J. Indones. Math. Soc.
Vol. 27, No. 01 (2021), pp. 75–89.

H_v-FIELD OF FRACTIONS AND H_v-QUOTIENT RINGS

Mansour Ghadiri

Department of Mathematics, Yazd University, Yazd, Iran,
mghadiri@yazd.ac.ir

Abstract. A larger class of algebraic hyperstructures satisfying the ring (field)-like axioms is the class of H_v-rings (H_v-fields). In this paper, we define the H_v-integral domain and introduce the H_v-field of fractions of an H_v-integral domain. Also, the H_v-quotient ring and some relative theorems are presented. Finally, some interesting results about the H_v-rings of fractions, H_v-quotient rings and the relations between them are proved.

Key words and Phrases: H_v-integral domain, H_v-field of fractions, H_v-normal subgroup, H_v-quotient ring, fundamental relation.

1. Introduction and preliminaries

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the non-empty subsets of H. A hyperoperation on H is a mapping $* : H \times H \longrightarrow \mathcal{P}^*(H)$. The pair $(H, *)$ is called a hypergroupoid. A semi-hypergroup is a hypergroupoid with associative law: $(x * y) * z = x * (y * z)$ for every $x, y, z \in H$; and a hypergroup is a semi-hypergroup with the reproduction axiom: $x * H = H * x = H$ for every $x \in H$. The theory of hyperstructures (hypergroup) was introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematics [7]. This theory has been studied in the following decades and nowadays by many mathematicians. There are applications to the following subjects: geomemtry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets. The concept of H_v-structures as a larger class than the well known hyperstructures was introduced by Vougiouklis in 1990 at Fourth Congress of AHA where the associative law was replaced by the non-empty intersections: $(x * y) * z \cap x * (y * z) \neq \emptyset$ for every $x, y, z \in H$. The basic definitions and results of H_v-structures can be found in [12]. We deal with H_v-rings and H_v-fields. H_v-rings are the largest class of algebraic systems that satisfy ring-like axioms. In

2020 AMS Mathematics Subject Classification: 20N20
Received: 24-08-2019, accepted: 16-02-2021.

[8], Spartalis studied a wide class of H_v-rings resulting from an arbitrary ring by using the p-hyperoperations. Ghadiri, et al. introduced the concepts of direct limit and direct system of H_v-modules on an H_v-rings in [4], and n-ary $P - H_v$-rings in [6]. Darafsheh and Davvaz defined the H_v-ring of fractions of a commutative hyperring which is a generalization of the concept of ring of fractions in [1]. In this paper, we define a zero divisor, an H_v-integral domain and an H_v-field of fractions which are generalization of concepts. If $x \in H$ and $A, B \subseteq H$ then $A * B = \bigcup_{a\in A, b\in B} a * b$, $A * x = A * \{x\}$, $x * B = \{x\} * B$. An H_v-group H is called weak-commutative if $(x * y) \cap (y * x) \neq \emptyset$ for every $x, y \in H$. A non-empty subset K of H is called an H_v-subgroup if $(K, *)$ is an H_v-group. A triple $(R, +, \cdot)$ is called an H_v-ring if $(R, +)$ is an H_v-group, $(R, \cdot)$ is a semi-H_v-group and $\cdot$ is weak distributive with respect to $+$, i.e., $(x \cdot (y + z)) \cap ((x \cdot y) + (x \cdot z)) \neq \emptyset$ and $((x + y) \cdot z) \cap ((x \cdot z) + (y \cdot z)) \neq \emptyset$. A mapping $f : R_1 \longrightarrow R_2$ on H_v-rings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ is called a weak homomorphism if for every $x, y \in R_1$ we have $(f(x +_1 y) \cap (f(x)) +_2 f(y)) \neq \emptyset$, $f(x \cdot_1 y) \cap (f(x) \cdot_2 f(y)) \neq \emptyset$ and is called strong homomorphism if $f(x +_1 y) = f(x) +_2 f(y)$, $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$. For more definitions, results and applications on H_v-rings and H_v-modules, see [1, 3, 4, 6, 8, 10, 11, 13]. The smallest equivalence relation γ^* such that the quotient R/γ^* is a ring, is called the fundamental relation that is the transitive closure of the relation γ defined as follows [10]: let N be the set of natural numbers and the set of all finite polynomials of elements R over N denoted by U (U_R). Now,

$$x\gamma y \Leftrightarrow \{x, y\} \subseteq u \in U.$$

$a\gamma^* b$ if and only if there exist $x_1, x_2, \cdots, x_{m+1}$ in R such that $x_1 = a, x_{m+1} = b$ and there exist $u_1, u_2, \cdots, u_m$ in U such that $\{x_i, x_{i+1}\} \subseteq u_i$ for all $i = 1, 2, \cdots, m$. Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$. On R/γ^*, the operations $\oplus$ and $\odot$ is defined as follows:

$$\begin{aligned} \gamma^*(x) \oplus \gamma^*(y) &= \gamma^*(c), \text{ for all } c \in \gamma^*(x) + \gamma^*(y), \\ \gamma^*(x) \odot \gamma^*(y) &= \gamma^*(d), \text{ for all } d \in \gamma^*(x) \cdot \gamma^*(y). \end{aligned}$$

If $\phi : R \longrightarrow R/\gamma^*$ is the canonical map, then the kernel of ϕ, $\omega_R = \{x \in R |\ \phi(x) = 0\}$ is called core of R and is denoted by ω_R, where 0 is the identity element of the group $(R/\gamma^*, \oplus)$. We have $\omega_R \oplus \gamma^*(x) = \gamma^*(x) \oplus \omega_R = \gamma^*(x)$ and

$$\gamma^*(x + y) = \gamma^*(x) \oplus \gamma^*(y),\ \gamma^*(x \cdot y) = \gamma^*(x) \odot \gamma^*(y),$$

for all $x, y \in R$ and so the map $\phi : R \longrightarrow R/\gamma^*$ defined by $\phi(x) = \gamma^*(x)$ is a strong homomorphism. An H_v-ring can be commutative with respective either " $+$ " or " $\cdot$ "; if it is in both commutative we call it commutative H_v-ring. The expression $(x \in x \cdot u = u \cdot x)$ defines a unit element. A scalar element u is such that $u \cdot x$ and $x \cdot u$ are single element subsets. Thus the scalar unit u is such that $u \cdot x = x \cdot u = \{x\}$. A non-empty subset I of R is called an H_v-ideal if $(I, +)$ is an H_v-subgroup of $(R, +)$ and $I \cdot R \subseteq I$, $R \cdot I \subseteq I$. A non-empty subset S of R is called a strong multiplicatively closed subset (s.m.c.s) if $1 \in S$ and $S \cdot a = a \cdot S \subseteq S$ for all $a \in S$. An H_v-ring is called H_v-field if it's fundamental ring R/γ^* is a field.

The H_v-ring of fractions with relative theorems and results are presented in section 2. Then in section 3, we define the H_v-integral domain and introduce the H_v-field of fraction of an H_v-integral domain. In section 4, it is considered an H_v-ideal I of an H_v-ring R and we introduce the H_v-quotient ring R/I, then find out the fundamental relation of R/I. Also, some theorems that present the relation between H_v-field of fractions and H_v-quotient ring are proved.

2. H_v-Ring of Fractions

Throughout this paper we let R be a commutative hyperring with scalar unit 1 and S is a s.m.c.s. of R. It is denoted the operations sum and product of all rings R, $S^{-1}R$ and quotient rings by $+$, $\cdot$ and we use index for the hyperoperations with the same symbols where is any ambiguity, like $\oplus_R$, $\oplus_{S^{-1}R}$, $+_R$ and $+_{S^{-1}R}$.

For $A \subseteq R$ and $B \subseteq S$, it is denoted the set $\{(a,b)|\ a \in A, b \in B\}$ by (A,B). The relation $\sim$ is defined on $\mathcal{P}^*(R) \times \mathcal{P}^*(S)$ as follows:
$(A,B) \sim (C,D)$ if and only if there exists a subset X of S such that

$$X \cdot (A \cdot D) = X \cdot (B \cdot C).$$

The relation $\sim$ is an equivalence relation on $\mathcal{P}^*(R)\times\mathcal{P}^*(S)$. Also, for $(r,s),(r_1,s_1) \in R\times S$, define $(r,s) \sim (r_1,s_1)$ if and only if there exists $A \subseteq S$ such that $A\cdot(r\cdot s_1) = A\cdot(r_1\cdot s)$, then the relation $\sim$ is an equivalence relation on $R\times S$. The equivalence class containing (r,s) in $R\times S$ is denoted by $[r,s]$ and the set of all the equivalence classes by $S^{-1}R$. The equivalence class containing (A,B) in $\mathcal{P}^*(R)\times\mathcal{P}^*(S)$ is denoted by $\|A,B\|$. It is defined:

$$\ll A,B \gg = \bigcup_{(A_1,B_1)\in\|A,B\|} \{[a_1,b_1]|\ a_1 \in A_1, b_1 \in B_1\},$$

and so $\ll r,s \gg = \ll r.s', s.s' \gg$.
The set $S^{-1}R$ with the following hyperoperation:

$$\begin{aligned}[r_1,s_1]\oplus[r_2,s_2] &= \textstyle\bigcup_{(A,B)\in\|r_1\cdot s_2+r_2\cdot s_1, s_1\cdot s_2\|}\{[r,s]|\ r\in A, s\in B\}\\ &= \ll r_1\cdot s_2 + r_2\cdot s_1, s_1\cdot s_2 \gg,\\ [r_1,s_1]\otimes[r_2,s_2] &= \textstyle\bigcup_{(A,B)\in\|r_1\cdot r_2, s_1\cdot s_2\|}\{[r,s]|\ r\in A, s\in B\}\\ &= \ll r_1\cdot r_2, s_1\cdot s_2 \gg .\end{aligned}$$

is an H_v-ring, which is called the H_v-ring of fractions of R [1].
For simplicity, we denote the γ^*_R, $\gamma^*_{S^{-1}R}$ and $U_{S^{-1}R}$ with γ^*, γ^*_s and U_s, respectively.

Lemma 2.1. (i) *If $u \in U_R$ then $\ll u,1 \gg \in U_S$,*
(ii) *For $r_1,r_2 \in R$; $\gamma^*(r_1) = \gamma^*(r_2)$ implies $\gamma^*_s([r_1,1]) = \gamma^*_s([r_2,1])$.*

Proof. The proofs of (i) follows from definitions. For (ii), let $\gamma^*(r_1) = \gamma^*(r_2)$, so there exist $x_1,\cdots,x_{m+1} \in R$ and $u_1,\cdots,u_m \in U$ such that $\{x_i,x_{i+1}\} \subseteq u_i$ for $i = 1,\cdots,m$. So $\{[x_i,1],[x_{i+1},1]\} \subseteq \ll u_i,1 \gg \in U_S$ and thus $\gamma^*_s([r_1,1]) = \gamma^*_s([r_2,1])$. $\square$

This Lemma is used in the proof of the following Theorem. Also we refer to this Lemma in the next sections.

Theorem 2.2. [1] *The following diagram is a commutative diagram of H_v-homomorphisms and H_v-rings,*

$$\begin{array}{ccc} R & \xrightarrow{\;h\;} & S^{-1}R \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi_s} \\ R/\gamma_R^* & \xrightarrow[h_s]{} & S^{-1}R/\gamma_{S^{-1}R}^* \end{array}$$

where φ and φ_s are canonical maps, $h(r) = [r, 1]$ and $h_s(\gamma^*(r)) = \gamma_s^*([r, 1])$.

Corollary 2.3. *If $r \in \omega_R$ and $s \in S$ then $[r, s] \in \omega_{S^{-1}R}$.*

Proof. For $r \in \omega_R$ and h_s in Theorem 2.2, we have $\gamma^*(r) = \omega_R$ and $\gamma_s^*([r, 1]) = h_s(\gamma^*(r)) = h_s(\omega_R) = \omega_{S^{-1}R}$, because h_s is a homomorphism of rings. So for $r \in \omega_R$ and $s \in S$,

$$\gamma_s^*([r, s]) = \gamma_s^*([r, 1] \cdot [1, s]) = \gamma_s^*([r, 1]) \odot \gamma_s^*([1, s]) = \omega_{S^{-1}R} \odot \gamma_s^*([1, s]) = \omega_{S^{-1}R}.$$

Therefore $[r, s] \in \omega_{S^{-1}R}$. □

3. H_v-INTEGRAL DOMAIN AND H_v-FIELD OF FRACTIONS

To introduce the H_v-field of fraction of a hyperring we need to define the concepts: of an H_v-zero divisor, an H_v-integral domain and an invertible element. Also we need an extension of H_v-ideal which is called weak ideal, for completeness of H_v-fields argument.

Lemma 3.1. *ω_R is an H_v-ideal of R.*

Proof. Let $x, y \in \omega_R$ and $r \in R$, we have:

$$\gamma^*(x + y) = \gamma^*(x) \oplus \gamma^*(y) = \omega_R \oplus \omega_R = \omega_R.$$

So $x+y \subseteq \omega_R$. Also there exits $z \in R$ such that $y \in x+z$. So $\gamma^*(y) = \gamma^*(x) \oplus \gamma^*(z)$, $\omega_R = \omega_R \oplus \gamma^*(z) = \gamma^*(z)$, then $z \in \omega_R$

On the other hand we have:

$$\gamma^*(r \cdot x) = \gamma^*(r) \odot \gamma^*(x) = \gamma^*(r) \odot \omega_R = \omega_R.$$

So $R \cdot \omega_R \subseteq \omega_R$ and similarly $\omega_R \cdot R \subseteq \omega_R$. □

Definition 3.2. *Let $a \in R$, a is called a zero-divisor of R if there exists $b \in R - \omega_R$ such that $a \cdot b \subseteq \omega_R$. The set of all zero-divisors of R denoted by $Z(R)$.*

Example 3.3. *For $x \in \omega_R$ we have*

$$\gamma^*(x \cdot a) = \gamma^*(x) \odot \gamma^*(a) = \omega_R \odot \gamma^*(a) = \omega_R, \text{ for every } a \in R,$$

so $x \cdot a \subseteq \omega_R$ and $x \in Z(R)$. Therefore $\omega_R \subseteq Z(R)$.

Definition 3.4. *A commutative hyper(H_v-)ring R where $1_R \notin \omega_R$* or *$R \neq \omega_R$ is called a non-trivial hyper(H_v-) ring. Also, a non-trivial hyper (H_v-)ring is called a hyper (H_v-) integral domain if $Z(R) = \omega_R$.*

Lemma 3.5.

(i) *R is an H_v-integral domain if and only if R/γ^* is an integral domain.*

(ii) *Every H_v-field is an H_v-integral domain.*

(iii) *R is an H_v-integral domain if and only if for every $a, b, c \in R - \omega_R$, $a \cdot b = a \cdot c \Rightarrow \gamma^*(b) = \gamma^*(c)$.*

Proof. (i) Suppose R is an H_v-integral domain, it is clear that $\gamma^*(1)$ is an unit in R/γ^*. Since $1_R \notin \omega_R$, then $\gamma^*(1) \neq \omega_R$ and $R/\gamma^* \neq \{\omega_R\}$. If $\gamma^*(r) \in Z(R/\gamma^*)$ then there exists $\gamma^*(x) \in R/\gamma^*$ such that $\gamma^*(x) \neq \omega_R$ and $\gamma^*(r \cdot x) = \gamma^*(r) \odot \gamma^*(x) = \omega_R$, thus $r \cdot x \subseteq \omega_R$. From $\omega_R \neq \gamma^*(x)$ we have $x \notin \omega_R$ and since R is an H_v-integral domain, $r \in Z(R) = \omega_R$, so $\gamma^*(r) = \omega_R$. Therefore $Z(R/\gamma^*) = \{\omega_R\}$ and R/γ^* is an integral domain. The converse is similarly. By using (i) the proofs of (ii) and (iii) are straightforward and omitted. □

Definition 3.6. *The element $x \in R$ is called invertible if there exists $y \in R$ such that $1 \in x \cdot y$.*

Example 3.7. *For every H_v-integral domain R;*

(i) *Every element of a s.m.c.s. of R is invertible,*

(ii) *If $t \notin \omega_R$, let us denote $t^0 = 1, t^1 = t, t^2 = t \cdot t, \cdots, t^n = t \cdot t^{n-1}$ and $S_t = \bigcup_{n \in N_0} t^n$, then S_t is a s.m.c.s. of R.*

Theorem 3.8. *Let R be a non-trivial hyperintegral domain and $S = R - \omega_R$. Then $S^{-1}R$ is an H_v-integral domain.*

Proof. First, we prove that $S = R - \omega_R$ is a s.m.c.s of R. For $x, y \in S$, $x, y \notin \omega_R$ then $\gamma^*(x) \neq \omega_R \neq \gamma^*(y)$. By (i) of Lemma 3.5 $x \cdot y \nsubseteq \omega_R$. Then $x \cdot y \cap \omega_R = \emptyset$ and $x \cdot y \subseteq R - \omega_R = S$; because, if $t \in x \cdot y \cap \omega_R$, we have $r^*(t) = r^*(x \cdot y) = \omega_R$ and so $x \cdot y \subseteq \omega_R$, that it is a contradiction by $x \cdot y \nsubseteq \omega_R$. Also by reproduction axiom there exists $z \in R$ such that $x \in y.z$. If $z \in \omega_R$, then $\omega_R = \gamma^*(y) \odot \gamma^*(z) = \gamma^*(y \cdot z) = \gamma^*(x)$, that is a contradiction. So $z \in R - \omega_R = S$, and $y \cdot S = S$. Also $1 \in R - \omega_R = S$.

Now we show that; if $[a, s] \cdot [b, t] \subseteq \omega_{S^{-1}R}$, then $[a, s]$ or $[b, t]$ is in $\omega_{S^{-1}R}$. $[a, s] \cdot [b, t] \subseteq \omega_{S^{-1}R}$ implies $\omega_{S^{-1}R} = \gamma_s^*([c, d])$ for $c \in \gamma^*(a \cdot b)$, $d \in \gamma^*(s \cdot t)$. For $c \in \gamma^*(a \cdot b)$, we consider two cases; $c \in \omega_R$ and $c \notin \omega_R$. The second case is not possible. Because if $c \notin \omega_R$, $c \in R - \omega_R = S$ and $\omega_{S^{-1}R} = \gamma_s^*([c, d]) \odot \gamma_s^*([d, c]) = \gamma_s^*([c, d] \cdot [d, c]) = \gamma_s^*([t, t])$, for t $\in \gamma^*(c \cdot d)$.

For t' as an invertible of t we have

$$\gamma_s^*([1, 1]) = \gamma_s^*([t' \cdot t'] \cdot [t, t]) = \gamma_s^*([t' \cdot t']) \odot \gamma_s^*([t, t]) = \omega_{S^{-1}R},$$

that is a contradiction and so $c \in \omega_R$.

If $c \in \omega_R$, then $\gamma^*(a \cdot b) = \gamma^*(c) = \omega_R$ and $a \cdot b \subseteq \omega_R$, so $a \in \omega_R$ or $b \in \omega_R$, since R is hyperintegral domain, therefore by corollary 2.3, $[a, s] \in \omega_{S^{-1}R}$ or $[b, t] \in \omega_{S^{-1}R}$ and $S^{-1}R$ is an H_v-integral domain. □

Lemma 3.9. *Let R be an H_v-ring, then*

(i) *$x \in R$ is invertible if and only if $\gamma^*(x)$ is invertible in R/γ^*,*

(ii) *R is an H_v-field if and only if every element of $R - \omega_R$ is invertible,*

(iii) *if R is an H_v-integral domain then $a, b \in R - \omega_R$ if and only if $a \cdot b \nsubseteq \omega_R$.*

Proof. The proof follows from definitions and (i) of Lemma 3.5 immediately. □

Theorem 3.10. *Let R be a hyper integral domain with scalar unit. If $S = R - \omega_R$ then $(S^{-1}R, \oplus, \otimes)$ is an H_v-field. This H_v-field is called the H_v-field of fractions of R.*

Proof. By Lemma 3.9 $S^{-1}R$ is an H_v-integral domain and an H_v-ring of fractions. So by Lemma 3.9 it is enough to prove $S^{-1}R$ has unit element and every element of $S^{-1}R - \omega_{S^{-1}R}$ is invertible. We know that $[a, b] \in \ll a, b \gg = \ll 1 \cdot a, 1 \cdot b \gg = \bigcup_{(A,B) \in \| 1 \cdot a, 1 \cdot b \|} \{[x, y] | \ x \in A, y \in B\} = [1, 1] \otimes [a, b]$. By corollary 2.3 and (iii) of Lemma 3.9, if $[a, b] \in S^{-1}R - \omega_{S^{-1}R}$ then $a, b \notin \omega_R$ and $a \cdot b \nsubseteq \omega_R$. So $[b, a] \in S^{-1}R$. Then $[1, 1] \in \ll 1, 1 \gg = \ll a \cdot b, a \cdot b \gg = [a, b] \otimes [b, a]$. Therefore, $[a, b]$ is invertible and $S^{-1}R$ is an H_v-field. □

Definition 3.11. *A subset L of H_v-ring R is called weak-ideal (w-ideal) of R if $\gamma^*(L)$ is an ideal of R/γ^*.*

Lemma 3.12. *Let R be an H_v-ring with fundamental relation γ^* and I be an H_v-ideal of R, then $\gamma^*(I)$ is an ideal of R/γ^*.*

Proof. If $\gamma^*(x), \gamma^*(y) \in \gamma^*(I)$ then there exist $i_1, i_2 \in I$ such that

$$\gamma^*(x) = \gamma^*(i_1),\ \gamma^*(y) = \gamma^*(i_2).$$

So, $\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(i_1) \oplus \gamma^*(i_2) = \gamma^*(i)$ for some $i \in i_1 + i_2$. Thus $\gamma^*(x) \oplus \gamma^*(y) \in \gamma^*(I)$.

For associativity law, let $\gamma^*(x), \gamma^*(y), \gamma^*(z) \in \gamma^*(I)$. We have:

$$\gamma^*(x) \oplus (\gamma^*(y) \oplus \gamma^*(z)) = \gamma^*(x + (y + z)),$$

$$(\gamma^*(x) \oplus \gamma^*(y)) \oplus \gamma^*(z) = \gamma^*((x + y) + z).$$

Since I is an H_v-group, we have $(x + (y + z)) \cap ((x + y) + z) \neq \emptyset$. On the other hand, the left sides of above equations are single, so we obtain:

$$\gamma^*(x) \oplus (\gamma^*(y) \oplus \gamma^*(z)) = (\gamma^*(x) \oplus \gamma^*(y)) \oplus \gamma^*(z).$$

Suppose $\gamma^*(x) = \gamma^*(i_1) \in \gamma^*(I)$ where $i_1 \in I$. By reproduction axiom of I there exists an $i \in I$ such that $i_1 \in i_1 + i$. Thus $\gamma^*(i_1) = \gamma^*(i_1) \oplus \gamma^*(i)$ and $\omega_R = \gamma^*(i) \in \gamma^*(I)$ and so $\gamma^*(I)$ has zero element. If $\gamma^*(y) = \gamma^*(i_2) \in \gamma^*(I)$ where $i_2 \in I$, there exists $i_3 \in I$ such that $i \in i_2 + i_3$. So $\omega_R = \gamma^*(i) = \gamma^*(i_2) \oplus \gamma^*(i_3)$ and $\gamma^*(i_3)$ is the inverse of $\gamma^*(i_2)$ in $\gamma^*(I)$. So $\gamma^*(I)$ is a subgroup of R/γ^*. Finally, since I is an H_v-ideal of R, for every $r \in R$ we have $\gamma^*(r) \odot \gamma^*(I) = \gamma^*(r \cdot I) \in \gamma^*(I)$. Therefore $\gamma^*(I)$ is an ideal of R/γ^*. □

Example 3.13. *(i) Every H_v-ideal is w-ideal,(ii) For $a \in R$, $Ra = \bigcup_{r \in R} r \cdot a$ is a w-ideal, that is not H_v-ideal necessary.*

Example 3.14. *By Theorem 3.2.2 of* [12], *if $(H, +)$ be an H_v-group, then for every hyperoperation "$\cdot$" such that $\{x, y\} \subset x \cdot y$ for every $x, y \in H$, the hyperstructure $(H, +, \cdot)$ is an H_v-ring. So $R = \{a, b, c, d\}$ with the following hyperoperations is an H_v-ring:*

$+$	a	b	c	d
a	a,b	a,b	c	d
b	a,b	a,b	c	d
c	c	c	d	a,b
d	d	d	a,b	c

$\cdot$	a	b	c	d
a	a	a,b	a,c	a,d
b	a,b	b	b,c	b,d
c	a,c	b,c	c	c,d
d	a,d	b,d	c,d	d

Now $I = \{a, b, c\}$ is a w-ideal ($\gamma^*(I) = R$) but it is not an H_v-ideal ($c + c = d \notin I$).

Definition 3.15. *The H_v-ring R is called strong H_v-ring (s-H_v-ring) if every w-ideal of R be an H_v-ideal of R.*

Theorem 3.16. *The non-trivial s-H_v-ring R is an H_v-field if and only if R and ω_R are only w-ideals of R.*

Proof. Suppose R and ω_R are only w-ideals of R, we show that every element of $R-\omega_R$ is invertible. For $a \in R-\omega_R$ we consider the w-ideal $R\cdot a$. If $R\cdot a=\omega_R$ then $\{a\}=1\cdot a \subset R\cdot a=\omega_R$ that is contradiction. So $R\cdot a\neq \omega_R$ and $R\cdot a=R$. Thus $1\in R\cdot a$ *i.e.* $1\in x\cdot a$ for some $x\in R$. Therefore, a is invertible and by Lemma 3.9 R is an H_v-field.

Conversely, suppose R is an H_v-field *i.e.* R/γ^* is a field and every element of $R-\omega_R$ is invertible. Let L be a w-ideal of R such that $\omega_R\subseteq L$. Suppose $a\in L-\omega_R$, then $a\in R-\omega_R$ and there exists $b\in R$ such that $1\in a\cdot b\subseteq R\cdot a\subseteq L$ (since R is a s-H_v-ring, every w-ideal is an H_v-ideal and so $R\cdot a\subseteq L$), so $L=R$. Therefore, R and ω_R are only w-ideals of R. □

Example 3.17. *Every H_v-field is a s-H_v-ring. The H_v-ring R in the Example 3.14 is not s-H_v-ring.*

4. H_v-Quotient Ring

In this section, we build the quotient H_v-ring by using γ^*, the fundamental relation of ring.

Theorem 4.1. *Let $(R,+,\cdot)$ be a commutative H_v-ring and I be an H_v-ideal of R. Define the hyperoperations $+$ and $\times$ on $R/I=\{r+I|\ r\in R\}$ as the following:*

$$\begin{aligned}(r+I)+(r^{'}+I) &= \{x+I|\ x\in\gamma^*(r+r^{'}+I)\}=\gamma^*(r+r^{'}+I)+I,\\ (r+I)\times(r^{'}+I) &= \{x+I|\ x\in\gamma^*((r\cdot r^{'})+I)\}=\gamma^*(r\cdot r^{'}+I)+I.\end{aligned}$$

then $(R/I,+,\times)$ is an H_v-ring, this is called H_v-quotient ring R on I.

Proof. We show that " $\times$ " is a well defined hyperoperation. First note that

$$\begin{aligned}\gamma^*(r_1\cdot r_2)\oplus\gamma^*(I) &= \{\gamma^*(r_1\cdot r_2)\oplus\gamma^*(i)|\ i\in I\}\\ &= \{\gamma^*(r_1\cdot r_2+i)|\ i\in I\}\\ &= \gamma^*(r_1\cdot r_2+I).\end{aligned}$$

Suppose $r_1+I=r_1^{'}+I$ and $r_2+I=r_2^{'}+I$ then $\gamma^*(r_i)\oplus\gamma^*(I)=\gamma^*(r_i^{'})\oplus\gamma^*(I)$ for $i=1,2$ and they are elements of ordinary quotient ring $\frac{R/\gamma^*}{\gamma^*(I)}$. Thus

$$\begin{gathered}(\gamma^*(r_1)\oplus\gamma^*(I))\otimes(\gamma^*(r_2)\oplus\gamma^*(I))=(\gamma^*(r_1^{'})\oplus\gamma^*(I))\otimes(\gamma^*(r_2^{'})\oplus\gamma^*(I))\\ (\gamma^*(r_1)\otimes\gamma^*(r_2))\oplus\gamma^*(I))=(\gamma^*(r_1^{'})\otimes\gamma^*(r_2^{'}))\oplus\gamma^*(I))\\ \gamma^*(r_1\cdot r_2)\oplus\gamma^*(I)=\gamma^*(r_1^{'}\cdot r_2^{'})\oplus\gamma^*(I)\\ \gamma^*(r_1\cdot r_2+I)=\gamma^*(r_1^{'}\cdot r_2^{'}+I)\\ \gamma^*(r_1\cdot r_2+I)+I=\gamma^*(r_1^{'}\cdot r_2^{'}+I)+I\\ (r_1+I)\times(r_2+I)=(r_1^{'}+I)\times(r_2^{'}+I).\end{gathered}$$

One can similarly investigate the other axioms in order to R/I is an H_v-ring. □

Proposition 4.2. *If I and J are H_v-ideals of the H_v-ring R and $\gamma^*(J) \subseteq I$, then $\frac{I}{J}$ is an H_v-ideal of $\frac{R}{J}$.*

Proof. If $i_1, i_2 \in I$, then $(i_1 + J) + (i_2 + J) = \{x + I|\ x \in \gamma^*(i_1 + i_2 + J)\} \subseteq \dfrac{I}{J}$. Let $x + J, y + J \in \frac{I}{J}$ by the reproduction axiom for I, there exists $z \in I$ such that $x \in y + z$, so $x + J \in (y + J) + (z + J)$. Now for every $r + J \in \frac{R}{J}$ and $i + J \in \frac{I}{J}$ we have $(r + J) \times (i + J) = \{x + J|\ x \in \gamma^*((r \cdot i) + J)\} \subseteq \frac{I}{J}$, since I is an H_v-ideal. Similarly $(i + J) \times (r + J) \subseteq \frac{I}{J}$. Therefore $\frac{I}{J}$ is an H_v-ideal of $\frac{R}{J}$. □

Definition and Lemma 4.3. [1] *An H_v-ideal I is called an H_v-isolated ideal if it satisfies the following axiom:*

$$\text{For all } X \subseteq I,\ Y \subseteq S \text{ if } (M, N) \in \|X, Y\|, \text{ then } M \subseteq I.$$

For H_v-isolated ideal I of R, $S^{-1}I = \{[a, s]|\ a \in I, s \in S\}$ is an H_v-ideal of $S^{-1}R$.

So, if I is an H_v-isolated ideal of R, then $\frac{S^{-1}R}{S^{-1}I}$ is an H_v-ring. Now by the following Lemma, we build a commutative diagram that relate the H_v-quotient rings and the H_v-ring of fractions.

It is straightforward to see that every element of $U_{R/I}$ is of the form $\gamma^*(u_i + I) + I$ for $u_i \in U$. So every expression of finite hyperoperations applied on finite subsets of R/I is equal to $\gamma^*(u + I) + I$ for some $u \in U$.

Lemma 4.4. *If γ^* and γ^*_I are the fundamental relations of H_v-rings R and R/I respectively, then $\gamma^*_I(r_1 + I) = \gamma^*_I(r_2 + I)$ if and only if $\gamma^*(r_1 + I) = \gamma^*(r_2 + I)$.*

Proof. For some $r_1, r_2 \in R$, suppose $\gamma^*_I(r_1 + I) = \gamma^*_I(r_2 + I)$ then there exist $u_1, u_2, \cdots, u_m \in U$ and $x_1, x_2, \cdots, x_{m+1} \in R$ such that

$$x_1 + I = r_1 + I,\ x_{m+1} + I = r_2 + I,$$

$$\{x_i + I, x_{i+1} + I\} \subseteq u_i + I \text{ for } i = 1, 2, \cdots, m.$$

Thus

$$\gamma^*(x_1) \oplus \gamma^*(I) = \gamma^*(r_1) \oplus \gamma^*(I),\ \gamma^*(x_{m+1}) \oplus \gamma^*(I) = \gamma^*(r_2) \oplus \gamma^*(I),$$

$$\{\gamma^*(x_1) \oplus \gamma^*(I), \gamma^*(x_{i+1}) \oplus \gamma^*(I)\} \subseteq \gamma^*(u_i + I) \oplus \gamma^*(I) \text{ for } u_i \in U.$$

Let for $i = 1, 2, \cdots, m,\ u_i \in \sum^{n_i}[r_{i1} \cdots r_{ik_i} \cdot \sum^{j_k} u_{ij}]$ where $u_{ij} \in R,\ j = 1, 2, \cdots, j_k,\ k = 1, 2, \cdots, n_i$. Note that in this combination for u_i the order of hyperoperation omitted because this order is not important in $\gamma^*(u_i)$. Now by properties of fundamental relation, we have

$$\gamma^*(u_i) = \oplus^{n_i}[\gamma^*(r_{i_1}) \odot \cdots \odot \gamma^*(r_{ik_i}) \odot (\oplus^{j_k} \gamma^*(u_{ij}))] = \gamma^*(t_i) \text{ for every } t_i \in u_i.$$

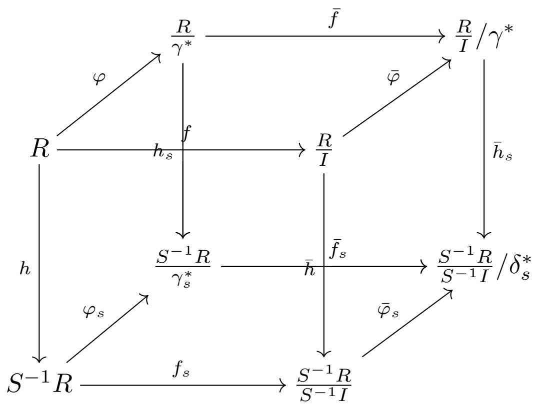
Since $\gamma^*(I)$ is an ideal of $\gamma^*(R)$ then $\gamma^*(x_i)+\gamma^*(I)$, $\gamma^*(u_i)\oplus\gamma^*(I)$ and $\gamma^*(t_i)+\gamma^*(I)$ are cosets of $\gamma^*(I)$ in R/γ^*, thus

$$\gamma^*(x_i)\oplus\gamma^*(I)=\gamma^*(x_{i+1})\oplus\gamma^*(I)=\gamma^*(t_i)\oplus\gamma^*(I) \text{ for } i=1,2,\cdots,m.$$

Therefore $\gamma^*(r_1)\oplus\gamma^*(I)=\gamma^*(r_2)\oplus\gamma^*(I)$.

Conversely, if $\gamma^*(r_1)\oplus\gamma^*(I)=\gamma^*(r_2)\oplus\gamma^*(I)$ then $\gamma^*(r_1+I)=\gamma^*(r_2+I)$. So for every $s_1\in r_1+I$ there exists $s_2\in r_2+I$ such that $\gamma^*(s_1)=\gamma^*(s_2)$. Thus there exist $x_1,x_2,\cdots,x_{m+1}\in I,\ u_1,u_2,\cdots,u_m\in U$ such that $x_1=s_1,\ x_{m+1}=s_2$ and $\{x_i,x_{i+1}\}\subseteq u_i$ for $i=1,2,\cdots,m$. Thus $x_1+I=s_1+I,\ x_{m+1}+I=s_2+I,\ \{x_i+I,x_{i+1}+I\}\subseteq u_i+I$ for $i=1,2,\cdots,m$. By definition of γ_I^*, we conclude that $\gamma_I^*(s_1+I)=\gamma_I^*(s_2+I)$ and so $\gamma_I^*(r_1+I)=\gamma_I^*(r_2+I)$. □

Theorem 4.5. *Let I be an H_v-isolated ideal of R. Then the following diagram of H_v-homomorphisms and H_v-rings are commutative.*



Proof. We prove that the left, up and front faces diagrams of cube are commutative diagrams of H_v-homomorphisms and H_v-rings. The left face diagram is the diagram in Theorem 2.2. For front face diagram we define the mappings in the diagram as the following; f by $f(r)=r+I$, h by $h(r)=[r,1]$, $\bar{h}$ by $\bar{h}(r+I)=[r,1]+S^{-1}I$ and f_s by $f_s([r,s])=[r,s]+S^{-1}I$. By the proof of Theorem 2.2, h is an H_v-homomorphisms. It is easy to see that f and f_s are H_v-homomorphisms. Now we have

$$\begin{aligned}\bar{h}((r_1+I)+(r_2+I)) &= \bar{h}(\{x+I|\ x\in\gamma^*(r_1+r_2+I)\})\\ &= \{[x,1]+S^{-1}I|\ x\in\gamma^*(r_1+r_2+I)\},\end{aligned}$$

and

$$\begin{aligned}\bar{h}(r_1+I)+\bar{h}(r_2+I) &= [r_1,1]+S^{-1}I \ + \ [r_2,1]+S^{-1}I \\ &= \{[x,s]+S^{-1}I|\ [x,s]\in \gamma_S^*([r_1,1]+[r_2,1]+S^{-1}I)\} \\ &= \Big\{[x,s]+S^{-1}I|\ [x,s]\in \gamma_S^*([r,1]+S^{-1}I), \\ &\quad r\in \gamma_S^*(r_1+r_2)\Big\}.\end{aligned}$$

By setting $x=r\in r_1+r_2$ and $s=1$ we have

$$[r,s]+S^{-1}I\in \bar{h}((r_1+I)+(r_2+I))\cap(\bar{h}(r_1+I)+\bar{h}(r_2+I))\neq\emptyset.$$

Similarly we obtain $\bar{h}((r_1+I)\times(r_2+I))\cap(\bar{h}(r_1+I)\times\bar{h}(r_2+I))\neq\emptyset$. Finally, for commutativity, for every $r\in R$ we have:

$$\bar{h}(f(r))=\bar{h}(r+I)=[r,1]+S^{-1}I,$$

$$f_s(h(r))=f_s([r,1])=[r,1]+S^{-1}I.$$

In the up face diagram; φ and $\bar{\varphi}$ are the canonical strong homomorphisms of R and R/I related to fundamental ring R/γ^* and $\frac{R}{I}/\gamma_I^*$, respectively. Define $\bar{f}$ by $\bar{f}(\gamma^*(r))=\gamma_I^*(r+I)$. For $r_1,r_2\in R$, we have:

$$\begin{aligned}\gamma^*(r_1)=\gamma^*(r_2) &\Rightarrow \gamma^*(r_1)+\gamma^*(I)=\gamma^*(r_2)+\gamma^*(I)\\ &\Rightarrow \gamma^*(r_1+I)=\gamma^*(r_2+I)\\ &\Rightarrow \gamma_I^*(r_1+I)=\gamma_I^*(r_2+I),\ \textit{by Lemma } 4.4.\end{aligned}$$

Therefore, $\bar{f}$ is well defined. Also

$$\begin{aligned}\bar{f}(\gamma^*(r_1)+\gamma^*(r_2)) &= \bar{f}(\gamma^*(r_1+r_2))\\ &= \bar{f}(\gamma^*(t))=\gamma_I^*(t+I),\ \text{for some } t\in r_1+r_2. \qquad (1)\end{aligned}$$

On the other hand

$$\begin{aligned}\bar{f}(\gamma^*(r_1))+\bar{f}(\gamma^*(r_2)) &= \gamma_I^*(r_1+I)+\gamma_I^*(r_2+I)\\ &= \gamma_I^*(t+I),\ \text{for some } t\in\gamma^*(r_1+r_2+I). \qquad (2)\end{aligned}$$

Since $r_1+r_2\subseteq\gamma^*(r_1+r_2)\subseteq\gamma^*(r_1+r_2+I)$, the statements in (1) and (2) are equal and $\bar{f}$ is a strong homomorphism. Also $\bar{\varphi}(f(r))=\bar{\varphi}(r+I)=\gamma_I^*(r+I)$ and $\bar{f}(\varphi(r))=\bar{f}(\gamma^*(r))=\gamma_I^*(r+I)$.

The diagram in other faces get from discussed diagrams by replacing R/I, $S^{-1}R$, $S^{-1}I$, γ_s^*, γ_{Is}^* instead of $R,R,I,\gamma^*,\gamma_I^*$, respectively and so these diagrams are commutative diagrams of H_v-homomorphisms and H_v-rings. □

Theorem 4.6. *Let I and J be H_v-ideals of H_v-rings R such that $I\subseteq L\subseteq R$ then*

(i) *L/I is a w-ideal of R/I,*

(ii) $\gamma_I^*(\frac{L}{I})\cong\frac{\gamma^*(L)}{\gamma^*(I)}$.

Proof. (i) We know $\frac{L}{I} = \{l + I|\ l \in L\}$, $\gamma_I^*(\frac{L}{I}) = \{\gamma_I^*(l + I)|\ l \in L\}$. Suppose $l_1 + I, l_2 + I \in \frac{L}{I}$ and $r + I \in \frac{R}{I}$, we show that

$$\gamma_I^*(l_1 + I) \oplus \gamma_I^*(l_2 + I) \in \gamma_I^*(\frac{L}{I}) \text{ and } \gamma_I^*(r + I) \otimes \gamma_I^*(l_1 + I) \in \gamma_I^*(\frac{L}{I}).$$

For $t \in \gamma^*(l_1 + l_2 + I)$ we have

$$\gamma^*(t) \in \gamma^*(l_1 + l_2 + I) = \gamma^*(l_1 + l_2) \oplus \gamma^*(I),$$

and

$$\gamma^*(t + I) = \gamma^*(t) \oplus \gamma^*(I) = \gamma^*(l_1 + l_2) \oplus \gamma^*(I) = \gamma^*(l) \oplus \gamma^*(I), \text{ where } l \in l_1 + l_2.$$

Thus for every $t \in \gamma^*(l_1 + l_2 + I)$ and $l \in l_1 + l_2$:

$$\begin{aligned} \gamma^*(t + I) &= \gamma^*(l + I), \\ \gamma_I^*(t + I) &= \gamma_I^*(l + I), \ \textit{by Lemma } 4.4. \end{aligned}$$

Therefore

$$\begin{aligned} \gamma_I^*(l_1 + I) \oplus \gamma_I^*(l_2 + I) &= \gamma_I^*(t + I), \ \textit{for some } t \in \gamma^*(l_1 + l_2 + I) \\ &= \gamma_I^*(l + I), \ \textit{for some } l \in l_1 + l_2 \\ &\in \gamma_I^*(\frac{L}{I}). \end{aligned}$$

And by similar argument, we conclude:

$$\gamma_I^*(r + I) \otimes \gamma_I^*(l_1 + I) \in \gamma_I^*(\frac{L}{I}).$$

(ii) Define $\theta : \gamma_I^*(\frac{L}{I}) \longrightarrow \frac{\gamma^*(L)}{\gamma^*(I)}$ by $\theta(\gamma_I^*(l + I)) = \gamma^*(l) \oplus \gamma^*(I)$. By Lemma 4.4, θ is an one to one mapping. Let $l_1 + I, l_2 + I \in \frac{L}{I}$, we have

$$\begin{aligned} \theta(\gamma_I^*(l_1 + I) \oplus \gamma_I^*(l_2 + I)) &= \theta(\gamma_I^*[(l_1 + I) + (l_2 + I)]) \\ &= \theta(\gamma_I^*[\gamma^*(l_1 + l_2 + I) + I]) \\ &= \theta(\gamma_I^*(x + I)), \ \textit{for some } x \in \gamma^*(l_1 + l_2 + I) \\ &= \gamma^*(x) \oplus \gamma^*(I), \ \textit{for some } x \in \gamma^*(l_1 + l_2 + I) \\ &= \gamma^*(l_1 + l_2) \oplus \gamma^*(I) \\ &= (\gamma^*(l_1) \oplus \gamma^*(l_2)) \oplus \gamma^*(I) \\ &= (\gamma^*(l_1) \oplus \gamma^*(I)) \oplus (\gamma^*(l_2) \oplus \gamma^*(I)) \\ &= \theta(\gamma^*(l_1 + I)) \oplus \theta(\gamma^*(l_2 + I)). \end{aligned}$$

And so

$$\begin{aligned} \theta(\gamma_I^*(r + I) \otimes \gamma_I^*(l_1 + I)) &= \theta(\gamma_I^*((r + I) \times (l_1 + I))) \\ &= \theta(\gamma_I^*(\gamma^*(r \cdot l_1 + I) + I)) \\ &= \theta(\gamma_I^*(x + I)), \ \textit{where } x \in \gamma^*(r \cdot l_1 + I) \\ &= \gamma^*(x) \oplus \gamma^*(I), \ \textit{for some } x \in \gamma^*(r \cdot l_1 + I) \\ &= \gamma^*(r \cdot l_1) \oplus \gamma^*(I) \\ &= \gamma^*(r) \odot \gamma^*(l_1) \oplus \gamma^*(I) \\ &= (\gamma^*(r) \oplus \gamma^*(I)) \otimes (\gamma^*(l_1) \oplus \gamma^*(I)) \\ &= \theta(\gamma_I^*(r + I)) \otimes \theta(\gamma_I^*(l_1 + I)). \end{aligned}$$

□

Corollary 4.7. *Let I and J are H_v-ideals of an H_v-ring R and $I \subseteq J$, then*

(i) $\frac{R/I}{\gamma_I^*} \cong \frac{\gamma^*(R)}{\gamma^*(I)}$,

(ii) $\omega_{\frac{R}{I}} = \gamma^*(I) + I$,

(iii) $\omega_{\frac{R}{\omega_R}} = \omega_R$.

Proof. (i) is immediate corollary of Theorem 4.6.
(ii) Consider the isomorphism $\theta : \frac{R/I}{\gamma_I^*} \longrightarrow \frac{\gamma^*(R)}{\gamma^*(I)}$ similar to Theorem 4.6 (ii), then by (i)

$$\begin{aligned} \omega_{R/I} &= \{r + I |\ \theta(\gamma_I^*(r + I)) = \gamma^*(I)\} \\ &= \{r + I |\ \gamma^*(r) \oplus \gamma^*(I) = \gamma^*(I)\} \\ &= \gamma^*(I) + I. \end{aligned}$$

(iii) By using the proof of (ii) we have $\omega_{\frac{R}{\omega_R}} = \gamma^*(\omega_R) + \omega_R = \omega_R + \omega_R = \omega_R$. □

Proposition 4.8. *Let M be a maximal H_v-ideal of an s-H_v-ring R then $\gamma^*(M)$ is a maximal ideal of R/γ^*.*

Proof. We prove that $\gamma^*(M) \oplus R/\gamma^* \otimes X = R/\gamma^*$ for every $X \in R/\gamma^* - \gamma^*(M)$. Suppose for some $x \in R$, $\gamma^*(x) = X \in R/\gamma^* - \gamma^*(M)$, so $x \notin M$. But $\gamma^*(M + R \cdot x) = \gamma^*(M) \oplus R/\gamma^* \otimes \gamma^*(x)$ is an ideal of R/γ^* and $M + R \cdot x$ is a w-ideal of R so $M + R \cdot x$ is an H_v-ideal of R. Therefore, $M + R \cdot x = R$ and $\gamma^*(M) + R/\gamma^* \otimes X = \gamma^*(R)$. □

Theorem 4.9. (First homomorphism theorem) *Let $f : R \longrightarrow S$ be a strong homomorphism of H_v-rings and $I = ker\ f$, then $\varphi : R/I \longrightarrow S/\omega_S$ where $\varphi(r+I) = f(r) + \omega_S$ is an H_v-homomorphism of H_v-rings.*

Proof. For $r_1 + I,\ r_2 + I \in R/I$;

$$\begin{aligned} r_1 + I = r_2 + I &\Rightarrow f(r_1) + f(I) + \omega_S = f(r_2) + f(I) + \omega_S \\ &\Rightarrow f(r_1) + \omega_S = f(r_2) + \omega_S,\ since\ f(I) \subseteq \omega_S. \end{aligned}$$

So φ is well defined.
For $t_0 \in r_1 + r_2$ we have:

$$f(t_0) \in f(r_1 + r_2) \subseteq \gamma^*(f(r_1 + r_2)) \oplus \omega_S = \gamma^*(f(r_1 + r_2) + \omega_S), \quad (3)$$

$$t_0 \in \gamma^*(t_0) \in \gamma^*(t_0) \oplus \gamma^*(I) = \gamma^*(r_1 + r_2) \oplus \gamma^*(I) = \gamma^*(r_1 + r_2 + I). \quad (4)$$

Also

$$\begin{aligned}\varphi((r_1+I)+(r_2+I)) &= \varphi(\gamma^*(r_1+r_2+I)+I)\\ &= \{f(t)+\omega_S|\ t\in\gamma^*(r_1+r_2+I)\},\end{aligned}$$

$$\begin{aligned}\varphi(r_1+I)+\varphi(r_2+I) &= (f(r_1)+\omega_S)+(f(r_2)+\omega_S)\\ &= \gamma^*((f(r_1)+f(r_2)+\omega_S)+\omega_S)\\ &= \gamma^*(f(r_1+r_2)+\omega_S)+\omega_S\\ &= \{s+\omega_S|\ s\in\gamma^*(f(r_1+r_2)+\omega_S)\}.\end{aligned}$$

Then by (3) and (4), for $t_0\in r_1+r_2$,

$$f(t_0)+\omega_S\in\varphi((r_1+I)+(r_2+I))\cap\big(\varphi(r_1+I)+\varphi(r_2+I)\big),$$

For $u_0\in r_1\cdot r_2$, we have

$$f(u_0)\in f(r_1\cdot r_2)\subseteq\gamma^*(f(r_1\cdot r_2))\oplus\omega_S=\gamma^*(f(r_1\cdot r_2)+\omega_S), \tag{5}$$

$$u_0\in\gamma^*(u_0)\in\gamma^*(u_0)\oplus\gamma^*(I)=\gamma^*(r_1\cdot r_2)\oplus\gamma^*(I)=\gamma^*(r_1\cdot r_2+I). \tag{6}$$

$$\begin{aligned}\varphi((r_1+I)\cdot(r_2+I)) &= \varphi(\gamma^*(r_1\cdot r_2+I)+I)\\ &= \{f(t)+\omega_S|\ t\in\gamma^*(r_1\cdot r_2+I)\},\end{aligned}$$

$$\varphi(r_1+I)\cdot\varphi(r_2+I)=\{s+\omega_S|\ s\in\gamma^*(f(r_1\cdot r_2)+\omega_S)\}.$$

Therefore, by (5) and (6), $f(u_0)+\omega_S\in\varphi((r_1+I)\cdot(r_2+I))\cap(\varphi(r_1+I)\cdot\varphi(r_2+I))$, and φ is an H_v-homomorphism of H_v-rings. □

REFERENCES

[1] M. R. Darafsheh, B. Davvaz, *H_v-ring of Fractions*, Italian J. Pure Appl. Math. **5** (1999), 25-34.

[2] B. Davvaz, *Remark on Weak Hypermodules*, Bull. Korean Math. Soc. **36** (1999), no. 3, 599-608.

[3] B. Davvaz and M. Ghadiri, *Weak equality and exact sequences in H_v-modules*, Southeast Asian Bull. Math. **25** (2001), no. 3, 403-411.

[4] M. Ghadiri and B. Davvaz, *Direct System and Direct Limit of H_v-modules*, Iran. J. Sci. Technol. Trans. A Sci. **28** (2004), no. 2, 267–275.

[5] M. Ghadiri and B. N. Waphare, *n-ary P-H_v-Groups*, J. Appl. Math. & Informatics **26** (2008), no. 5-6, 445-959.

[6] M. Ghadiri, B. N. Waphare and B. Davvaz, *Quotient n-ary P-H_v-ring*, Carpathian J. Math. **28** (2012), no. 2, 239-246.

[7] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm. (1934), 45-49.

[8] S. Spartalis, *On the number of H_v-rings with P-hyperoperations*, Discrete Math. **155** (1996), no. 1-3, 225-231.

[9] S. Spartalis and T. Vougiouklis, *The Fundamental Relations in H_v-rings*, Rivista Math. Pura ppl. **14** (1994) 7-20.

[10] T. Vougiouklis, *The Fundamental Relations in Hyperrings, The General Hyperfield*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), Scientific (1991) 203-211.

[11] T. Vougiouklis, *Convolutions on WASS hyperstructures*, Discrete Math. **174** (1997), 347-355.

[12] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press Inc., Florida, 1994.

[13] T. Vougiouklis, *On H_v-rings and H_v-representations*, Discrete Math. **208-209** (1999) 615-620.