J. Indones. Math. Soc.
Vol. 25, No. 02 (2019), pp. 128–138.

ON β-PRIME SUBMODULES

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Abstract. We introduce the concepts of β-prime submodules and weakly β-prime submodules of unital left modules over a commutative ring with nonzero identity. Some properties of these concepts are investigated. We use the notion of the product of two submodules to characterize β-prime submodules of a multiplication module. Characterization of β-prime and weakly β-prime submodules of arbitary modules are also given.

Key words and Phrases: β-prime submodules, weakly β-prime submodules

Abstrak. Pada paper ini diperkenalkan konsep submodul β-prima and submodul β-prima lemah dari modul kiri *unital* atas ring komutatif dengan unsur identitas tak-nol. Beberapa sifat dari konsep ini akan dikaji. Pada paper ini, digunakan notasi produk dari dua submodul untuk mengkarakterisasi submodul β-prima dari modul perkalian. Selain itu, akan dikarakterisasi juga submodul β-prima dan β-prima lemah dari modul sebarang yang diberikan.

Kata kunci: submodul β-prima, submodul β-prima lemah

1. INTRODUCTION

Throughout this paper all rings are assumed to be commutative with nonzero identity and all left modules are unital. Let $(G, +)$ be a group and $H \subseteq G$. We denote the symbol $\beta(H)$ by $\{h + h \mid h \in H\}$ and $\alpha(H)$ by $\{h \mid h + h \in H\}$. It is clear that $\beta(H) \subseteq H \subseteq \alpha(H)$. Let M be a left R-module. If N is a submodule of an R-module M, by $(N : M)$ we mean $\{r \in R \mid rM \subseteq N\}$. For an element $x \in M$

2010 Mathematics Subject Classification: 16D99.
Received: 24-04-2018, accepted: 10-10-2018.

and a submodule N of M, we will denote $\{r \in R \mid rx \in N\}$ with the short form $(N : x)$.

The first Lemma obtains properties of $\alpha(H)$ and $\beta(H)$ where H are ideals and submodules, respectively. The proof of the following lemma is routine and so we omit it.

Lemma 1.1. *Let I be an ideal of R and N be a submodule of an R-module M. Then*

(i) $\beta(I)$ and $\alpha(I)$ are ideals of R.
(ii) $\beta(N)$ and $\alpha(N)$ are submodules of M.

Let R be a commutative ring with identity. A unitary left R-module M is a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$. Recall that a proper submodule P of a module M over a commutative ring R is said to be prime submodule if whenever $rm \in P$ for some $r \in R$ and $m \in M$, then $r \in (P : M)$ or $m \in P$. Historically, Z. El-Bast and P. Smith [3] introduced the notion of multiplication modules and gave a characterization of prime submodules of a unital module. The definition of a multiplication module leads to the product of two submodules which is showed by R. Ameri [1] that this product is well-defined and is used to characterize prime submodule of a multiplication module.

S. Atani and F. Farzalipour in [2] defined a weakly prime submodules, i.e., a proper submodule P of an R-module M with the property that for $r \in R$ and $m \in M$, $0 \neq rm \in P$ implies $r \in (P : M)$ or $m \in P$. Every prime submodule of a module is a weakly prime submodule. However, a weakly prime submodule need not be prime. This result obtains that a weakly prime submodule is a generalization of a prime submodule. Various properties of prime submodules and weakly prime submodules are considered (see [1] and [2]).

The major objective of this paper is to study a generalization of prime submodules. Our idea is to shrink and stretch a submodule of a module by taking β and α, respectively. That is, we shrink a submodule N of a module M to a submodule $\beta(N)$ and we stretch a submodule N of a module M to a submodule $\alpha(N)$. By shrinking and stretching, we get another generalization of a prime submodule, namely, β-prime submodules.

2. β-PRIME SUBMODULES

The aim of this section is to introduce the generalization of prime submodules in a different way which is motivated by the literature.

Definition 2.1. *Let P be a proper submodule of M. We call P is β-**prime** if for any element $r \in R$ and $m \in M$ such that $rm \in P$, we have $r + r \in (P : M)$ or $m + m \in P$.*

Clearly, every prime submodule of M is a β-prime submodule of M. In $\mathbb{Z}$ as $\mathbb{Z}$-module, $8\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ and $8\mathbb{Z}$ is not a prime submodule of $\mathbb{Z}$, (see Example 5.1), a β-prime submodule need not to be a prime submodule.

Theorem 2.2. *Let P be a proper submodule of an R-module M. The following statements are equivalent.*

(i) P is a β-prime submodule of M.

(ii) For each ideal I of R and for each submodule N of M,

$$\text{if } IN \subseteq P, \text{ then } \beta(I) \subseteq (P : M) \text{ or } \beta(N) \subseteq P.$$

(iii) For each $a \in R$ and for each submodule N of M,

$$\text{if } aN \subseteq P, \text{ then } a + a \in (P : M) \text{ or } \beta(N) \subseteq P.$$

(iv) For each ideal I of R and for each $m \in M$,

$$\text{if } Im \subseteq P, \text{ then } \beta(I) \subseteq (P : M) \text{ or } m + m \in P.$$

(v) For each $a \in R$ and for each $m \in M$,

$$\text{if } aRm \subseteq P, \text{ then } a + a \in (P : M) \text{ or } m + m \in P.$$

(vi) For each $x \in M$, if $x + x \notin P$, then $(P : x) \subseteq \alpha((P : M))$.

Proof. The proof is straightforward. □

Proposition 2.3. *Let $\phi : M_1 \to M_2$ be an R-module homomorphism. Then*

(i) If ϕ is an epimorphism and P is a β-prime submodule of M_1 containing $\ker \phi$, then $\phi(P)$ is a β-prime submodule of M_2.

(ii) If K is a β-prime submodule of M_2, then $\phi^{-1}(K)$ is a β-prime submodule of M_1.

(iii) If N is a β-prime submodule of M_1 and K is a submodule of M_1 contained in N, then N/K is a β-prime submodule of M_1/K.

Proof. These proof are trivial. □

Let R_1 and R_2 be commutative rings with identity, M_i be a unital R_i-module where $i = 1, 2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module under the operation $(r_1, r_2)(m_1, m_1) = (r_1 m_1, r_2 m_2)$ for all $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$. We have the following results.

Lemma 2.4. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and P be an R_1-submodule of M_1. If $r_1 \in R_1$ and $r_2 \in R_2$ with $r_1 + r_1 \in (P : M_1)$, then $(r_1, r_2) + (r_1, r_2) \in (P \times M_2 : M_1 \times M_2)$.*

Proof. It is evident. □

Proposition 2.5. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let P be an R_1-submodule of M_1. Then P is a β-prime submodule of M_1 if and only if $P \times M_2$ is a β-prime submodule of $M_1 \times M_2$.*

Proof. ($\rightarrow$) Assume that P is a β-prime submodule of M_1. Let $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ be such that $(r_1, r_2)(m_1, m_2) \in P \times M_2$. Then $(r_1m_1, r_2m_2) \in P \times M_2$. This means $r_1m_1 \in P$ and $r_2m_2 \in M_2$. Since P is a β-prime submodule of M_1, $r_1 + r_1 \in (P : M_1)$ or $m_1 + m_1 \in P$. By Lemma 2.4, we have $(r_1, r_2) + (r_1, r_2) \in (P \times M_2 : M_1 \times M_2)$ or $(m_1, m_2) + (m_1, m_2) \in P \times M_2$. Therefore $P \times M_2$ is a β-prime submodule of $M_1 \times M_2$.

($\leftarrow$) Assume that $P \times M_2$ is a β-prime submodule of $M_1 \times M_2$. Let $r \in R$ and $m \in M_1$ be such that $rm \in P$. Then $(r, 0)(m, 0) = (rm, 0) \in P \times M_2$. This implies that $(r, 0) + (r, 0) \in (P \times M_2 : M_1 \times M_2)$ or $(m, 0) + (m, 0) \in P \times M_2$. Therefore $r + r \in (P : M_1)$ or $m + m \in P$. This proves that P is a β-prime submodule of M_1. □

Similarly ways, we have

Proposition 2.6. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let P be an R_2-submodule of M_2. Then P is a β-prime submodule of M_2 if and only if $M_1 \times P$ is a β-prime submodule of $M_1 \times M_2$.*

Multiplication module play an important role in studying prime submodules. In [3], Z. El-Bast and P. Smith proved that a module M is a multiplication module if and only if $N = (N : M)M$ for all submodule N of M.

The definition of the product of two submodules was given by R. Ameri [1] as follows. Let N and K be submodules of a multiplication module M. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$. R. Ameri cleverly used the concept of product of submodules to characterize prime submodules in a multiplication module. Also, we use this notion to characterize β-prime submodules. Before doing that, we give a useful Lemma.

Lemma 2.7. *Let U and P be submodules of an R-module M and I be an ideal of R such that $U = IM$. If $\{u + u \mid u \in U\} \nsubseteq P$, then there are $r \in I$ and $y \in M \backslash P$ such that $ry + ry \notin P$.*

Proof. Assume that $\{u + u \mid u \in U\} \nsubseteq P$. Then $u + u \notin P$ for some $u \in U$. Since $U = IM$, $u = \sum_{i=1}^{k} r_im_i$ for some $r_i \in I$, $m_i \in M$ and integer k. Note that $u + u = \sum_{i=1}^{k}(r_im_i + r_im_i)$. Since $u + u \notin P$, we get that $r_im_i + r_im_i \notin P$ for some $i \in \{1, 2, \ldots, k\}$. □

Note that for all $m, n \in M$, we denote mn the product of the submodules Rm and Rn of M.

Theorem 2.8. *Let P be a proper submodule of a multiplication module M. Then*

(i) P is a β-prime submodule of M.
(ii) For each submodules U and V of M, if $UV \subseteq P$, then $\beta(U) \subseteq P$ or $\beta(V) \subseteq P$.
(iii) For every $m, n \in M$, if $m \cdot n \subseteq P$, then $m + m \in P$ or $n + n \in P$.

Proof. $(i) \to (ii)$ Assume that P is a β-prime submodule of M. Let U and V be submodules of M such that $UV \subseteq P$. Suppose that $\{u + u \mid u \in U\} \nsubseteq P$ and $\{v + v \mid v \in V\} \nsubseteq P$. Let I and J be ideals of R such that $U = IM$ and $V = JM$. There exist $r \in I, s \in J$ and $x, y \in M \backslash P$ such that $ry + ry \notin P$ and $sx + sx \notin P$. Also, $rsx \in IJM \subseteq P$. Since P is a β-prime submodule of M and $sx + sx \notin P$, we have $(r + r)M \subseteq P$ which leads to a contradiction to $ry + ry \notin P$.

$(ii) \to (iii)$ Clearly.

$(iii) \to (i)$ Assume that (iii) holds. Let $r \in R$ and $m \in M$ be such that $rm \in P$ and $m + m \notin P$. To show that $(r + r)M \subseteq P$, let $n \in M$. Then there are ideals I and J of R such that $Rm = IM$ and $Rn = JM$. This implies that $Rrn = rJM$. Then $rn \cdot m = rJIM = JrRm = JRrm \subseteq JRP \subseteq P$. By (iii) and $m + m \notin P$, we have $rn + rn \in P$. This shows that $(r + r)M \subseteq P$. □

3. β-MULTIPLICATIVE SYSTEM

Definition 3.1. *Let R be a ring and M be an R-module. A nonempty set $S \subseteq M \backslash \{0\}$ is called a β-**multiplicative system** if for all ideals I of R and for all submodules K and N of M, if $\Big(K + \beta(I)M\Big) \cap S \neq \emptyset$ and $\Big(K + \beta(N)\Big) \cap S \neq \emptyset$, then $\Big(K + IN\Big) \cap S \neq \emptyset$.*

Proposition 3.2. *Let P be a submodule of an R-module M. Then P is a β-prime submodule of M if and only if $M \backslash P$ is a β-multiplicative system.*

Proof. $(\to)$ Assume that P is a β-prime submodule of M. Let I be an ideal of R and let K and N be submodules of M such that $\Big(K + IN\Big) \cap M \backslash P = \emptyset$. Then $K + IN \subseteq P$. It follows that $K \subseteq P$ and $IN \subseteq P$. Since P is a β-prime submodule of M, $\beta(I)M \subseteq P$ or $\beta(N) \subseteq P$. Hence $K + \beta(I)M \subseteq P$ or $K + \beta(N) \subseteq P$. This leads to $\Big(K + \beta(I)M\Big) \cap M \backslash P = \emptyset$ or $\Big(K + \beta(N)\Big) \cap M \backslash P = \emptyset$. This obtains that $M \backslash P$ is a β-multiplicative system.

($\leftarrow$) Assume that $M\backslash P$ is a β-multiplicative system. Let I be an ideals of R and N be a submodule of M such that $IN \subseteq P$. Hence $\Big(IN\Big) \cap M\backslash P = \emptyset$. Since $M\backslash P$ is a β-multiplicative system, $\Big(\beta(I)M\Big) \cap M\backslash P = \emptyset$ or $\Big(\beta(N)\Big) \cap M\backslash P = \emptyset$. That is, $\beta(I)M \subseteq P$ or $\beta(N) \subseteq P$. Therefore P is a β-prime submodule of M. $\square$

Proposition 3.3. *Let P be a submodule of an R-module M. The following statements are equivalent.*

(i) P is a β-prime submodule of M.
(ii) $M\backslash P$ is a β-multiplicative system.
(iii) For every ideal I of R and for every $m \in M$,

$$\text{if } \beta(I)M \cap M\backslash P \neq \emptyset \text{ and } m + m \in M\backslash P, \text{ then } Im \cap M\backslash P \neq \emptyset.$$

(iv) For every $r \in R$ and for every $m \in M$,

$$\text{if } (r + r)M \cap M\backslash P \neq \emptyset \text{ and } m + m \in M\backslash P, \text{ then } rRm \cap M\backslash P \neq \emptyset.$$

Proof. $(i) \leftrightarrow (ii)$ by Proposition 3.2.

$(ii) \rightarrow (iii)$ Assume that $M\backslash P$ is a β-multiplicative system. Let I be an ideal of R and $m \in M$ such that $\beta(I)M \cap M\backslash P \neq \emptyset$ and $m + m \in M\backslash P$. Since $m \in Rm$, we have $m + m \in \beta(Rm)$. This means $\beta(Rm) \cap M\backslash P \neq \emptyset$. Since $M\backslash P$ is a β-multiplicative system, $Im \cap M\backslash P = IRm \cap M\backslash P \neq \emptyset$.

$(iii) \rightarrow (iv)$ Assume that (iii) holds. Let $r \in R$ and $m \in M$ be such that $(r+r)M \cap M\backslash P \neq \emptyset$ and $m+m \in M\backslash P$. Then $\beta(Rr)M \cap M\backslash P \neq \emptyset$. By (iii), we have $rRm \cap M\backslash P \neq \emptyset$.

$(iv) \rightarrow (i)$ Let $r \in R$ and $m \in M$ be such that $(r+r)M \nsubseteq P$ and $m+m \notin P$. This implies that $(r + r)M \cap M\backslash P \neq \emptyset$. By (iv), we have $rRm \cap M\backslash P \neq \emptyset$. If $rm \in P$, then $rRm \subseteq P$ which is a contradiction. Therefore $rm \notin P$. This shows that P is a β-prime submodule of M. $\square$

Proposition 3.4. *Let M be an R-module and X be a β-multiplicative system. If P is a submodule of M maximal with respect to the property that $P \cap X = \emptyset$, then P is a β-prime submodule of M.*

Proof. Assume that P is a submodule of M maximal with respect to the property that $P \cap X = \emptyset$. Let I be an ideal of R and N be a submodule of M. Assume that $\beta(I)M \nsubseteq P$ and $\beta(N) \nsubseteq P$. Then $\Big(P+\beta(I)M\Big)\cap X \neq \emptyset$ and $\Big(P+\beta(N)\Big)\cap X \neq \emptyset$. Since X is a β-multiplicative system, $\Big(P+IN\Big)\cap X \neq \emptyset$. Since $P\cap X = \emptyset$, $IN \nsubseteq P$. This implies that P is a β-prime submodule of M. $\square$

Definition 3.5. *Let M be an R-module and N be a submodule of M. If there is a β-prime submodule of M containing N, then we define*

$$\sqrt[\beta]{N} = \{x \in M \mid \text{ every } \beta\text{-multiplicative system containing } x \text{ meets } N\}.$$

If there is no a β-prime submodule of M containing N, then we define $\sqrt[\beta]{N} = M$.

Theorem 3.6. *Let M be an R-module and N be a submodule of M. Then either $\sqrt[\beta]{N} = M$ or $\sqrt[\beta]{N}$ is the intersection of all β-prime submodule of M containing N.*

Proof. Assume that $\sqrt[\beta]{N} \neq M$. Let $x \in \sqrt[\beta]{N}$ and P be a β-prime submodule of M containing N. Then $M \backslash P$ is a β-multiplicative system and $N \cap (M \backslash P) = \emptyset$. Hence $x \in P$. Conversely, let $x \in M$ be such that $x \notin \sqrt[\beta]{N}$. Let S be a β-multiplicative system such that $x \in S$ and $S \cap N = \emptyset$. We apply Zorn's lemma on the set of submodule J of M containing N and $S \cap J = \emptyset$. Then we have a submodule K of M that is maximal with respect to the property $S \cap K = \emptyset$. By Proposition 3.4, K is a β-prime submodule of M. Hence $x \notin K$. □

4. WEAKLY β-PRIME SUBMODULES

In 2007, E. Atani and F. Farzalipour [2] gave the notion of weakly prime submodules as the generalization of prime submodules. A proper submodule P of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in P$ for some $r \in R$ and $m \in M$, then $r \in (P : M)$ or $m \in P$. In this section we extend weakly prime submodules to weakly β-prime submodules. Some of its properties are also investigated.

Definition 4.1. *Let P be a proper submodule of M. We call P is **weakly β-prime** if for any $r \in R$ and $m \in M$ such that $rm \in P \backslash \{0\}$, we have $r + r \in (P : M)$ or $m + m \in P$.*

It is clear that every β-prime submodule is a weakly β-prime submodule. Also, every weakly prime submodule is a weakly β-prime submodule. However, weakly β-prime submodules need not to be β-prime submodules or weakly prime submodules.

Theorem 4.2. *If P is a weakly β-prime submodule of M and $(P : M)\beta(P) \neq 0$, then P is a β-prime submodule of M.*

Proof. Assume that P is a weakly β-prime submodule of M and $(P : M)\beta(P) \neq 0$. Let $r \in R$ and $m \in M$ be such that $rm \in P$. If $rm \neq 0$, then $r + r \in (P : M)$ or $m + m \in P$. Assume that $rm = 0$. We have the following two cases.

Case 1. $r\beta(P) \neq 0$.

Let $n_0 \in P$ be such that $r(n_0+n_0) \neq 0$. Then $r(m+n_0+n_0) = r(n_0+n_0) \in P$. Since P is a weakly β-prime submodule of M and $n_0 \in P$, we have $r+r \in (P : M)$ or $m + m \in P$.

Case 2. $r\beta(P) = 0$. We divide this case into two subcases as follows.

Subcase 2.1. $(P : M)m \neq 0$.

Let $t \in (P : M)$ be such that $tm \neq 0$. Then $(r + t)m = rm + tm = tm \in P$. Since P is a weakly β-prime submodule of M and $t \in (P : M)$, $r + r \in (P : M)$ or $m + m \in P$.

Subcase 2.2. $(P : M)m = 0$.

Since $(P : M)\beta(P) \neq 0$, we have $k(n + n) \neq 0$ for some $k \in (P : M)$ and $n \in P$. Then $(r + k)(m + n + n) = k(n + n) \in P$. Since P is a weakly β-prime submodule of M, $r + k + r + k \in (P : M)$ or $m + n + n + m + n + n \in P$. Since $k \in (P : M)$ and $n \in P$, $r + r \in (P : M)$ or $m + m \in P$. This proves that P is a β-prime submodule of M. □

Theorem 4.3. *If P is a weakly β-prime submodule of a multiplication module M and P is not a β-prime submodule of M, then $\beta(P)^2 = \{0\}$.*

Proof. Assume that P is a weakly β-prime submodule of a multiplication module M and P is not a β-prime submodule of M. By Theorem 4.2, $(P : M)\beta(P) = \{0\}$. Then $\beta(P)^2 = (\beta(P) : M)(\beta(P) : M)M = (\beta(P) : M)\beta(P) \subseteq (P : M)\beta(P) = \{0\}$. Hence $\beta(P)^2 = \{0\}$. □

Theorem 4.4. *Let M be an R-module and P be a submodule of M. The following statements are equivalent.*

(i) P is a weakly β-prime submodule of M.
(ii) $(P : m) \subseteq \alpha((P : M)) \cup (0 : m)$ for all $m \in M\backslash\alpha(P)$.
(iii) $(P : m) \subseteq \alpha((P : M))$ or $(P : m) \subseteq (0 : m)$ for all $m \in M\backslash\alpha(P)$.

Proof. $(i) \to (ii)$ Assume that P is a weakly β-prime submodule of M. Let $m \in M\backslash\alpha(P)$ and $r \in (P : m)$. Then $rm \in P$. If $rm = 0$, then $r \in (0 : m)$. Assume that $rm \neq 0$. Since P is a weakly β-prime submodule of M and $m + m \notin P$, $r + r \in (P : M)$. Hence $r \in \alpha((P : M))$.

$(ii) \to (i)$ Assume that $(P : m) \subseteq \alpha((P : M)) \cup (0 : m)$ for all $m \in M\backslash\alpha(P)$. Let $r \in R$ and $m \in M$ be such that $rm \in P\backslash\{0\}$ and $m + m \notin P$. Then $(P : m) \subseteq \alpha((P : M)) \cup (0 : m)$. Since $rm \neq 0$ and $r \in (P : m)$, we have $r \in \alpha((P : M))$. Hence $r + r \in (P : M)$. That is P is a weakly β-prime submodule of M.

It is clear that $(ii) \leftrightarrow (iii)$. □

Let M_1 and M_2 be R-modules. Then $M_1 \times M_2$ is an R-module under the operations $(a, b) + (c, d) = (a + c, b + d)$ and $r(a, b) = (ra, rb)$ for all $a, c \in M_1$, $b, d \in M_2$ and $r \in R$. We denote this module by $M_1 \oplus M_2$.

Proposition 4.5. *Let N_1 be a submodule of M_1 and N_2 be a submodule of M_2. If $N_1 \times N_2$ is a weakly β-prime submodule of $M_1 \oplus M_2$, then N_1 is a weakly β-prime submodule of M_1 and N_2 is a weakly β-prime submodule of M_2.*

Proof. The proof is straightforward. □

Let R_1 and R_2 be commutative rings with identity and M_i be a unital R_i-module where $i = 1, 2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module under the operation $(r_1, r_2)(m_1, m_1) = (r_1m_1, r_2m_2)$ for all $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$. We have the following results.

Proposition 4.6. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let P be an R_1-submodule of M_1. Consider the following statements.*

(i) P is a β-prime submodule of M_1.
(ii) $P \times M_2$ is a β-prime prime submodule of $M_1 \times M_2$.
(iii) $P \times M_2$ is a weakly β-prime submodule of $M_1 \times M_2$.

Then $(i) \to (ii) \to (iii)$. Moreover, if $M_2 \neq \{0\}$, then $(i), (ii)$ and (iii) are equivalent.

Proof. We have $(i) \leftrightarrow (ii)$ from proposition 2.5 and the part $(ii) \to (iii)$ and $(iii) \to (i)$ are obvious. □

5. EXAMPLES

Example 5.1. *To show that $8\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$, let r and m be integers such that $8 \mid rm$ and $8 \nmid r + r$. Then $8k = rm$ for some integer k and $4 \nmid r$. We have the following two cases.*

Case 1. $2 \mid r$ *and* $4 \nmid r$.

Then $2t = r$ for some odd integers t. Hence $8k = rm = 2tm$. Therefore $4k = tm$. This implies that $2g = m$ for some integer g. Thus $4k = tm = 2gt$. So $2k = gt$. Hence $2h = g$ for some integers h. That is $4 \mid m$. This shows that $8 \mid m + m$.

Case 2. $2 \mid m$ *and* $4 \nmid r$.

Then $2t = m$ for some integers t. We have $4k = rt$. If $2 \mid t$, then $4 \mid m$. Assume that $2 \mid r$. Then $2x = r$ for some odd integers x. Thus $2k = xt$. This implies that $2 \mid t$. Therefore $8 \mid m + m$.

Example 5.2. *Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module and let $p \in \mathbb{Z}$. Then $p\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ if and only if $p = 0$ or $p = 8$ or p is a prime number or $p = 2q$ where q is a prime number.*

Proof. $(\to)$ Assume that $p\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$. Suppose that $p \neq 0$ and $p \neq 8$ and p is not prime number. Then $p = ab$ for some integers a and b with $1 < a, b < p$. Then $p \mid ab$. This implies that $p \mid a + a$ or $p \mid b + b$. If $p \mid a + a$, then $ab = p \leq 2a$. Hence $b = 2$. Assume that a is not a prime integer. Then $a = cd$ for some integers c and d with $1 < c, d < a$. Hence $p = 2a = 2cd$ and $p > 8$. This means $p \mid 4c$ or $p \mid d + d$. If $p \mid 2d$, then $a \mid d$ which is a contradiction. Assume that

$p \mid 4c$. Then $p \mid 8$ or $p \mid 2c$ which is a contradiction with the fact that $p > 8$ and $a > c$. Hence $p = 2q$ for some prime integers q.

($\leftarrow$) It is easy to see that if $p = 0$ or $p = 8$ or p is a prime number or $p = 2q$ where q is a prime number, then $p\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$. □

In the following, $M_2(\mathbb{Z})$ denotes the ring of 2×2 matrices over $\mathbb{Z}$.

Example 5.3. *We have that* $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ *is a weakly β-prime submodule of $M_2(\mathbb{Z})$ as $M_2(\mathbb{Z})$-module. However,* $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ *is not a β-prime submodule of $M_2(\mathbb{Z})$ because of* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Example 5.4. *(i) This follows from Example 5.2 that $8\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ and $8\mathbb{Z}$ is not a prime submodule of $\mathbb{Z}$.*

(ii) This follows from Proposition 4.6 that $8\mathbb{Z} \times \mathbb{Z}$ is a weakly β-prime submodule of $\mathbb{Z} \times \mathbb{Z}$ but $8\mathbb{Z} \times \mathbb{Z}$ is not a weakly prime submodule of $\mathbb{Z} \times \mathbb{Z}$ as $\mathbb{Z} \times \mathbb{Z}$-module because of $(0,0) \neq (2,1)(4,1) = (8,1) \in 8\mathbb{Z} \times \mathbb{Z}$ and $(2,1)(\mathbb{Z} \times \mathbb{Z}) \nsubseteq 8\mathbb{Z} \times \mathbb{Z}$ and $(4,1) \notin 8\mathbb{Z} \times \mathbb{Z}$.

The following implication directly follows from Definition 2.1 and 4.1.

$$\begin{array}{ccc} \text{prime} & \Rightarrow & \beta - \text{prime} \\ \Downarrow & & \Downarrow \\ \text{weakly prime} & \Rightarrow & \beta - \text{weakly prime} \end{array}$$

However, Example 5.3 and 5.4 obtain that the converse of each part is not true. The following Example shows that the condition $M_2 \neq \{0\}$ is necessary for Proposition 4.6.

Example 5.5. *Consider $M_2(\mathbb{Z})$ as a $M_2(\mathbb{Z})$-module and $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $M_2(\mathbb{Z}) \times \mathbb{Z}$ is a $(M_2(\mathbb{Z}) \times \mathbb{Z})$-module under the operation in Proposition 4.6. We have* $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \times \{0\}$ *is a weakly β-prime submodule of $M_2(\mathbb{Z}) \times \mathbb{Z}$. However,* $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ *is not a β-prime submodule of $M_2(\mathbb{Z})$.*

Example 5.6. *Let $\mathbb{Z}$ be an $\mathbb{Z}$-module. Then*

(i) We have that $6\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ and $\beta(6\mathbb{Z}) = 12\mathbb{Z}$ is not a β-prime submodule of $\mathbb{Z}$. Next, we have $\beta(\mathbb{Z}) = 2\mathbb{Z}$ is a β-prime submodule of $\mathbb{Z}$ and $\mathbb{Z}$ is not a β-prime submodule of $\mathbb{Z}$. This example shows that the β-prime submodule condition between P and $\beta(P)$ do not depent on others.

(ii) We know that $\beta(3\mathbb{Z}) = 6\mathbb{Z}$ *is not a prime submodule of* $\mathbb{Z}$ *and* $3\mathbb{Z}$ *is a prime submodule of* $\mathbb{Z}$*. On the other hand,* $\beta(\mathbb{Z}) = 2\mathbb{Z}$ *is a prime submodule of* $\mathbb{Z}$ *and* $\mathbb{Z}$ *is not a prime submodule of* $\mathbb{Z}$*. This obtains that the prime submodule condition between* P *and* $\beta(P)$ *do not depent on others.*

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

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