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ON SOME PROPERTIES OF ABIAN'S SEMIRING AND ITS ZERO DIVISOR GRAPHS

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Abstract. In this paper, we have studied some properties of Abian's semiring, especially about the supremum of a subset of an Abian's semiring. We have also considered the zero divisor graphs of Abian's semiring and found some properties of those graphs.

Key words and Phrases: Semiring, Abian's relation, annihilator, ideal, zero divisor graph.

1. INTRODUCTION

Abian's relation on a ring R is the relation '$\leq$' defined by $a \leq b$ if and only if $a^2 = ab$. This relation was first introduced by Abian [1] and Chacron [5]. Abian's relation on reduced commutative ring has been studied by Anderson and LaGrange [2]. Recently, the present authors have introduced this relation in a semiring [10]. As the relation is defined in terms of multiplicative operation, some results are analogous to those in a ring. But while studying supremum of a subset of an Abian's semiring, the absence of additive inverse in it is a motivating factor towards the embedding of the semiring into an Abian's ring with a necessary condition.

In this paper, our aim is to extend the following result that holds in an Abian's ring. Let X be a subset of an Abian's ring R. Then the following two statements are equivalent for any $c \in R$.

(i) c is supremum of X,

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(ii) c is an upper bound of X and annihilator of $\{c\}$ in R contains the annihilator of X in R.

We have shown that in case of a commutative, additively cancellative Abian's semiring, statement (ii) follows from statement (i); but to establish (i) from (ii), some additional conditions are required to be imposed on the semiring.

In Section 4, we have considered the zero divisor graphs of an Abian's semiring and have found some properties of those graphs.

2. PRELIMINARIES

First we give some basic definitions and results on semirings and semigroups.

Definition 2.1. *A non-empty set S is said to form a semiring with respect to two associative binary compositions, addition $(+)$ and multiplication $(.)$ defined on it, if the following conditions are satisfied:*

(1) $(S,+)$ is a commutative monoid with identity element '0'. i.e., $\forall\, a,b,c \in S$
  (i) $(a+b)+c = a+(b+c)$
  (ii) $0+a = a+0 = a$
  (iii) $a+b = b+a$.
(2) Multiplication over addition is left distributive and right distributive.
(3) Multiplication by '0' annihilates S, i.e., $0.a = a.0 = 0\ \forall\ a \in S$.

Definition 2.2. [4] *A semiring $(S,+,.)$ is a partially ordered semiring if there is a partial order relation $\leq$ on S satisfying the following conditions for all $a,b,c \in S$.*

(i) $a \leq b \Rightarrow a+c \leq b+c$, and
(ii) $a \leq b$ and $0 \leq c \Rightarrow ac \leq bc$, and $ca \leq cb$, where '0' is the additive identity of S.

Definition 2.3. *Let $(S,.)$ be a semigroup. Then, S is said to be left separative if and only if $x^2 = xy$ and $y^2 = yx$ together imply $x = y$. Similarly, S is said to be right separative if and only if $x^2 = yx$ and $y^2 = xy$ together imply $x = y$. S is said to be separative, if it is both left separative and right separative.*

Definition 2.4. *An element 'a' in a semigroup S is said to be nilpotent element of index k, if k is the least positive integer such that $a^k = 0$ (additive identity of S).*

Definition 2.5. *A semigroup is said to be a reduced semigroup if it does not contain any non-zero nilpotent element.*

Since a semiring $(S, +, .)$ is obviously a semigroup with respect to '.', the above definitions are applicable to $(S, +, .)$ also. Note that, every separative semiring is a reduced semiring.

Definition 2.6. *A semiring $(S, +, .)$ is said to be additively cancellative if for all $a, b, c \in S$, $a + b = a + c$ implies that $b = c$.*

Definition 2.7. *A left semiring ideal (right semiring ideal) of a semiring S is a non empty subset I of S such that $a+b \in I \ \forall \ a, b \in I$ and $s\,a \in I$ $(a\,s \in I)$ $\forall \ s \in S$ and $a \in I$. I is said to be a semiring ideal of S if it is both left semiring ideal and right semiring ideal of S.*

Definition 2.8. *A semiring ideal I of a semiring S is called a semiring k-ideal of the semiring S, if $a \in I$ and $x \in S$ together with $x + a \in I$ imply $x \in I$.*

Definition 2.9. *A ring $(R, +, .)$ is called an Abian's ring if $(R, +, .)$ is a partially ordered set (poset) with respect to Abian's relation.*

Definition 2.10. *A semiring $(S, +, .)$ is called Abian's semiring if $(S, +, .)$ is a poset with respect to Abian's relation. In other words, Abian's semiring is a semiring where Abian's relation is a partial order relation.*

It may be noted that, an Abian's semiring is not necessarily a partially ordered semiring.

Definition 2.11. *Let $(S, +, .)$ be a semiring with a partial order $\leq$. A non empty subset I of S is said to be a poset ideal of S if*

(i) for every $x \in I$, $y \leq x$ implies that $y \in I$,
(ii) for every $x, y \in I$, there is some element $z \in I$ such that $x \leq z$, $y \leq z$.

Definition 2.12. *Given a semiring $(S, +, .)$ and $X \subseteq S$. The semiring left annihilator of X, denoted by ${}^{L}Ann_S(X)$, is defined by ${}^{L}Ann_S(X) = \{s \in S : s\,x = 0 \ \forall \ x \in X\}$. Similarly, a semiring right annihilator of X, denoted by ${}^{R}Ann_S(X)$, is defined by ${}^{R}Ann_S(X) = \{s \in S : x\,s = 0 \ \forall \ x \in X\}$. A semiring annihilator of X is the subset of S which is both semiring left annihilator and semiring right annihilator of X and is denoted by $Ann_S(X)$. If X is a singleton $\{c\}$, then we denote the semiring annihilator by $Ann_S(c)$.*

If $(S, +, .)$ is a ring instead of a semiring in the above definition, we call $Ann_S(X)$ a ring annihilator of X in S.

Definition 2.13. *Let $(S, +, .)$ be a semiring with additive identity '0' and $\leq$ be a partial order in it. For any two elements $x, y \in S$, we define the lower cone of $\{x, y\}$ as $L(x, y) = \{z \in S : z \leq x \text{ and } z \leq y\}$. Let X be a subset of S. A poset annihilator of X, denoted by $Ann_P(X)$, is defined by $Ann_P(X) = \{y \in S : L(x, y) = \{0\} \ \forall\ x \in X\}$.*

Definition 2.14. *An upper bound of a subset X of a partially ordered set $(S, \leq)$ is an element z of S such that $x \leq z$ for all $x \in X$. An upper bound u of X is called a supremum or least upper bound of X if for all upper bounds z of X, $u \leq z$. Supremum of X is denoted by* $\sup(X)$.

Remark 2.15. *In a reduced semiring $(S, +, .)$, if $a \neq 0$, $b \neq 0$ and $ab = 0$ then, $ba = 0$. Indeed, $(ba)^2 = baba = b(ab)a = b0a = 0$. Therefore, $ba = 0$, as S is a reduced semiring.*

In the following we state some results on semigroups, which will be used in our work.

Theorem 2.16. [8] *Let G be a left separative semigroup and $a, b, x \in G$. Then,*

(i) $xa = xb$ *implies* $ax = bx$,
(ii) $x^2a = x^2b$ *implies* $ax = bx$,
(iii) G *is a reduced semigroup.*

Corollary 2.17. *If G be a left separative partially ordered semigroup with respect to Abian's relation '$\leq$' and $a, b \in G$. Then, $a \leq b$ implies $ab = ba$.*

Theorem 2.18. [8] *Let G be a separative semigroup and $a, b, x \in G$. Then, $xa = xb$ if and only if $ax = bx$.*

3. PROPERTIES OF ABIAN'S SEMIRING

Throughout this section, we assume that S is a separative semiring. Note that, every separative semiring is a reduced semiring and in a reduced semiring, semiring left annihilator is same as semiring right annihilator and so ${}^{L}Ann_S(X) = {}^{R}Ann_S(X) = Ann_S(X)$.

3.1. **Relation between annihilators and ideals of Abian's semiring.**

Proposition 3.1. *Let S be a reduced semiring and $X \subseteq S$. Then $Ann_S(X)$ is a semiring k-ideal of S.*

PROOF. We know that $Ann_S(X) \neq \phi$, as $0 \in Ann_S(X)$ $(0.x = x.0 = 0\ \forall x \in X)$.

Let $s_1, s_2 \in Ann_S(X)$. Then, $s_1x = 0$ and $s_2x = 0\ \forall\ x \in X$. Therefore, $(s_1+s_2)x = s_1x+s_2x = 0$, i.e., $s_1, s_2 \in Ann_S(X)$ implies that $(s_1+s_2) \in Ann_S(X)$. Also, if $t \in S$ then, $ts_1x = t.0 = 0$. Therefore, $ts_1 \in Ann_S(X)$ for $t \in S$. Again, $s_1x = 0$ implies that $xs_1 = 0$ (as S is reduced semiring) $\Rightarrow x(s_1t) = (xs_1)t = 0 \Rightarrow s_1t \in Ann_S(X)$. This shows that $Ann_S(X)$ is a semiring ideal of S.

Let $s_1 + s_2 \in Ann_S(X)$, where $s_1 \in Ann_S(X)$ and $s_2 \in S$. Then, $s_1x = 0$ and $(s_1 + s_2)x = 0\ \forall\ x \in X$. Now, $(s_1 + s_2)x = 0 \Rightarrow s_1x + s_2x = 0 \Rightarrow 0 + s_2x = 0 \Rightarrow s_2x = 0 \Rightarrow s_2 \in Ann_S(X)$.

Therefore, $Ann_S(X)$ is a semiring k-ideal of S. □

Proposition 3.2. *Let S be a reduced semiring. For any $a \in S$, $Ann_S(a) \cap Sa = \{0\}$.*

PROOF. Let $t \in Ann_S(a) \cap Sa$. Then, $t \in Ann_S(a)$ and $t \in Sa$. Therefore, $ta = 0$ and hence $at = 0$, as S is a reduced semiring. Again, $t \in Sa$ implies that $t = s_1a$, for some $s_1 \in S$. Now, $t = s_1a \Rightarrow t^2 = s_1at \Rightarrow t^2 = s_10 \Rightarrow t^2 = 0 \Rightarrow t = 0$ as S is a reduced semiring.

This shows that $Ann_S(a) \cap Sa = \{0\}$. □

Proposition 3.3. *Let $(S, +, .)$ be a partially ordered semiring with respect to Abian's order '$\leq$' and $X \subseteq S$. Then, semiring annihilator $Ann_S(X)$ is a poset ideal of S.*

PROOF. Let $I = Ann_S(X)$. Now, $I \neq \phi$, as $0 \in I$.

Let $a \in I$ and $b \leq a$. Now, $a \in I$ gives $ax = 0\ \forall x \in X$ and $b \leq a$ gives $b^2 = ba$. Therefore, $b^2x = bax \Rightarrow b^2x = b.0 \Rightarrow bbx = 0 \Rightarrow bxb = 0$ (by Theorem 2.16).

Therefore, $(bx)^2 = bxbx = 0x = 0$, i.e., $bx = 0\ \forall x \in X$, since S is reduced. Thus $b \in I$.

We know that in an Abian's semiring $S, 0 \leq s\ \forall s \in S$. Let $a, b \in I$. Then, $ax = 0$ and $bx = 0\ \forall\ x \in X$, and so $(a + b)x = ax + bx = 0 + 0 = 0\ \forall\ x \in X$. Therefore, $a + b \in I$. Now, S is a partially ordered semiring and $0 \leq a$, $0 \leq b$. Therefore, $0 \leq a \Rightarrow b \leq a + b$ and $0 \leq b \Rightarrow a \leq a + b$, i.e., $a \leq a + b$ and $b \leq a + b$. Therefore, $I = Ann_S(X)$ is a poset ideal of S. □

Proposition 3.4. *Let $(S, +, .)$ be a partially ordered semiring with respect to Abian's order '$\leq$' and $X \subseteq S$. If poset annihilator $Ann_P(X)$ is closed under addition, then it is a poset ideal of S.*

PROOF. Let $I = Ann_P(X)$.

Now, $I \neq \phi$, as $0 \in I$, since for any $x \in S$, $0 \leq x\ \forall\ x \in S$ and so $L(0, x) = \{0\}$.

Let $a \in I$ and $b \leq a$. Then $L(a, x) = \{0\}\ \forall x \in X$. Let x be a particular element of X. We can show that $L(b, x) = \{0\}$. If possible let $0 \neq y \in L(b, x)$. Then, $y \leq b$ and $y \leq x$. Now $y \leq b$ and $b \leq a$ together imply $y \leq a$. Again, $y \leq x$

and $y \leq a$ together imply that $y \in L(a,x)$, which is a contradicts that $L(a,x) = \{0\}$. Since x can be an arbitrary element of X, we have $L(b,x) = \{0\}$ $\forall$ $x \in X$. Hence, $a \in I$ and $b \leq a$ together imply that $b \in I$.

Since I is closed under addition, $a, b \in I \Rightarrow a + b \in I$. Again since, S is a partially ordered semiring, therefore as before, $0 \leq a \Rightarrow b \leq a + b$ and $0 \leq b \Rightarrow a \leq a + b$, i.e., $a \leq a + b$ and $b \leq a + b$. Hence, $Ann_P(X)$ is a poset ideal of the partially ordered Abian's semiring S. □

Example 3.5. *The division semiring* $(\mathbb{N}, lcm, gcd)$, *where* $\mathbb{N}$ *is the set of positive integers, is a partially ordered semiring with respect to Abian's relation* [10]. *Here additive identity is* 1 *and for any two elements* $a, b \in S, a \leq b$ *if and only if* b *is a multiple of* a. *Let* X *be any subset of* S. *Then, poset annihilator* $Ann_P(X)$ *exists and is closed under addition. Indeed,* $Ann_P(X)$ *contains those elements which are not multiple of any* $x \in X$. *If* $a, b \in Ann_P(X)$ *then* a *is not a multiple of any* $x \in X$, b *is not multiple of any* $x \in X$ *and* $a + b$ *i.e.,* $lcm(a, b)$ *is also not multiple of any* $x \in X$. *Therefore,* $a + b \in Ann_P(X)$. *Hence,* $Ann_P(X)$ *is a poset ideal of* S.

3.2. **Supremum of a subset of Abian's semiring.** Here we recall the following theorem which we wish to extend for Abian's semiring.

Theorem 3.6. [8] *Let* S *be an Abian's ring and* X *be a subset of* S. *Then, the following are equivalent for any* $c \in S$.

(i) $c = \sup(X)$,
(ii) *c is an upper bound for X and* $Ann_S(X) \subseteq Ann_S(c)$. *In fact,* $Ann_S(X) = Ann_S(c)$.

In the following, we show that if S is an additively cancellative Abian's semiring then (ii) of the above theorem follows from (i).

Lemma 3.7. *Let* S *be an additively cancellative Abian's semiring and* $X \subseteq S$. *If* $c = \sup(X)$, *then* c *is an upper bound of* X *satisfying* $Ann_S(X) \subseteq Ann_S(c)$. *In fact,* $Ann_S(X) = Ann_S(c)$.

Proof. Suppose $c = \sup(X)$. Then, c is obviously an upper bound of X.

Let $t \in Ann_S(X)$. Then, $tx = 0$ $\forall$ $x \in X$. So, $xt = 0$, as S is reduced. Since c is an upper bound of X, we have $x \leq c$ $\forall$ $x \in X \Rightarrow x^2 = xc$ $\forall$ $x \in X \Rightarrow x^2 = x(c+t)$ $\forall$ $x \in X$ (as $xt = 0$) $\Rightarrow x \leq c + t$ $\forall$ $x \in X \Rightarrow c + t$ is an upper bound of X. Again since, $c = \sup(X)$ then, $c \leq c + t \Rightarrow c^2 = c(c+t) \Rightarrow c^2 = c^2 + ct \Rightarrow ct = 0 \Rightarrow tc = 0 \Rightarrow t \in Ann_S(c)$. Therefore, $t \in Ann_S(X)$ implies that $t \in Ann_S(c)$. So,

$$Ann_S(X) \subseteq Ann_S(c). \tag{1}$$

Let us denote $Ann_S(X^2) = \{t : tx^2 = 0 \forall x \in X\}$. Now let $t \in Ann_S(c)$. Then, $tc = 0 = ct$. So, $x^2t = xct = x0 = 0$ $\forall$ $x \in X$. Therefore, $tx^2 = 0$ $\forall$ $x \in X$, i.e., $t \in Ann_S(X^2)$. Therefore,

$$Ann_S(c) \subseteq Ann_S(X^2). \tag{2}$$

Let $p \in Ann_S(X^2)$. Then, $px^2 = 0 \Rightarrow x^2p = 0 \Rightarrow x(xp) = 0 \Rightarrow (xp)x = 0$. So, $(xp)^2 = (xp)(xp) = (xpx)p = 0x = 0$. Therefore, $xp = 0 \ \forall \ x \in X$. So, $px = 0 \ \forall \ x \in X$ and hence $p \in Ann_S(X)$. Therefore,

$$Ann_S(X^2) \subseteq Ann_S(X). \tag{3}$$

From (1), (2), and (3), we get $Ann_S(X) \subseteq Ann_S(c) \subseteq Ann_S(X^2) \subseteq Ann_S(X)$. Therefore, $Ann_S(X) = Ann_S(c)$. □

To establish (i) from (ii) of Theorem 3.6 for an Abian's semiring S, it is embedded in a ring R as follows.

Proposition 3.8. [4] *Let S be a commutative, additively cancellative semiring with '0'. Then, S can be embedded in a ring R.*

Proof. Define ρ on $S \times S$ by $(a, b)\rho(c, d)$ if and only if $a + d = b + c$. Clearly, ρ is an equivalence relation.

Let R be the quotient ring induced by the equivalence relation ρ, i.e., $R = S|\rho$ where ' + ' and '.' are defined as $\overline{(a, b)} + \overline{(c, d)} = \overline{(a + c, b + d)}$ and $\overline{(a, b)}.\overline{(c, d)} = \overline{(ac + bd, ad + bc)}$.

Let us define $f : S \to R$ by $f(a) = \overline{(a, 0)}$. Then, $f(ab) = f(a)f(b)$ and $f(a + b) = f(a) + f(b)$.

Thus S can be embedded in the ring $R = S|\rho$. □

Proposition 3.9. *Let S be a commutative, additively cancellative Abian's semiring with '0' and $R = S|\rho$ as described in the above proposition. Then, $a \leq b$ in S if and only if $\overline{(a, 0)} \leq \overline{(b, 0)}$ in R.*

Proof. We have,

$$\begin{aligned} a \leq b \text{ in } S \quad &\Leftrightarrow \quad a^2 = ab \text{ in } S, \\ &\Leftrightarrow \quad a^2 + 0 = 0 + ab \text{ in } S, \\ &\Leftrightarrow \quad \overline{(a^2, 0)} = \overline{(ab, 0)} \text{ in } R, \\ &\Leftrightarrow \quad \overline{(a, 0)}.\overline{(a, 0)} = \overline{(a, 0)}.\overline{(b, 0)} \text{ in } R, \\ &\Leftrightarrow \quad \overline{(a, 0)} \leq \overline{(b, 0)} \text{ in } R. \end{aligned}$$

□

Using the above described embedding of the Abian's semiring S in the ring $R = S|\rho$, we have the following lemma.

Lemma 3.10. *Let S be a commutative, additively cancellative Abian's semiring, $X \subseteq S$ and c is an upper bound of X. If for any two elements p, q of S, $px = qx \ \forall \ x \in X$ implies that $pc = qc$ then, $c = \sup(X)$.*

Proof. Let $R = S|\rho$ as described in Proposition 3.8. Firstly, we show that $Ann_S\overline{(X, 0)} \subseteq Ann_S\overline{(c, 0)}$, and $\overline{(c, 0)} = \sup \overline{(X, 0)}$, where $\overline{(X, 0)}$ means the set $\{\overline{(x, 0)} \ \forall \ x \in X\}$.

Let $\overline{(p,q)} \in Ann_S\overline{(X,0)}$. Then,

$$\begin{aligned}
& \overline{(p,q)}.\overline{(x,0)} = \overline{(0,0)} = \overline{(x,0)}.\overline{(p,q)} \ \forall\ x \in X \\
\Rightarrow\ & \overline{(px,qx)} = \overline{(0,0)} \text{ and } \overline{(xp,xq)} = \overline{(0,0)} \ \forall\ x \in X \\
\Rightarrow\ & \overline{(px,qx)}\rho\overline{(0,0)} \text{ and } \overline{(xp,xq)}\rho\overline{(0,0)} \ \forall\ x \in X \\
\Rightarrow\ & px = qx \text{ and } xp = xq \ \forall\ x \in X \\
\Rightarrow\ & pc = qc \text{ and } cp = cq \text{ (as } px = qx \ \forall\ x \in X \text{ implies } pc = qc, \text{ and } S \text{ is commutative)} \\
\Rightarrow\ & \overline{(p,q)} \in Ann_S\overline{(c,0)} \\
\Rightarrow\ & Ann_S\overline{(X,0)} \subseteq Ann_S\overline{(c,0)}.
\end{aligned}$$

Since c is an upper bound of $X \subseteq S$, by Proposition 3.9, $x \leq c \Rightarrow \overline{(x,0)} \leq \overline{(c,0)}$. Again, by Theorem 3.6, $\overline{(c,0)}$ is $\sup\overline{(X,0)}$, i.e., $\overline{(c,0)} = \sup\overline{(X,0)}$. Let d be an upper bound of X. Therefore, $x \leq d\ \forall\ x \in X$. Therefore, $\overline{(x,0)} \leq \overline{(d,0)}\ \forall\ x \in X$. But, $\overline{(c,0)} = \sup\overline{(X,0)}$. Therefore, $\overline{(d,0)} \leq \overline{(c,0)}$. Therefore, $d \leq c$. Hence, $c = \sup(X)$. □

Using Lemma 3.7 and Lemma 3.10, finally we have the following theorem.

Theorem 3.11. *Let S be a commutative, additively cancellative Abian's semiring and $X \subseteq S$. Moreover, for any two elements p, q of S, $px = qx\ \forall\ x \in X$ implies that $pc = qc$ for a fixed $c \in S$. Then, the following are equivalent.*

(i) $c = \sup(X)$,

(ii) *c is an upper bound for X and $Ann_S(X) \subseteq Ann_S(c)$. In fact, $Ann_S(X) = Ann_S(c)$.*

4. ZERO DIVISOR GRAPHS OF ABIAN'S SEMIRING

Relating a graph with an algebraic structure has been studied by a large number of researchers in the recent past. The present authors have studied the comparability graph of an Abian's semiring in [10]. In this section we obtain the partial order based zero divisor graph, ideal based zero divisor graph with respect to a poset ideal, and ideal based zero divisor graph with respect to a semiring ideal of an Abian's semiring and list some properties of those graphs.

Let G be a simple connected graph. The shortest path between two vertices u, v of G is called a geodesic. The length of the largest geodesic in G is called the diameter of G and is denoted by $diam(G)$. The length of the smallest cycle in G is called its girth and is denoted by $gr(G)$. A clique in G is a complete subgraph of G. The number of vertices of the largest clique in G is called the clique number of G and is denoted by $\omega(G)$.

Let S be a poset with respect to the partial order $\leq$. Comparability graph of S is the graph with vertex set S and two vertices x and y are adjacent if and only if $x \leq y$ or $y \leq x$, i.e., x, y are comparable. Comparability graph of S is denoted by $(S, \perp)$. Now we define different types of zero divisor graphs in the following.

Definition 4.1. *Let P be a poset. The zero divisor graph of P, denoted by $G(P)$, is the graph with vertex set $V = \{x \in P \setminus \{0\} : L(x, y) = \{0\}$ for some $y \in P \setminus \{0\}\}$ and two vertices x and y are adjacent if and only if $L(x, y) = \{0\}$.*

Definition 4.2. *Let I be a poset ideal of a poset P. The ideal based zero divisor graph of P with respect to the poset ideal I is the graph $G_I(P)$ whose vertex set is $V(G_I(P)) = \{x \in P \setminus I : L(x, y) \subseteq I$ for some $y \in P \setminus I\}$ and two vertices x, y are adjacent if and only if $L(x, y) \subseteq I$.*

Definition 4.3. *Let $(S, +, .)$ be a semiring and I be a semiring ideal of S. We define an undirected graph $\Gamma_I(S)$ with vertices $V(\Gamma_I(S)) = \{x \in S \setminus I : xy \in I$ for some $y \in S \setminus I\}$, and two distinct vertices x and y are adjacent if and only if $xy \in I$. $\Gamma_I(S)$ is called the ideal based zero divisor graph of S with respect to the semiring ideal I.*

Theorem 4.4. [9] *If I be a poset ideal of a poset P, then $G_I(P)$ is connected with $diam(G_I(P)) \leq 3$. Furthermore, if $G_I(P)$ contains a cycle then $gr(G_I(P)) \leq 7$.*

Using the above propositions, we have the following.

Proposition 4.5. *Let $(S, +, .)$ be an Abian's semiring and $X \subseteq S$. Let $I_1 = Ann_S(X)$ and $I_2 = Ann_P(X)$ be semiring annihilator and poset annihilator of X in S respectively. Then, $G_{I_1}(S), G_{I_2}(S), \Gamma_{I_1}(S)$ are connected graphs with diameter less than or equal to 3. Furthermore, if they contain cycles then girth of each graph will be less than or equal to 7.*

Proposition 4.6. *If S be a finite Abian's semiring with $|S| = n, (S, \perp)$ be its comparability graph and m be the second maximum degree of $(S, \perp)$ then, $\omega(G(S)) > n - m$.*

Proof. Since S is an Abian's semiring, $0 \leq s\ \forall s \in S$. So, in $(S, \perp), 0$ is the maximum degree vertex with degree $n - 1$. Let p be the second maximum degree vertex of $(S, \perp)$. Then there are $m - 1$ non zero vertices adjacent to p. So, there are $n - m$ numbers of non zero vertices which are not adjacent to p but adjacent to 0. For each of these vertices x, $L(x, p) = \{0\}$. Therefore, in $G(S)$, p is a vertex of degree $n-(m-1)$ and so, each vertex of $G(S)$ will be adjacent to at least $n-(m-1)$ number of vertices. This shows that $G(S)$ has a clique of order $n - m + 1$. $\square$

Proposition 4.7. *Let $(S, +, .)$ be an Abian's semiring, $0 \in X \subseteq S$ and $I = Ann_P(X)$. Then $G_I(S)$ will be complete if the comparability graph $(S, \perp)$ is a star.*

Proof. Since the comparability graph $(S, \perp)$ is a star, for every pair of non zero vertices x and y, $L(x, y) = \{0\}$. Hence every two non zero vertices of $G_I(S)$ are adjacent. Now, $0 \in X \Rightarrow I = Ann_P(X) = \{0\}$ and so 0 is not a vertex of $G_I(S)$. Therefore, $G_I(S)$ is complete. $\square$

5. CONCLUSION

In this paper, we have investigated the relationships between ideals and annihilators of an Abian's semiring. A result related to supremum of a subset of an Abian's ring has been extended for an Abian's semiring. We have also given some properties of different zero divisor graphs of an Abian's semiring.

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