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SPLIT DOMINATION VERTEX AND EDGE CRITICAL GRAPHS

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Abstract. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set* if the induced graph $\langle V - D\rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set.* A graph G is called *vertex split domination critical* if $\gamma_s(G-v) < \gamma_s(G)$ for every vertex $v \in G$. A graph G is called *edge split domination critical* if $\gamma_s(G+e) < \gamma_s(G)$ for every edge e in $\overline{G}$. In this chapter, whether for some standard graphs are split domination vertex critical or not are investigated and then characterized 2-γ_{ns}-critical and 3-γ_{ns}-critical graphs with respect to the diameter of a graph G with vertex removal. Further, it is shown that there is no existence of γ_s-critical graph for edge addition.

1. INTRODUCTION

The graphs considered in this paper are finite, connected, undirected and without loops or multiple edges. Terminology not defined here will conform to that in [4]. Let $P_n, C_n, K_{1,n}, K_n, K_{m,n}, W_n, D_n^{(m)}$ denote the *path, cycle, star, complete graph, bipartite graph, wheel graph, and dutch wind mill graph.* An *end vertex* of a graph G is a vertex of degree 1 and an *support vertex* of a graph G is a vertex adjacent to an end vertex.

The neighborhood of a vertex v in G is denoted by $N(v)$ and is given by $N(v) = \{u \in V(G)/u$ is adjacent to v in $G\}$. The *diameter of the graph* G is a measure of the length of the longest minimal path and is denoted by $diam(G)$.

The graph with $n+1$ vertices labeled $u_1, u_2, u_3, u_4, \ldots, u_{n+1}$ and edges u_1u_2, $u_2u_3, u_3u_4, \ldots, u_nu_{n+1}$ is called a path of length n, denoted by P_{n+1}. We call u_1 and u_{n+1} the end-vertices of the path. The cycle of length of n, C_n, is the graph with n vertices $u_0, u_1, u_2, u_3, \ldots, u_{n-1}$ and the edges $u_0u_1, u_1u_2, u_2u_3, \ldots, u_{n-1}u_0$.

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A simple graph of order n with all possible edges is called a *complete graph* of order n, denoted by K_n.

A graph $G = (V, E)$ is said to be r-partite (where r is an positive integer) if its $V(G)$ can be partitioned as $V = V_1 \cup V_2 \cup, \ldots, \cup V_r$ such that if uv is an edge of G then u is in some V_i and v is in some other V_j; that is, everyone of the induced subgraphs $\langle V_i \rangle$ is an empty graph. The graph obtained from G by subdividing each edge of G exactly once is called the *subdivision graph* [1] of G and it denoted by $S(G)$.

A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with v adjacent to u. The minimum cardinality of any dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. For complete review on domination refer [5]. The concept of split domination has been studied by V.R. Kulli and B. Janikiram [6]. A *dominating set* D of a graph $G = (V, E)$ is a *split dominating set* if the induced graph $\langle V - D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ is the minimum cardinality of a *split domination set.*

A graph G is called *vertex split critical* if $\gamma_s(G-v) < \gamma_s(G)$ for every vertex v in G. Thus, G is *k-γ_s-critical* if $\gamma_s(G) = k$, for each vertex $v \in V(G)$, $\gamma_s(G-v) < k$. A graph G is called *edge split critical* if $\gamma_s(G + e) < \gamma_s(G)$ for every edge e in $\overline{G}$. Thus, G is k-γ_s-critical if $\gamma_s(G) = k$ and for each edge $e \in \overline{G}$, $\gamma_s(G + e) < k$.

Let us consider the two terrorist groups say A and B which are interconnected with each other with group members as the vertices and the edges as the communication between them. Among these two groups, their are minimum number of people who had the communication with all the members of the two groups called domination members, among them few may have communication between two groups. Since the two terrorist groups are connected their terrorist activity may increase. Suppose the military people wants to make the terrorists activity to be inactive, it is better to destroy the domination members and also the groups members in such a way that the two groups gets separated so that there is no communication between the groups and also between the members of the groups. This is the motivation for studying split domination.

The domination vertex and edge critical graphs studied extensively in [7, 2, 8, 3, 9]. Since the dominating members are more strong, instead of dominating members, we can destroy the others members along with the dominating members in such a way that, no communication between the two groups. The numbers of people to be destroyed will be less than or equal to the domination members. This motivated us to study split domination vertex and edge critical graphs. First we discuss whether some particular classes of graphs are γ_s-vertex critical or not, $2-\gamma_s$-vertex critical and $3-\gamma_s$-vertex critical graphs are characterized with respect to *diameter* of the graph G for vertex removal and then we had shown that there is no γ_s-critical graph for edge addition.

2. Split domination vertex critical graphs

In this entire section, the graph $G \neq K_n$, considered should be a simple graph and $G-\{v\}, v \in V(G)$ having n components, either contains a non-complete component or at least two non-trivial components.

Theorem 2.1. *(Kulli and Janakiram [6]) For any cycle C_n, $\gamma_s(C_n) = \lceil \frac{n}{3} \rceil, n \geq 4$.*

Theorem 2.2. *(Chelvam and Chellathurai [10]) $\gamma_s(P_n) = \lceil \frac{n}{3} \rceil, n > 2$, where P_n is a path of length $n-1$.*

Theorem 2.3. *(Kulli and Janakiram [6]) For any connected graph G with an end-vertex, $\gamma_s(G) = \gamma(G)$. Furthermore, there exists a γ_s-set of G containing all vertices adjacent to end-vertices.*

Theorem 2.4. *For any connected graph G,*

$$\gamma_s(G) - 1 \leq \gamma_s(G-v) \leq \gamma_s(G) + deg(v) - 1, v \in V(G).$$

Proof. Let G be a connected graph and $v \in V(G)$. Since the domination number will increase by more than one and decreases by at most one when a vertex is removed from G, thus $\gamma_s(G) - 1 \leq \gamma_s(G-v)$. For the upper bound, let H be a γ_s-set of G.
Case 1. Let $v \notin H$, then $\gamma_s(G-v) \leq \gamma_s(G)$.
Case 2. Let $v \in H$, and let $B = \{v_i / v_i \in N(v), v_i \notin N(H-v)\}$.
(i) If $|B| = \phi$, then $\gamma_s(G-v) = |H| - |v|$. Hence $\gamma_s(G-v) < \gamma_s(G)$.
(ii) If $|B| \neq \phi$, then $\gamma_s(G-v) \leq |H| + |B| - |\{v\}| = \gamma_s(G) + |B| - 1$.
Since $|B| \leq deg(v)$, $\gamma_s(G-v) \leq \gamma_s(G) + deg(v) - 1$.
Hence the proof.

Theorem 2.5. *The graph $G = C_n$, $n > 3$ is γ_s-vertex critical for $n = 3p+1, p \geq 1$ and is not γ_s-vertex critical for $n \neq 3p+1$.*

Proof. By Theorem 2.1, $\gamma_s(C_n) = \lceil \frac{n}{3} \rceil$ and in $G - \{v\}$, $v \in V(G)$ will be a path with $n-1$ vertices and by Theorem 2.2, for $n-1$ vertices, $\gamma_s(G-v) = \gamma_s(P_{n-1}) = \lceil \frac{n-1}{3} \rceil$.
Case 1. When $n = 3p+1, p \geq 1$.
Then, $\gamma_s(G-v) = \gamma_s(P_{n-1}) = \lceil \frac{3p+1-1}{3} \rceil = \lceil p \rceil$. Since $\lceil p \rceil < \lceil p + \frac{1}{3} \rceil$ for any $p \geq 1$. Thus $\gamma_s(G-v) < \gamma_s(G)$. Hence $G = C_n$ is γ_s-vertex critical for $n = 3p+1, p \geq 1$.
Case 2. When $n \neq 3p+1, p \geq 1$.
Since $\lceil \frac{n}{3} \rceil = \lceil \frac{n-1}{3} \rceil$ for $n = 3p+2$ or $n = 3p+3$, $p \geq 1$. Thus $\gamma_s(G-v) = \gamma_s(G)$. Hence $G = C_n$ is not γ_s-vertex critical for $n \neq 3p+1, p \geq 1$.

Theorem 2.6. *The path P_n, $n \geq 5$ is not γ_s-vertex critical.*

Proof. Case 1. If v_i is not an end-vertex, then $P_n - \{v_i\}$ is disconnected into two components G_1 and G_2 with $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$ such that $n_1 + n_2 + 1 = n$. For a disconnected graph, $\gamma_s(G) = \gamma(G)$, thus $\gamma_s(P_n - v_i) = \gamma(G_1) + \gamma(G_2)$. Since G_1 and G_2 are paths.
Subcase 3.1. When ($n_1 \neq 3r$ and $n_2 \neq 3r$) or ($n_1 \neq 3r$ and $n_2 = 3r$) or ($n_1 = 3r$ and $n_2 \neq 3r, r \geq 1$),

$$\begin{aligned}\gamma_s(P_n - v_i) &= \lceil \tfrac{n_1}{3} \rceil + \lceil \tfrac{n_2}{3} \rceil \\ &\geq \lceil \tfrac{n_1+n_2}{3} \rceil \\ &\geq \lceil \tfrac{n_1+n-1-n_1}{3} \rceil \\ &\geq \lceil \tfrac{n-1}{3} \rceil = \lceil \tfrac{n}{3} \rceil\end{aligned}$$

Thus $\gamma_s(P_n - v_i) \geq \gamma_s(P_n)$.

Subcase 3.2. When $n_1 = 3r$ and $n_2 = 3r, r \geq 1$,

$$\begin{aligned}\gamma_s(P_n - v_i) &= \lceil \tfrac{n_1}{3} \rceil + \lceil \tfrac{n_2}{3} \rceil \\ &= \tfrac{n_1+n_2}{3} \\ &= \tfrac{n_1+n-1-n_1}{3} \\ &= \tfrac{n-1}{3}\end{aligned}$$

Since $\frac{n-1}{3} < \lceil \frac{n}{3} \rceil$. Thus $\gamma_s(P_n - v_i) < \gamma_s(P_n)$. P_n in not γ_s-critical because $\gamma_s(P_n - v_i) < \gamma_s(P_n)$ only if v_i is not an end-vertex and $n_1 = 3r, n_2 = 3r$ not for any vertex $v_i \in V(P_n)$

Case 2. Suppose v_i is an end-vertex of P_n and $n = 3p + 1, p \geq 2$. By Theorem 2.2, $\gamma_s(P_n) = \lceil \frac{3p+1}{3} \rceil = \lceil p + \frac{1}{3} \rceil$. Thus $\gamma_s(P_n - v_i) = \gamma_s(P_{n-1}) = \lceil \frac{n-1}{3} \rceil = \lceil \frac{3p+1-1}{3} \rceil = \lceil p \rceil$. Since $\lceil p \rceil < \lceil p + \frac{1}{3} \rceil = \lceil \frac{n}{3} \rceil$ for $p \geq 2$. Hence $\gamma_s(P_n - v_i) < \gamma_s(P_n)$. P_n in not γ_s-critical because $\gamma_s(P_n - v_i) < \gamma_s(P_n)$ only if v_i is end-vertex not for any vertex $v_i \in V(P_n)$

Case 3. Suppose v_i is an end-vertex and $n \neq 3p + 1, p \geq 2$. By Theorem 2.2, $\gamma_s(P_n) = \lceil \frac{n}{3} \rceil$. Thus $\gamma_s(P_n - v_i) = \gamma_s(P_{n-1}) = \lceil \frac{n-1}{3} \rceil$. Since $\lceil \frac{n}{3} \rceil = \lceil \frac{n-1}{3} \rceil$ for $n = 3p + 2$ or $n = 3p + 3$, $p \geq 1$. Hence $\gamma_s(P_n - v_i) = \gamma_s(P_n)$.

From all the cases above, $P_n, n \geq 5$ is not γ_s-vertex critical.

Theorem 2.7. *If a graph G is 2-γ_s-vertex critical, then $diam(G) = 2$.*

PROOF. Let G be a connected 2-γ_s-vertex critical graph and suppose G has diameter at least 3. Assume that $P = v_1, v_2, \ldots, v_d$ be the longest diametrical path with the distance equal to the $diam(G)$. Let D^1 be a γ_s-set of $G - \{v\}, v \in$ diametrical path. Since G is 2-γ_s-vertex critical graph, then $\gamma_s(G - v) = 1$ for any vertex $v \in V(G)$ and $|D^1| = 1$. If $v_k \in D^1 \cap P$, then there exists at least one vertex say $v_j \in$ diametrical path in $G - \{v\}$ which is not covered by v_k. Thus v_j is not dominated by any vertex of D^1, which is a contradiction. If $diam(G)$ is one, then G is not γ_s-vertex critical.

Theorem 2.8. *If a connected graph G is 3-γ_s-vertex critical, then $diam(G) \leq 4$.*

PROOF. Let G be a connected 3-γ_s-vertex critical graph and suppose G has diameter at least 5. Assume that $P = v_i, i = 1, 2, 3, \ldots, d$ be the longest diametrical path with its distance equal to the $diam(G)$. Let D_1 be a γ_s-set of $G - \{v\}, v \in$ diametrical path. Since G is 3-γ_s-vertex critical, then $\gamma_s(G - v) \leq 2$ for any vertex $v \in V(G)$. The set D_1 contains at most two vertices say v_1 and v_2.

Case 1: If both v_1 and v_2 are in D_1, then there exists at least one vertex say $v_r \in$ diametrical path in $G - \{v\}$ which is not adjacent to v_1 or v_2 and hence a contradiction.

Case 2: If either of v_1 or v_2 is in D_1, then there exists at least one vertex say v_k in

$G-\{v\}$ which is not covered by v_1 or v_2, which is a contradiction.
This completes the proof.

Theorem 2.9. *No tree $T, n \geq 4$ is γ_s-vertex critical with respect to split domination.*

PROOF. Let D be a γ_s-set of tree T, $A = \{v_i/v_i$ is a support vertex of $T\}$, then by Theorem 2.3, the set D will not contain an end-vertices, $A \subseteq D$. Now consider the graph $T-\{v\}, v \in A$. The graph $T-\{v\}$ is disconnected into $k \geq 2$ components and let $C = \{v_m/v_m \in N(v), v_m \notin N(D-v)\}$.
Case 1. If $|C| = 1$, then $\gamma_s(T-v) = |D| - |\{v\}| + |C| = |D| = \gamma_s(T)$.
Case 2. if $|C| \neq 1$, then $\gamma_s(T-v) = |D| - |\{v\}| + |C| > |D| = \gamma_s(T)$.
Hence the tree T is not γ_s-vertex critical.

Proposition 2.10. *For any Wheel graph W_n, $\gamma_s(W_n) = 3$.*

Theorem 2.11. *The graph $G = W_n$ is γ_s-vertex critical for $n = 5, 6, 7$ and not γ_s-vertex critical for $n \geq 8$.*

PROOF. Let $v_j \in V(G)$ such that $deg(v_j) = n-1$ and let D be a γ_s-set of $G = W_n$.
Case 1. $n = 5, 6, 7$.
Let $v \in V(G), v \neq v_j$, then $\gamma_s(G-v) = |\{v_j\} \cup \{v_k\}| = 2, v_k \notin N(v)$ in G. By Proposition 2.10, $\gamma_s(G-v) < \gamma_s(G)$. If $v = v_j$, then $G-\{v_j\}$ will be a cycle C_4 or C_5 or C_6 and $\gamma_s(C_4$ or C_5 or $C_6) = 2$. By Proposition 2.10, $\gamma_s(G-v) < \gamma_s(G)$.
Hence $G = W_n$ is γ_s-vertex critical for $n = 5, 6, 7$.
Case 2. $n \geq 8$.
Let $v \in V(G), v \neq v_j$, then $\gamma_s(G-v) = |\{v_j\} \cup \{v_k\}| = 2, v_k \notin N(v)$ in G. By Proposition 2.10, $\gamma_s(G-v) < \gamma_s(G)$. If $v = v_j$, then $G-\{v_j\}$ will be a cycle $C_n, n \geq 7$ and $\gamma_s(C_n) \geq 3$. By Proposition 2.10, $\gamma_s(G-v) \geq \gamma_s(G)$. Hence $G = W_n$ is not γ_s-vertex critical for $n \geq 8$. This completes the proof.

Definition 2.12. *The* Dutch wind mill graph $D_n^{(m)}$ *is the graph obtained by taking m copies of the cycle C_n with a vertex in common.*

Proposition 2.13. *For any Dutch wind mill graph $D_n^{(m)}, m \geq 2, n \geq 4$, $\gamma_s(D_n^{(m)}) = m\lceil \frac{n-3}{3} \rceil + 1$.*

Theorem 2.14. *The Dutch wind mill graph $G = D_n^{(m)}, m \geq 2, n \geq 4$, is γ_s-vertex critical for $n = 3p+1, p \geq 1$ and not γ_s-vertex critical for $n \neq 3p+1, p \geq 1$.*

PROOF. Let $v_k \in V(D_n^{(m)})$ with $deg(v_k) = 2m$.
Case 1. If $v = v_k$, then $G-\{v_k\}$ is disconnected into m components, where each component is of P_{n-1}. By Theorem 2.2, $\gamma_s(G-v_k) = m\lceil \frac{n-1}{3} \rceil$.
Subcase 1.1. If $n = 3p+1, p \geq 1$, then $m\lceil \frac{n-1}{3} \rceil = m\lceil \frac{n-3}{3} \rceil, m \geq 2, n \geq 4$. Hence by Proposition 2.13, $\gamma_s(G-v_k) < \gamma_s(G)$.
Subcase 1.2. If $n \neq 3p+1, p \geq 1$, then $m\lceil \frac{n-1}{3} \rceil > m\lceil \frac{n-3}{3} \rceil, m \geq 2, n \geq 4$. Hence by Proposition 2.13, $\gamma_s(G-v_k) > \gamma_s(G)$.
Case 2. If $v \neq v_k$, then $G-v = G_1 \cup G_2$, where $G_1 = D_n^{(m-1)}$ and $G_2 = P_{n-1}$ with

$V(G-\{v\}) = V(G_1-\{v_1\}) \cup V(G_2-\{v_2\}) \cup v_3, v_1 = v_2 = v_3, v_1 \in V(G_1), v_2 \in V(G_2)$ and $E(G-\{v\}) = E(G_1) \cup E(G_2)$. Let A and B be a γ_s-set of G_1 and G_2.
Subcase 2.1. If $n = 3p+1, p \geq 1$.
The set B has to contain a vertex v_k or $N(v_k), v_k \in V(D_n^{(m)})$, then
$\gamma_s(G_1) = |A| = (m-1)\gamma_s(P_{n-1}) = m\lceil\frac{n-1}{3}\rceil$. Thus $\gamma_s(G-v) = |A|+|B| = m\lceil\frac{n-1}{3}\rceil + \lceil\frac{n-1}{3}\rceil = m\lceil\frac{n-1}{3}\rceil$. Thus $\gamma_s(G-v) = m\lceil\frac{n-1}{3}\rceil < m\lceil\frac{n-1}{3}\rceil + 1 = \gamma_s(G)$. Hence $G = D_n^{(m)}$ is γ_s-vertex critical for $n = 3p+1, p \geq 1$.
Subcase 2.2. If $n \neq 3p+1, p \geq 1$.
The set B has to contain a vertex v_k, then
$\gamma_s(G_1) = |A| = (m-1)\gamma_s(D_n^{(m-1)}) = (m-1)\lceil\frac{n-3}{3}\rceil + 1$. Thus $\gamma_s(G-v) = |A|+|B| = (m-1)\lceil\frac{n-3}{3}\rceil + 1 + \lceil\frac{n-3}{3}\rceil = m\lceil\frac{n-3}{3}\rceil + 1$. Thus by Proposition 2.13, $\gamma_s(G-v) = \gamma_s(G)$. Hence $G = D_n^{(m)}$ is not γ_s-vertex critical for $n \neq 3p+1, p \geq 1$.
The result follows from the cases above.

Theorem 2.15. *If G is γ_s-vertex critical, then there is no support vertex in G which is adjacent to one or more end-vertices.*

PROOF. Suppose v_1 is a support vertex which is adjacent to at least one end-vertex, say x_1 of a graph G and let $G_1 = G - \{v_1\}$. Let D_1 be a γ_s-set of G. Now consider the graph G_1, since v_1 is a support vertex in G by Theorem 2.3, $v_1 \in D_1$ and the graph G_1 is disconnected. Let $A_1 = \{v_m/v_m \in N(v_1), v_m \notin N(D_1 - v_1)\}$.
Case 1. If $|A_1| = 1$, then $\gamma_s(G_1) = |D_1| - |\{v_1\}| + |\ A_1| = |D_1|$. Thus $\gamma_s(G_1) = \gamma_s(G)$. Which contradicts our assumption.
Case 2. If $|A_1| > 1$, then $\gamma_s(G_1) = |D_1| - |\{v_1\}| + |A_1| > |D_1|$. Thus $\gamma_s(G_1) > \gamma_s(G)$. Which contradicts our assumption.
Hence the proof.

Corollary 2.16. *If G is γ_s-vertex critical, then no two support vertices are adjacent.*

Proposition 2.17. *For any subdivision graph of K_n, $\gamma_s(S(K_n)) = n-1, n \geq 3$.*

Theorem 2.18. *The graph $G = S(K_n)$ is not γ_s-vertex critical for $n \geq 3$.*

PROOF. Let us assume that $G = S(K_n)$ is γ_s-vertex critical, then for each vertex $v \in V(G)$, $\gamma_s(G-v) < \gamma_s(G)$. Let us consider the graph $G-\{v\}$, $v \in V(K_n)$ and $B = \{v_i/v_i$ is a support vertex in $G-\{v\}\}$ with $|B| = n-1$ and by Theorem 2.3, B belongs to γ_s-set of $G-\{v\}$ and $\langle G-\{v\}-B\rangle$ is disconnected. Thus $\gamma_s(G-v) = |B| = n-1$. By Proposition 2.17, $\gamma_s(G-v) = \gamma_s(G)$. Which contradicts our assumption. Hence the proof.

Theorem 2.19. *The graph $G = S(D_n^{(m)})$ is γ_s-vertex critical for $n = 3p+2, p \geq 1$ and not γ_s-vertex critical for $n \neq 3p+2, p \geq 1, m \geq 2, n \geq 4$.*

PROOF. Case 1. When $n = 3p+2, p \geq 1$.
The graph $S(D_n^{(m)}), n = 3p+2, p \geq 1$ is the graph which is obtained by a subdivision of $D_n^{(m)}$ for $n = 3k+2, k \geq 1$. By Theorem 2.14, $G = S(D_n^{(m)})$ is γ_s-vertex critical for $n = 3p+2, p \geq 1.$,

Case 2. When $n \neq 3p+2, p \geq 1$.
The graph $S(D_n^{(m)}), n \neq 3p+2, p \geq 1$ is the graph which is obtained by a subdivision of $D_n^{(m)}$ for $n \neq 3k+2, k \geq 1$. By Theorem 2.14, $G = S(D_n^{(m)})$ is not γ_s-vertex critical for $n \neq 3p+2, p \geq 1$.
The result follows from the cases above.

Proposition 2.20. *For any subdivision graph of $K_{m,n}$,*

$$\gamma_s(S(K_{m,n})) = m+n-1, m \geq n, m, n \geq 2.$$

Theorem 2.21. *The connected graph $G = S(K_{m,n})$ is not γ_s-vertex critical for $m \geq n, m, n \geq 2$.*

Proof. Let us assume that the graph $G = S(K_{m,n})$ is γ_s-vertex critical, then for each vertex $v \in V(G)$, $\gamma_s(G-v) < \gamma_s(G)$. Let $V(K_{m,n}) = V_1 \cup V_2$ where $|V_1| = m, |V_2| = n, A = V(S(K_{m,n})) - V(K_{m,n})$. Let us consider the graph $G - \{v\}$, $v \in V_2$ and $E = \{v_k / v_k$ is a support vertex in $G - \{v\}\}$ with $|E| = m$ and by Theorem 2.3, E belongs to γ_s-set of $G - \{v\}$ and $\langle G - \{v\} - E \rangle$ is disconnected. Thus $\gamma_s(G-v) = |E| + |V_2| - |\{v\}| = m+n-1$. By Proposition 2.20, $\gamma_s(G-v) = \gamma_s(G)$. Which contradicts our assumption. Hence the proof.

Proposition 2.22. *For any subdivision graph of W_n,*

$$\gamma_s(S(W_n)) = \lceil \tfrac{2(n-1)}{3} \rceil + 1, n \geq 4.$$

Theorem 2.23. *The graph $G = S(W_n)$ is not γ_s-vertex critical for $n \geq 4$.*

Proof. Let us assume that $G = S(W_n)$ is γ_s-vertex critical, then for each vertex $v \in V(G)$, $\gamma_s(G-v) < \gamma_s(G)$. Let us consider the graph $G - \{v\}$, $v \in V(W_n), deg(v) = n-1$ and let $B = \{v_i / v_i$ is an end-vertex $\in G - \{v\}\}$ with $|B| = n-1$ and by Theorem 2.3, $N(B)$ belongs to γ_s-set of $G - \{v\}$ and $\langle G - \{v\} - N(B) \rangle$ is disconnected. Thus $\gamma_s(G-v) = |B| = n-1$ and $\lceil \frac{2(n-1)}{3} \rceil + 1 \geq n-1$ for $n \geq 4$, by Proposition 2.22, $\gamma_s(G-v) \geq \gamma_s(G)$. Which contradicts our assumption. Hence the proof.

3. Split domination edge critical graphs

Definition 3.1. *A graph G is called* edge split critical, *if $\gamma_s(G+e) < \gamma_s(G)$ for every edge e in $\overline{G}$. Thus G is k-γ_s-*critical *if $\gamma_s(G) = k$ for each edge $e \in E(\overline{G})$, $\gamma_s(G+e) < k$.*

In this entire section, the graph considered should be a connected non-complete graph.

Lemma 3.2. *For any connected graph G, the cardinality of the cut-set will remain equal or increases, if an edge e_1 is added to a graph G, where $e_1 \in E(\overline{G})$, $(n \geq 4)$.*

Proof. Let $V(G) = V(\overline{G}) = \{v_{11}, v_{12}, \ldots, v_{1n}\}$ and C be a minimum cut-set of a graph G. Then the graph $G - C$ is disconnected into at least two components say G_1 and G_2 with $V(G_1) = \{v_{21}, v_{22}, \ldots, v_{2n}\}$ and $V(G_2) = \{v_{31}, v_{32}, \ldots, v_{3n}\}$. Let us consider the graph $G_1 = G + e_1$, where $e_1 = (v_{2i}, v_{2j}) \in E(\overline{G})$ for $i \neq j$. Then the graph G_1 is disconnected and hence the cut-set remains same. Otherwise if $e_1 = (v_{2i}, v_{3j}) \in E(\overline{G})$, then the graph G_1 is connected and thus we need a vertex of G to make the graph G_1 disconnected. Hence the cardinality of the cut-set will remains same or increase, if an edge e_1 is added to a graph G, where $e_1 \in E(\overline{G})$. This completes the proof.

Lemma 3.3. *Every minimum split domination set must contains the cut-set of a graph G.*

Theorem 3.4. *There is no edge critical graph with respect to split domination.*

PROOF. Let D_1 be a γ_s-set of G. Let us consider the graph $G+e$, where $e \in E(\overline{G})$. By Lemma 3.1 and Lemma 3.2, $\gamma_s(G+e) \geq |D_1 + 1|$. Hence with respect split domination, G is not edge critical.

Theorem 3.5. *For any connected graph G,*

$$\gamma_s(G) \leq \gamma_s(G+e) \leq \gamma_s(G) + 1, e \in E(\overline{G}),\ (n \geq 4).$$

PROOF. Let D and D_1 be a γ_s-set of G and $G+e, e \in E(\overline{G})$. From Lemma 3.1 and 3.2, $\gamma_s(G) \leq \gamma_s(G+e)$. Let A be a minimum cut-set of a graph G. By Lemma 3.2, $A \subseteq D$. The graph $G - A$ will be disconnected into k number of components $G_1, G_2, G_3, \ldots, G_k$.
Case 1. If $e = v_1v_2 \in E(\overline{G}), \{v_1, v_2\} \in V(G_1)$, then $\gamma_s(G) = \gamma_s(G+e)$.
Case 2. If $e = v_1v_2 \in E(\overline{G}), v_1 \in V(G_1), v_2 \in V(G_2)$, then at least one of $\{v_1\}$ or $\{v_2\}$ must belong to D_1 to make the graph $G+e$ disconnected. Thus $\gamma_s(G+e) = |D_1| = |D| + |\{v_1\}$ or $\{v_2\}| = \gamma_s(G) + 1$. Hence $\gamma_s(G+e) \leq \gamma_s(G) + 1$.
This completes the proof.

4. CONCLUSION

In this paper, it is verified that some standard graphs along with any connected graphs are split domination vertex critical or not. In the military network database, they will have an idea what type of network they can have in such a way that the network or part of the network is critical. In the network if any of the domination members is not available, they can reduce the domination members by at least one.

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