Y. Let $\mathcal{T} : C \to P(C)$ be a mapping defined by*

$$\mathcal{T}(s) = \left\{\frac{3s+4}{5}\right\} \qquad \forall s \in C. \tag{34}$$

It is clear that the mapping $\mathcal{T}$ is nonexpansive.

TABLE 1. Numerical values of s_m, $\|s_m - s_{m-1}\|_2$ and $h(s_m)$

No. of iterations	s_m	$\|s_m - s_{m-1}\|_2$	$h(s_m)$
m=1	50	-	1269.2
m=2	2.59511	47.4049	3.60534
m=3	1.99745	0.59766	1.99389
m=4	1.99841	0.00095	1.99618
m=5	1.99903	0.00062	1.99767
m=6	1.99931	0.00028	1.99835
m=7	1.99947	0.00015	1.99872
m=8	1.99956	0.00009	1.99895
m=9	1.99962	0.00006	1.99910
m=10	1.99967	0.00004	1.99921
m=11	1.99970	0.00003	1.99929
m=12	1.99973	0.00003	1.99935
m=13	1.99975	0.00002	1.99940
m=14	1.99977	0.00002	1.99944
m=15	1.99978	0.00001	1.99947
m=16	1.99979	0.00001	1.99950
m=17	1.99980	0.00000	1.99952
m=18	1.99981	0.00000	1.99954
m=19	1.99981	0.00000	1.99956
m=20	1.99981	0.00000	1.99956

Now we define a mapping $h : Y \to (-\infty, \infty]$ such that

$$h(s) = \|s\|_1 + \frac{1}{2}\|s\|_2^2 - \frac{3}{5}s - \frac{4}{5}$$

It is easy to check that h is a proper convex and lower semi-continuous function and consider the nearly Lipschitzian mappings A, B, C from definitions (2.8), (2.9) and (2.10), respectively and $\mathcal{T}$ is nonexpansive mapping with $G(A) \cap G(B) \cap G(C) \cap G(\mathcal{T}) = \{2\}$. Suppose that $a_m = \frac{15m-3}{16m}, b_m = \frac{m+5}{22m}$ and $c_m = \frac{33m-7}{34m}$ and $s_1 = 50$ is the initial value. We obtain the numerical results with the errors values in Table 1. From Table 1, Figure 1 and Figure 2, it is clear that the sequence $\{s_m\}$ converges to $1.99999 \equiv 2$ which is common fixed point of solution of a minimizer of a function h, multivalued mapping $\mathcal{T}$ and three nearly Lipschtzian mappings A, B and C.

5. CONCLUSION

In this paper, we proved the $\Delta-$convergence, strong and weak convergence results for the modified proximal point algorithm for three nearly Lipschitzian asymptotically nonexpansive mappings and multivalued mapping in CAT(0) space. Also, we illustrated the efficiency of modified proximal point algorithm by numerical

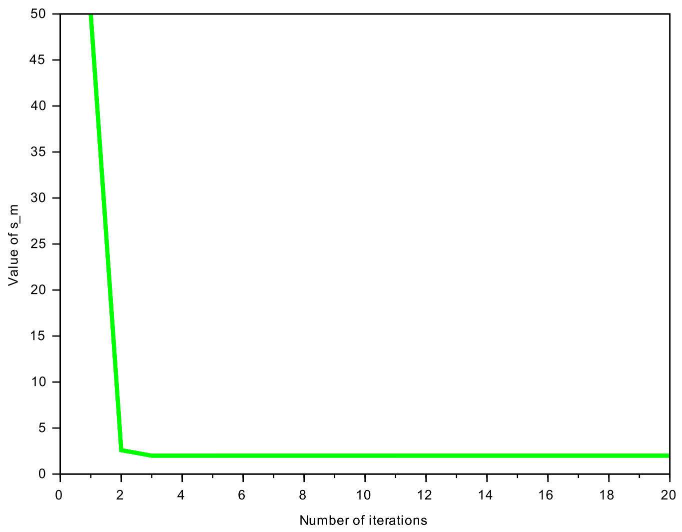
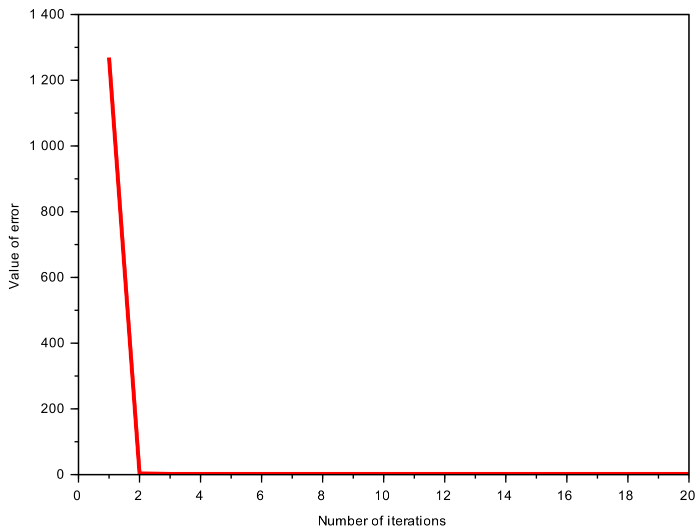


FIGURE 1. Value of s_m



]

FIGURE 2. Value of error

example in CAT(0) space for supporting our results. The results in this paper generalized the results of Hussain *et al.* [22] and Pakkaranang *et al.* [33].

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