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LONG TIME EXISTENCE OF HYPERBOLIC RICCI-BOURGUIGNON FLOW ON RIEMANNIAN SURFACES

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Abstract. We consider the *hyperbolic Ricci-Bourguignon flow*$(HRBF)$ equation on Riemannian surfaces and we find a sufficient and necessary condition to this flow has global classical solution. Also, we show that the scalar curvature of the solution metric g_{ij} convergence to the flat curvature.

Key words and Phrases: Hyperbolic geometric flow; quasilinear hyperbolic equation; strict hyperbolicity

1. Introduction

Let (M, g) be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. The general variation equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}(g, \frac{\partial g}{\partial t}) = 0, \tag{1}$$

was introduced by Kong and Liu ([4]) and called the generalized *hyperbolic geometric flow* (denoted by HGF). Here $\mathcal{F}$ are some smooth functions of the Riemannian metric and its first derivative with respect to t, and we consider R_{ij} as the components of Ricci curvature tensor. Liu and Zhang in ([8]) have shown that the *hyperbolic geometric flow* (HGF) has global classical solution on Riemannian surfaces. In this paper, we would like to prove that the global solution of hyperbolic Ricci-Bourguignon flow (HRBF) exists on Riemannian surfaces.
The present work investigates the variation of a Riemannian metric g_{ij} on a Riemannian surface M by its Ricci curvature tensor R_{ij} and scalar curvature R under

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the following equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + 2\rho R g_{ij} \tag{2}$$

where ρ is a real constant. When $\rho = 0$, this equation is hyperbolic geometric flow and the global existence and blowup phenomenon of smooth solutions to this flow on Riemannian surface have been investigated in [8]. The Ricci-Bourguignon flow is $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + 2\rho R g_{ij}$ and the short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ were showed by Catino et al ([1]) for $\rho < \frac{1}{2(n-1)}$.

This study regards the initial metric as follows

$$ds^2 = u_0(x)(dx^2 + dy^2) \qquad \text{at } t = 0 \tag{3}$$

on a surface of topological type $\mathbb{R}^2$, where $u_0(x)$ is a function from C^2 class with bounded C^2 norm and the following inequality is hold

$$0 < k \le u_0(x) \le m < \infty \tag{4}$$

where k and m are positive constants.

Since all the information about curvature is contained in the scalar curvature function R, we can simplify the HRBF equation on this surface. According to our notation, $R = 2K$, where K denotes Gauss curvature and also the Ricci curvature is given by $R_{ij} = \frac{1}{2} R g_{ij}$, so the (HRBF) equation simplifies to

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -R(1 - 2\rho) g_{ij}. \tag{5}$$

At least locally the metric for a surface can be written as $g_{ij} = u(t, x, y)\delta_{ij}$, where $u(t, x, y) > 0$, and δ_{ij} is Kronecker's symbol. Hence, we have

$$R = -\frac{\Delta \ln u}{u} \tag{6}$$

as a result, the aforementioned equation (5) reduces to $u_{tt} - (1 - 2\rho)\Delta \ln u = 0$. The initial data $u_0(x)$ depends only on x and not y; thus, we can consider the Cauchy problem as below

$$\begin{cases} u_{tt} - (1 - 2\rho)(\ln u)_{xx} = 0, \\ u = u_0(x), \qquad \text{at } t = 0 \\ u_t = u_1(x), \qquad \text{at } t = 0 \end{cases} \tag{7}$$

where $u_1(x) \in C^1$ with bounded C^1 norm. By using the transformation

$$\phi = \ln u, \tag{8}$$

Kong and Liu in ([5]) proved a theorem as follows

Theorem 1.1. *Suppose that $u_1(x) \ge |u_0'(x)|/\sqrt{u_0(x)}$ for all $x \in \mathbb{R}$. Then, the Cauchy problem (7) admits a unique global solution for all $t \in \mathbb{R}$.*

Moreover, if $u_1(x) \equiv u_0'(x)/\sqrt{u_0(x)}$, and there exists a point $x_0 \in \mathbb{R}$ such *that* $u_0'(x_0) < 0$*, then the Cauchy problem (7) admits a unique classical solution only in* $[0,T) \times \mathbb{R}$*, where*

$$T = -\frac{2}{inf_x\{u_0'(x)u_0^{-3/2}(x)\}}. \tag{9}$$

The following theorem will proven without using (6) in our investigate.

Theorem 1.2. *Let*

$$u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \geq 0 \qquad \text{at all } x \in \mathbb{R}, \tag{10}$$

and

$$u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \geq 0 \qquad \text{at all } x \in \mathbb{R}. \tag{11}$$

Hence, the Cauchy problem (7) has a unique global solution for all $t \in \mathbb{R}$*.*

Theorem 1.3. *If a point* $x_0 \in \mathbb{R}$ *exists, which satisfy*

$$u_1(x_0) + \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} < 0 \tag{12}$$

or there exists a point $x_0 \in \mathbb{R}$*, such that*

$$u_1(x_0) - \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} < 0 \tag{13}$$

thus, the Cauchy problem (7) has a unique classical solution only in $[0,T) \times \mathbb{R}$*.*

Note. Based on Theorem 1.2, we can conclude the Cauchy problem

$$\begin{cases} \frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + 2\rho R g_{ij}, & \text{for } i,j = 1,2 \\ g_{ij} = u_0(x)\delta_{ij}, & \text{for } t = 0 \\ \frac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij}, & \text{for } t = 0 \text{ and } i,j = 1,2. \end{cases}$$

has a unique smooth solution for all $t \in \mathbb{R}$. Besides we can consider the solution metric g_{ij} as below

$$g_{ij} = u(x,t)\delta_{ij} \qquad \text{for } i,j = 1,2. \tag{14}$$

We will prove the above mentioned Theorems 1.2 and 1.3, in the subsequent sections (3 and 4, respectively). Moreover, using Theorem 1.2, the following theorem will be proven in Section 5.

Theorem 1.4. *Let be*

$$inf_x\{u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}}\} > 0 \qquad \text{and} \qquad inf_x\{u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}}\} > 0. \tag{15}$$

Hence, a unique classical solution of (1) is exist as the form (14) for all time. Furthermore, the scalar curvature $R(x,t)$ *relates to the solution metric* g_{ij} *admits*

$$R(x,t) \to 0 \qquad \text{as } t \to +\infty,$$

and $R(x,t) \leq k_1$ *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$*, where* k_1 *is a positive constant and independent of* t *and* x*.*

2. Preliminaries

In this section we require only to discuss the classical solution on $t \geq 0$. The result for $t \leq 0$ can be easily obtained.

Suppose that

$$u_t = v \qquad \text{and} \qquad u_x = w. \tag{16}$$

Thus, from the above equations and Cauchy problem (7) we have

$$u_t = v, \qquad w_t - v_x = 0, \qquad \text{and} \qquad v_t - (\frac{1-2\rho}{u})w_x = (2\rho - 1)\frac{w^2}{u^2}. \tag{17}$$

Eigenvalues of equations (17) can be easily calculated as follows

$$\lambda_1 = -\lambda, \qquad \lambda_2 = 0, \qquad \lambda_3 = \lambda, \qquad \lambda = \sqrt{\frac{1-2\rho}{u}} \tag{18}$$

and we have the matrices $L(U)$ and $R(U)$(where $U = (u, w, v)$) of left and right eigenvectors, respectively as below

$$L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & 1 \end{pmatrix},$$

$$R(U) = \big(r_1(u), r_2(u), r_3(u)\big) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & -\lambda \\ 1 & 0 & 1 \end{pmatrix}.$$

Equation system (17) is a linear degenerate strict hyperbolic system because of $\nabla\lambda_i(U)r_i(U) \equiv 0$ for $i = 1, 2, 3$.

Define p and q as follows

$$p = v + \lambda w \qquad \text{and} \qquad q = v - \lambda w. \tag{19}$$

Lemma 2.1. *p and q satisfy the following equations:*

$$p_t - \lambda p_x = \frac{1}{4(1-2\rho)}\lambda^2(q-p)p, \tag{20}$$

$$u_t = \frac{1}{2}(p+q), \tag{21}$$

$$q_t + \lambda q_x = \frac{1}{4(1-2\rho)}\lambda^2(p-q)q. \tag{22}$$

Proof. By differentiating of the function λ with respect to t and x, λ_t *and* λ_x can easily be obtained as,

$$\lambda_t = -\frac{1}{2(1-\rho)}\lambda^3 v,$$

and

$$\lambda_x = -\frac{1}{2(1-2\rho)}\lambda^3 w.$$

Therefore,

$$\begin{aligned} p_t - \lambda p_x &= (v+\lambda w)_t - \lambda(v+\lambda w)_x \\ &= v_t - \lambda v_x + \lambda(w_t - \lambda w_x) + w(\lambda_t - \lambda\lambda_x) \\ &= v_t - \lambda^2 w_x - \lambda(v_x - w_t) + w(\lambda_t - \lambda\lambda_x) \\ &= -\frac{1}{2(1-2\rho)}\lambda^3 w(v+\lambda w) = \frac{1}{4(1-2\rho)}\lambda^2(q-p)p. \end{aligned}$$

We can prove (22) in the same way as above, and it is obvious that (21) is hold. □

For the next lemma, consider

$$r = p_x + \frac{1}{8(1-2\rho)}\lambda pq \qquad \textit{and} \qquad s = q_x - \frac{1}{8(1-2\rho)}\lambda pq.$$

Lemma 2.2. *r and s satisfy*

$$r_t - \lambda r_x = \frac{\lambda^2}{4(1-2\rho)}(2q-3p)r + \frac{\lambda^3}{32(1-2\rho)}(2p-3q)pq + \frac{\lambda^3 p(q-p)}{32(1-2\rho)^2}(p+5q), \tag{23}$$

$$s_t + \lambda s_x = \frac{\lambda^2}{4(1-2\rho)}(2q-3p)r + \frac{\lambda^3}{32(1-2\rho)}(2p-3q)pq + \frac{\lambda^3 p(q-p)}{32(1-2\rho)^2}(p+5q). \tag{24}$$

Proof. Suppose

$$L_1 = \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial x} \qquad \textit{and} \qquad L_2 = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}.$$

Hence, by a direct computation we can get

$$\begin{aligned} L_1 p_x &= \frac{1}{4(1-2\rho)}\lambda^2((2q-3p)p_x + pq_x) + \frac{1}{8(1-2\rho)^2}\lambda^3 p(q-p)^2, \\ L_2 q_x &= \frac{1}{4(1-2\rho)}\lambda^2((2p-3q)q_x + qp_x) - \frac{1}{8(1-2\rho)^2}\lambda^3 q(p-q)^2. \end{aligned}$$

Now we can easily prove (23) and (23). □

Notice from Cauchy problem (7), (16) and (19), we can write following equations at $t = 0$.

$$p = p_0(x) \equiv u_1(x) + \lambda_0(x)u_0'(x), \qquad u = u_0(x) \tag{25}$$

$$q = q_0(x) \equiv u_1(x) - \lambda_0(x)u_0'(x), \tag{26}$$

where $\lambda_0(x) = \sqrt{\frac{1-2\rho}{u_0(x)}}$. Now in following theorem we show that the Cauchy problem (7) has a unique global solution under some conditions.

Theorem 2.3. *Let M_1 be a positive constant satisfying*

$$0 \le p(x,t) \le M_1 \qquad \textit{and} \qquad 0 \le q(x,t) \le M_1, \tag{27}$$

then, on D(T),

$$|u(x,t)| \le M(T), \qquad |u_x(x,t)| \le M(T), \qquad |u_t(x,t)| \le M(T),$$

$$|r(x,t)| \leq M(T), \qquad |s(x,t)| \leq M(T),$$

where $M(T)$ is a positive constant, and

$$D(T) = \{(x,t)|\, x \in \mathbb{R},\, 0 \leq t \leq T\,,\, T > 0\,\}.$$

Hence, the Cauchy problem (7) has a unique global classical solution on $t \geq 0$, by the local existence theorem of the classical solution to quasilinear hyperbolic systems.

Proof. Consider any point (t,x), and let

$$x = x_1(t,\beta_1), \qquad x = x_2(t,\beta_2), \qquad x = x_3(t,\beta_3)$$

be the λ_1, λ_2 and λ_3 characteristics, respectively, that satisfy

$$(x_1)_t = \lambda_1 = -\lambda, \qquad (x_2)_t = \lambda_2 = 0, \qquad (x_3)_t = \lambda_3 = \lambda$$

$$x_1(0,\beta_1) = \beta_1, \qquad x_2(0,\beta_2) = \beta_2, \qquad x_3(0,\beta_3) = \beta_3.$$

By integrating (21) along the λ_2 characteristics we can obtain

$$u(x,t) = u_0(\beta_2) + \frac{1}{2}\int_0^t (p+q)(x_2(\tau,\beta_2),\tau)d\tau. \tag{28}$$

Thus, as a result of (21), (27) and (28) we have

$$|u_t| \leq M_1(T) \qquad \textit{and} \qquad 0 < inf_x u_0(x) \leq u(x,t) \leq M_2(T).$$

Using a same method, by integrating (23) along the λ_1 characteristics $x = x_1(t,\beta_1)$

$$|r(x,t)| \leq M_2(T) + M_3(T)\int_0^t R(\tau)d\tau, \qquad \text{where } R(t) = sup_x\,|r(x,t)|\,.$$

Therefore, we have $|r(x,t)| \leq M_4(T)$ by the Bellman lemma. As a similar way, holds $|s(x,t)| \leq M_5(T)$. Since $(u_x)_t = \frac{1}{2}(r+s)$, it is obvious that

$$|u_x(x,t)| \leq M_6(T),$$

which $M_i(t)$ for $i = 1,2,3,...$ denote positive constants. □

3. Proof of Theorem 1.2

On the basis of the local existence and uniqueness theorems of the classical solutions to the quasilinear hyperbolic systems ([7]), to prove Theorem (1.2) it suffices to establish uniform *a priori* estimates of the C^1 norms of p,q and u. We have following lemma from [2, 3].

Lemma 3.1. *Suppose*

$$u_t + \lambda_1(x,t)u_x = A(x,t)(u-v),$$
$$v_t + \lambda_2(x,t)v_x = B(x,t)(v-u),$$

where λ_1, λ_2, A and B are continuous functions, and $\lambda_1 \leq \lambda_2$. If A and B are both non positive, then

$$min(u_0(x),v_0(x)) \leq u(x,t), v(x,t) \leq max(u_0(x),v_0(x)).$$

For prove Theorem 1.2 we need the following lemma.

Lemma 3.2. *On the existence domain of the classical solution to the Cauchy problem (7) and (25), if (10) and (11) hold, then*

$$0 \leq p(x,t) \leq sup_{x\in\mathbb{R}} p_0(x), \tag{29}$$

$$0 \leq q(x,t) \leq sup_{x\in\mathbb{R}} q_0(x), \tag{30}$$

$$0 < inf_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq sup_x u_0(x) + Ct, \tag{31}$$

where $C > 0$ is a constant.

Proof. Along λ_1 characteristics, we can obtain

$$p(x,t) = p_0(\beta_1) exp(\int_0^t \frac{1}{4(1-2\rho)} \lambda^2 (q-p)(x_1(\tau,\beta_1),\tau) d\tau).$$

By (25) and (10), $p_0(x) \geq 0$ for all $x \in \mathbb{R}$. Therefore, we have $p(x,t) \geq 0$. In a similar way we can prove $q(x,t) \geq 0$. As a result of these inequalities, we have

$$\frac{1}{4(1-2\rho)} \lambda^2 p \geq 0 \qquad \textit{and} \qquad \frac{1}{4(1-2\rho)} \lambda^2 q \geq 0.$$

Hence, by Lemma 3.1 we can easily see that

$$p(x,t) \leq sup_x p_0(x) \qquad \text{and} \qquad q(x,t) \leq sup_x q_0(x).$$

Also, we can get following equality by integrating (21)

$$u(x,t) = u_0(\beta_2) + \frac{1}{2} \int_0^t (p+q)(x_2(\tau,\beta_2),\tau) d\tau.$$

Thus, we can get to result since $p(x,t) \geq 0$ and $q(x,t) \geq 0$. □

Proof of Theorem 1.2. Now from aforementioned Lemma 3.2 and Theorem 2.3, Theorem 1.2 is obvious. □

Note. By (6) and (29), and based on the hypotheses of Theorem 1.2, we have

$$|R(x,t)| \leq M_7(T).$$

4. Proof of Theorem 1.3

The blow-up phenomena of the *hyperbolic geometric flow* will be discussed in this section.

Suppose

$$m = \sqrt{\lambda} p \qquad \textit{and} \qquad n = \sqrt{\lambda} q. \tag{32}$$

We have

$$\frac{1}{4}\lambda^2 q = \frac{1-2\rho}{4}((\ln u)_t - \lambda(\ln u)_x) \qquad \text{and} \qquad \frac{1}{4}\lambda^2 p = \frac{1-2\rho}{4}((\ln u)_t + \lambda(\ln u)_x).$$

By the use of (16), (19) and Lemma 2.1 the following lemma can be proven

Lemma 4.1. *m and n satisfy*

$$m_t - \lambda m_x = -\frac{1}{4(1-2\rho)}\lambda^{3/2}m^2, \tag{33}$$

$$n_t - \lambda n_x = -\frac{1}{4(1-2\rho)}\lambda^{3/2}n^2. \tag{34}$$

Observe that at $t = 0$, set

$$m = m_0(x) = \sqrt[4]{\frac{1-2\rho}{u_0(x)}}(u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}}), \tag{35}$$

$$n = n_0(x) = \sqrt[4]{\frac{1-2\rho}{u_0(x)}}(u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}}). \tag{36}$$

Proof of Theorem 1.3. Without loss of generality, we assume that (12) holds; in the same way we can proceed if (13) holds.

As a result of (33) and (34) we have $m_t - \lambda m_x \le 0$ and $n_t - \lambda n_x \le 0$. Thus, we can easily see that

$$m(x,t) + n(x,t) \le M_0 \qquad \text{and} \qquad M_0 \equiv sup m_0(x) + sup n_0(x). \tag{37}$$

Notice that $u_0(x) \ge k > 0$, and also from (12) and (35) we have $m_0(x) < 0$. Next, the get following equation is obtained from (33) by integrating along λ_1 characteristics. That is,

$$m(x_0,t) = m_0(x_0)/F(t,x_0), \tag{38}$$

where

$$F(t,x_0) = 1 + m_0(x_0)/4(1-2\rho)\int_0^t \lambda^{3/2}(x_1(x_0,\tau),\tau)d\tau \text{ and } \lambda^{3/2} = (\frac{u}{1-2\rho})^{-3/4}. \tag{39}$$

By (21) and (32), it is easy to see that $((\frac{u}{1-2\rho})^{-3/4})_t = \frac{3}{8(1-2\rho)}(m+n)$. Hence, we have

$$(\frac{u}{1-2\rho})^{3/4}(x,t) = (\frac{u_0}{1-2\rho})^{3/4}(x_0) + \frac{3}{8(1-2\rho)}\int_0^t (m+n)(x_2(x_0,\tau),\tau)d\tau. \tag{40}$$

By (4), (37) and (40), we get

$$u^{3/4}(x,0) \ge k^{3/4} \qquad \text{and} \qquad u^{3/4}(x,t) \ge M^{3/4} + \frac{3}{8(1-2\rho)}M_0 t. \tag{41}$$

We consider three cases.
Case(i). If $M_0 < 0$, then there exists $\tau_0 = 8(1-2\rho)M^{3/4}/(3(-M_0)) > 0$, such that

$$u(x,t) \le 0 \qquad \textit{and} \qquad t \ge \tau_0.$$

This imply the system in (7) is meaningless for $t \ge \tau_0$, that is, it admits a unique local classical solution.

Case(ii). If $M_0 = 0$, then, by (39) and (41), following inequality can be easily obtained

$$F(x_0, t) \le 1 + \frac{1-2\rho}{4} m_0(x_0) M^{-3/4} t.$$

When $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, we can find $t_0 = 4M^{3/4}/(1-2\rho)(-m_0(x_0)) > 0$, such that

$$F(x_0, t) \to 0^+ \qquad \text{as } t \to t_0^-. \tag{42}$$

Thus, the finite time $T = T(x_0) > 0$ exists such that

$$F(x_0, t) \to -\infty \qquad \text{as } t \to T^-. \tag{43}$$

Case(iii). If $M_0 > 0$, then, by (39) and (41) we can get

$$F(x_0, t) \le 1 + \frac{2m_0(x_0)}{3M_0} \ln(1 + \frac{3M_0}{8(1-2\rho)M^{3/4}} t).$$

Therefore, since, $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_* > 0$ such that $F(x_0, t) \to 0^+$ as $t \to t_*^-$, and then (43) follows. □

5. PROOF OF THEOREM 1.4

In this section, we will study the asymptotic behaviour of the scalar curvature $R(x, t)$.

Proof of Theorem 1.4. We assume that (15) holds. Using Lemmas 3.1 and 3.2, we have

$$K_1 \le p(x, t) \le K_2 \qquad \text{and} \qquad K_1 \le q(x, t) \le K_2,$$

where here after C_i for $i = 1, 2, ...$ denote positive constants independent of t and x. Therefore, we can obtain the following inequality as a result of (4) and (28)

$$K_3(1 + t) \le u(x, t) \le K_4(1 + t), \tag{44}$$

and then

$$\frac{K_3^{3/4}(1+t)^{3/4}}{1-2\rho} \le \frac{u^{3/4}(x, t)}{1-2\rho} \le \frac{K_4^{3/4}(1+t)^{3/4}}{1-2\rho} \le \frac{K_4^{3/4}(1+t)}{1-2\rho}.$$

It follows from(40) and (44) that

$$K_5(1 + \ln(1 + t)) \le F(x, t) \le K_6(1 + (1 + t)^{1/4}).$$

Thus, by (15) and (38), we get

$$0 \le \frac{K_7}{1 + (1+t)^{1/4}} \le m(x, t) \le \frac{K_8}{1 + \ln(1 + t)}. \tag{45}$$

Similarly, we can obtain

$$0 \le \frac{K_7}{1 + (1+t)^{1/4}} \le m(x, t) \le \frac{K_8}{1 + \ln(1 + t)}. \tag{46}$$

Hence, $m(x, t) \to 0$ and $n(x, t) \to 0$ as $t \to +\infty$. Noting (44), (45) and (46), we get

$$p = \frac{1}{\sqrt{\lambda}} m = \frac{u}{1-2\rho}^{1/4} m, \quad q = \frac{1}{\sqrt{\lambda}} n = \frac{u}{1-2\rho} n, \quad u_x = \frac{p-q}{2\lambda} = \frac{1}{2} u^{3/4}(m - n),$$

and

$$-\frac{K_8}{1+\ln(1+t)} \leq m(x,t) - n(x,t) \leq \frac{K_8}{1+\ln(1+t)}. \quad (47)$$

Then, we can easily obtain

$$|u_x| \leq K_9 \frac{(1+t)^{3/4}}{1+\ln(1+t)}. \quad (48)$$

Next an easy derivation gives

$$u_{xx} = \frac{p_x - q_x}{2\lambda} + \frac{1}{2}\lambda^2 u_x^2 = \frac{1}{2}u^{1/2}(p_x - q_x) + \frac{1}{2u}u_x^2. \quad (49)$$

Let $\bar{p} = p\ u$ and $\bar{q} = q\ u$. Thus, based on [5] we have

$$L_1 \bar{p_x} = -A_1 \bar{p_x} - B_1 \bar{q_x},$$
$$L_2 \bar{q_x} = -A_2 \bar{p_x} - B_2 \bar{q_x},$$

where

$$A_1 = \frac{1}{4}(2\bar{q} + 3\bar{q}), \qquad B_1 = \frac{3}{4}\bar{q}, \qquad A_2 = \frac{3}{4}\bar{q}, \qquad B_2 = \frac{1}{4}(2\bar{p} + 3\bar{q}).$$

Therefore, by [5] holds

$$|\bar{p}_x(x,t)|\,, |\bar{q}_x(x,t)| \leq K_{10}. \quad (50)$$

Noting that $p_x - q_x = u_x(p-q)/u + u(\bar{p_x} - \bar{q_x})$, and from (6) and (49), we have

$$R = \frac{1}{u^3}(u_x^2 - u u_{xx}) = \frac{u_x^2}{2u^3} - \frac{p_x - q_x}{2u^{3/2}} = \frac{u_x^2}{2u^3} - \frac{\bar{p_x} - \bar{q_x}}{2u^{1/2}} - \frac{u_x(m-n)}{2u^3}. \quad (51)$$

Thus, from (41), (47), (48), (50) and (51) we conclude that

$$|R(x,t)| \leq \frac{K_{11}}{(1+\ln(1+t))^2(1+t)^{3/2}} + \frac{K_1 2}{(1+t)^{1/2}} + \frac{K_{13}(1+t)^{3/4}}{(1+t)^3(1+ln(1+t))^2}.$$

Hence, $R(x,t) \to 0$ as $t \to +\infty$. □

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