J. Indones. Math. Soc.
Vol. 25, No. 02 (2019), pp. 92–107.

FRACTIONAL OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE MIXED DERIVATIVES ARE PREQUASIINVEX AND α-PREQUASIINVEX FUNCTIONS

Badreddine Meftah[1], Meriem Merad[2], Abdourazek Souahi[3]

[1]Laboratoire des télécommunications, Faculté des Sciences et de la Technologie, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria, badrimeftah@yahoo.fr

[2] Département des Mathématiques, Faculté des mathématiques, de l'informatique et des sciences de la matière, Université 8 mai 1945 Guelma, Algeria, mrad.meriem@gmail.com

[3] Laboratory of Advanced Materials, Faculty of Sciences University Badji Mokhtar-Annaba P.O. Box 12, 23000, Annaba, Algeria, arsouahi@yahoo.fr

Abstract. The aim of this paper is to establish a new fractional Ostrowski type inequalities involving functions of two independent variable whose mixed derivatives are prequasiinvex and α-prequasiinvex functions which are two novel classes of generalized convex functions. These estimates are relying on a new integral identity.

Key words and Phrases: integral inequality, prequasiinvex functions, α-prequasiinvex functions, Hölder inequality.

Abstrak. Tujuan utama dari makalah ini adalah untuk membuktikan suatu ketaksamaan tipe Ostrowski fraksional yang melibatkan fungsi-fungsi dua peubah yang turunan campurannya adalah fungsi *prequasiinveks* dan α-*prequasiinveks*, yang mana kedua kelas ini merupakan contoh fungsi konveks yang diperumum. Ketaksamaan yang diperoleh bergantung pada suatu indentitas integral yang baru.

Kata kunci: Ketaksamaan integral, fungsi α-*prequasiinveks*, ketaksamaan Hölder

In 1938, A. M. Ostrowski proved the following result

Theorem 0.1. [27] *Let $f : I \to \mathbb{R}$, be a differentiable mapping in the interior I° of I, where $I \subseteq \mathbb{R}$ is an interval and $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all*

2000 Mathematics Subject Classification: 26D15, 26D20, 26A51.
Received: 03-03-2018, revised: 17-06-2018, accepted: 14-10-2018.

$x \in [a,b]$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \tag{1}$$

The above inequality did not stop attracting the attention of researchers, various generalizations, refinements, extensions and variants have appeared in the literature see [1, 4, 11–16, 28, 29, 31, 32, 34, 37] and references therein.

Also the concept of convexity has been extended and generalized in several directions. One of the most significant generalization is that introduced by Hanson [5] where he introduced the concept of invexity, In [3] the authors gave the notion of preinvex functions which is special case of invexity. Many authors have study the basic properties of invex set and preinvex functions, and their role in optimization, variational inequalities and equilibrium problems, see [19, 20, 30, 35, 38].

It is important to remember that the credit goes to Professor Noor, who was the first to have had the opportunity to study the integral inequalities in the context of the preinvex functions see [21–26].

Barnett et al. [2] gave the following Ostrowski's inequality involving functions of two independent variable

$$\begin{aligned}&\Bigg|\int_a^b\int_c^d f(t,s)dsdt - \left((b-a)\int_c^d f(x,s)ds + (d-c)\int_a^b f(t,y)dt\right)\\ &- (b-a)(d-c)f(x,y)\Bigg|\\ &\le \left[\tfrac{1}{4}(b-a)^2 + \left(x-\tfrac{a+b}{2}\right)^2\right]\left[\tfrac{1}{4}(d-c)^2 + \left(y-\tfrac{d+c}{2}\right)^2\right]\left\|\tfrac{\partial^2 f(t,s)}{\partial s\partial t}\right\|_\infty .\end{aligned}$$

Latif et al. [8] established the following fractional Ostrowski's inequality for double variables

$$\begin{aligned}&\left|\tfrac{[(b-x)^\alpha+(x-a)^\alpha]\left[(d-y)^\beta+(y-c)^\beta\right]}{(b-a)(d-c)} f(x,y) + V\right|\\ &\le \tfrac{(\alpha\beta+2\alpha+2\beta+4)\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]\left[(d-y)^{\beta+1}+(y-c)^{\beta+1}\right]}{(b-a)(d-c)(1+\alpha)(2+\alpha)(1+\beta)(2+\beta)}\left\|\tfrac{\partial^2 f(x,y)}{\partial y\partial x}\right\|_\infty ,\end{aligned}$$

where

$$\begin{aligned}V =&\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)} J^{\alpha,\beta}_{x^-,y^-} f(a,c) + J^{\alpha,\beta}_{x^-,y^+} f(a,d) + J^{\alpha,\beta}_{x^+,y^-} f(b,c) + J^{\alpha,\beta}_{x^+,y^+} f(b,d)\\ &- \tfrac{[(b-x)^\alpha+(x-a)^\alpha]\Gamma(\beta+1)}{(b-a)(d-c)}\left(J^\beta_{y^-} f(x,c) + J^\beta_{y^+} f(x,d)\right)\\ &- \tfrac{\left[(d-y)^\beta+(y-c)^\beta\right]\Gamma(\alpha+1)}{(b-a)(d-c)}\left(J^\alpha_{x^-} f(a,y) + J^\alpha_{x^+} f(b,y)\right).\end{aligned}$$

The aim of this paper is to establish a new fractional Ostrowski type inequalities involving functions of two independent variable whose mixed derivatives are prequasiinvex and α-prequasiinvex functions which are two novel classes of generalized convex functions. These estimates are relying on a new integral identity.

1. Preliminaries

In this sections we begin by giving some definitions

Definition 1.1. [7] *A function $f : \Delta \to \mathbb{R}$ is said to be co-ordinated quasi-convex on Δ, if*

$$f\left(tx + (1-t)\, u, \lambda y + (1-\lambda)\, v\right) \leq \max\left\{f(x,y), f(x,v), f(u,y), f(u,v)\right\}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (u,v) \in \Delta$.

Definition 1.2. [36] *A function $f : \Delta \to \mathbb{R}$ is said to be co-ordinated (α, QC)-convex on Δ, for some fixed $\alpha \in (0,1]$, if*

$$\begin{aligned} f\left(tx + (1-t)\, u, \lambda y + (1-\lambda)\, v\right) \leq & t^{\alpha} \max\left\{f(x,y), f(x,v)\right\} \\ & + (1-t^{\alpha}) \max\left\{f(u,y), f(u,v)\right\} \end{aligned}$$

holds for all $\lambda, t \in [0,1]$ and $(x,y), (u,v) \in \Delta$.

Let K_1 and K_2 be two nonempty subsets of $\mathbb{R}^n$ and $(u,v) \in K_1 \times K_2$

Definition 1.3. [10] *$K_1 \times K_2$ is said to be an invex set at (u,v) with respect to η_1 and η_2, if for each $(x,y) \in K_1 \times K_2$ and $t, s \in [0,1]$, we have*

$$\left(u + t\eta_1\left(x,u\right), v + s\eta_2\left(y,v\right)\right) \in K_1 \times K_2.$$

We note that the set $K_1 \times K_2$ is an invex set with respect to η_1 and η_2, if $K_1 \times K_2$ is invex at each $(u,v) \in K_1 \times K_2$.

In [9] Latif and Dragomir introduced the class of co-ordinated preinvex functions

Definition 1.4. [9] *A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be co-ordinated preinvex on $K_1 \times K_2$, if the following inequality:*

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \leq & (1-\lambda)(1-t) f(u,v) + (1-\lambda) t f(u,y) \\ & + \lambda(1-t) f(x,v) + \lambda t f(x,y) \end{aligned}$$

holds for $(x,y), (x,v), (u,y), (u,v) \in K_1 \times K_2$ and $\lambda, t \in [0,1]$.

Using this new class they have established some Hermite-Hadamard type inequalities, of which certain results are recalled as follows:

for any function twice partially differentiable on the invex set $K_1 \times K_2$ such that $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is co-ordinated preinvex and integrable on $[a, a + \eta_1(b,a)] \times [c, c + \eta_2(d,c)] \subset K_1 \times K_2$, then we have

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dydx\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|}{4}\right).$$

And if $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated preinvex with $q\geq 1$ we have

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dydx\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right)^{\frac{1}{q}}$$

and

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dydx\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$A=\frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}\left[f(x,c)+f(x,c+\eta_2(d,c))\right]dx$$
$$+\frac{1}{2\eta_2(d,c)}\int_c^{c+\eta_2(d,c)}\left[f(a,y)+f(a+\eta_1(b,a),y)\right]dy.$$

Definition 1.5. [6] *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_x^b (t-x)^{\alpha-1} f(t)dt, \qquad b > x$$

respectively, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

Definition 1.6. [6] *Let $f \in L([a,b]\times[c,d])$. The Riemann-Liouville fractional integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$, and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta > 0$ where $a,c \geq 0$ with $a < b$ and $c < d$ are defined by*

$$J^{\alpha,\beta}_{a^+,c^+}f(b,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_a^b\int\limits_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} f(x,y)\,dydx, \tag{2}$$

$$J^{\alpha,\beta}_{a^+,d^-}f(b,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_a^b\int\limits_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} f(x,y)\,dydx, \tag{3}$$

$$J^{\alpha,\beta}_{b^-,c^+}f(a,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_a^b\int\limits_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f(x,y)\,dydx, \tag{4}$$

and

$$J^{\alpha,\beta}_{b^-,d^-}f(a,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_a^b\int\limits_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f(x,y)\,dydx, \tag{5}$$

where Γ is the Gamma function, and

$$J^{0,0}_{a^+,c^+}f(b,d) = J^{0,0}_{a^+,d^-}f(b,c) = J^{0,0}_{b^-,c^+}f(a,d) = J^{0,0}_{b^-,d^-}f(a,c) = f(x,y).$$

Definition 1.7. [33] *Let $f \in L([a,b]\times[c,d])$. The Riemann-Liouville fractional integrals $J^{\alpha}_{b^-}f(a,c)$, $J^{\alpha}_{a^+}f(b,c)$, $J^{\beta}_{d^-}f(a,c)$, and $J^{\alpha}_{c^+}f(a,d)$ of order $\alpha,\beta > 0$ where $a,c \geq 0$ with $a < b$ and $c < d$ are defined by*

$$J^{\alpha}_{b^-}f(a,c) = \frac{1}{\Gamma(\alpha)}\int\limits_a^b (x-a)^{\alpha-1} f(x,c)\,dx, \tag{6}$$

$$J^{\alpha}_{a^+}f(b,c) = \frac{1}{\Gamma(\alpha)}\int\limits_a^b (b-x)^{\alpha-1} f(x,c)\,dx, \tag{7}$$

$$J_{d^-}^{\beta} f(a,c) = \frac{1}{\Gamma(\beta)} \int_c^d (y-c)^{\beta-1} f(a,y)\, dy, \tag{8}$$

and

$$J_{c^+}^{\beta} f(a,d) = \frac{1}{\Gamma(\beta)} \int_c^d (d-y)^{\beta-1} f(a,y)\, dy, \tag{9}$$

where Γ *is the Gamma function.*

2. Main results

We first present these two new classes of generalized convex functions called co-ordinated prequasiinvex and co-ordinated α-prequasiinvex, Throughout this paper we assume that $K = [a, a+\eta(b,a)] \times [c, c+\eta_2(d,c)] \subset \mathbb{R}^2$ is an invex set.

Definition 2.1. *A function* $f : K \to \mathbb{R}$ *is said to be co-ordinated prequasiinvex on* K*, if the following inequality:*

$$f(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)) \leq \max\{f(a,c), f(a,d), f(b,c), f(b,d)\}$$

holds for all $t, \lambda \in [0,1]$ *and* $(a,c),(b,d) \in K$*.*

Definition 2.2. *A function* $f : K \to \mathbb{R}$ *is said to be co-ordinated* α*-prequasiinvex on* K*, for some fixed* $\alpha \in (0,1]$*, if the following inequality:*

$$\begin{aligned} f(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)) \leq & (1-t^\alpha)\max\{f(a,c), f(a,d)\} \\ & + t^\alpha \max\{f(b,c), f(b,d)\} \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ *and* $(a,c),(b,d) \in \Delta$*.*

Lemma 2.3. *Let* $f : K \to \mathbb{R}$ *be a differentiable mapping on* K *with* $\eta_1(b,a), \eta_2(d,c) > 0$*. If* $\frac{\partial^2 f}{\partial t \partial \lambda} \in L(K)$*, then the following equality holds*

$$\begin{aligned} & O(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A) \\ = & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \left(\int_0^1 \int_0^1 kh \tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\, d\lambda dt \right. \\ & - \int_0^1 \int_0^1 (t^\tau - (1-t)^\tau)\left(\lambda^\beta - (1-\lambda)^\beta\right) \\ & \left. \times \tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\, d\lambda dt \right), \end{aligned} \tag{10}$$

where

$$\begin{aligned}
&O\left(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A\right)\\
=&f(x,y)-\tfrac{f(x,c)+f(x,c+\eta_2(d,c))+f(a,y)+f(a+\eta_1(b,a),y)}{2}+A\\
&-\tfrac{\Gamma(\tau+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\tau}(\eta_2(d,c))^{\beta}}\Big(J^{\tau,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&+J^{\tau,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))+J^{\tau,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)\\
&+J^{\tau,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\Big),
\end{aligned}\tag{11}$$

$$k=\begin{cases}1 \ \mathit{if}\ 0\le t<\frac{x-a}{\eta_1(b,a)},\\ -1\ \mathit{if}\ \frac{x-a}{\eta_1(b,a)}\le t<1,\end{cases}\tag{12}$$

$$h=\begin{cases}1 \ \mathit{if}\ 0\le \lambda<\frac{y-c}{\eta_2(d,c)},\\ -1\mathit{if}\ \frac{y-c}{\eta_2(d,c)}\le \lambda<1,\end{cases}\tag{13}$$

and

$$\begin{aligned}
A=&\tfrac{\Gamma(\alpha+1)}{4(\eta_1(b,a))^{\tau}}\Big(J^{\tau}_{(a+\eta_1(b,a))^-}f(a,c)+J^{\tau}_{a^+}f(a+\eta_1(b,a),c)\\
&+J^{\tau}_{(a+\eta_1(b,a))^-}f(a,c+\eta_2(d,c))+J^{\tau}_{a^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\\
&+\tfrac{\Gamma(\beta+1)}{4(\eta_2(d,c))^{\beta}}\Big(J^{\beta}_{(c+\eta_2(d,c))^-}f(a,c)+J^{\beta}_{c^+}f(a,c+\eta_2(d,c))\\
&+J^{\beta}_{(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\beta}_{c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\Big).
\end{aligned}\tag{14}$$

Proof. Let

$$I=\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(I_1-I_2\right),\tag{15}$$

where

$$I_1=\int_0^1\int_0^1 kh\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)d\lambda dt,\tag{16}$$

and

$$I_2=\int_0^1\int_0^1\left(t^{\tau}-(1-t)^{\tau}\right)\left(\lambda^{\beta}-(1-\lambda)^{\beta}\right)\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)d\lambda dt\tag{17}$$

k,h are defined by (12) and (13) respectively.

Clearly,

$$I_1 = \int_0^1 k \left(\int_0^{\frac{y-c}{\eta_2(d,c)}} \frac{\partial^2 f}{\partial t \partial \lambda} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c)\right) d\lambda - \int_{\frac{y-c}{\eta_2(d,c)}}^1 \frac{\partial^2 f}{\partial t \partial \lambda} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c)\right) d\lambda \right) dt$$

$$= \frac{1}{\eta_2(d,c)} \int_0^1 k \left(2\frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), y\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \eta_2(d,c)\right) \right) dt$$

$$= \frac{1}{\eta_2(d,c)} \left(\int_0^{\frac{x-a}{\eta_1(b,a)}} \left(2\frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), y\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \eta_2(d,c)\right) \right) dt - \int_{\frac{x-a}{\eta_1(b,a)}}^1 \left(2\frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), y\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c\right) - \frac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \eta_2(d,c)\right) \right) dt \right)$$

$$= \frac{4}{\eta_1(b,a)\eta_2(d,c)} \left(f(x,y) + \frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} - \frac{1}{2} \left(f(x,c) + f(x, c+\eta_2(d,c)) + f(a,y) + f(a+\eta_1(b,a), y) \right) \right). \tag{18}$$

Now, by integration by parts, I_2 gives

$$\int_0^1 \int_0^1 \left(t^\tau - (1-t)^\tau\right) \left(\lambda^\beta - (1-\lambda)^\beta\right) \frac{\partial f}{\partial t \partial \lambda} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c)\right) d\lambda dt$$

$$= \int_0^1 \left(t^\tau - (1-t)^\tau\right)$$

$$\times \left(\int_0^1 \left(\lambda^\beta - (1-\lambda)^\beta \right) \tfrac{\partial f}{\partial t \partial \lambda} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c) \right) d\lambda \right) dt$$

$$= \int_0^1 \left(t^\tau - (1-t)^\tau \right)$$

$$\times \left(\tfrac{1}{\eta_2(d,c)} \left(\tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \eta_2(d,c) \right) + \tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c \right) \right) \right.$$

$$\left. - \tfrac{\beta}{\eta_2(d,c)} \int_0^1 \left(\lambda^{\beta-1} + (1-\lambda)^{\beta-1} \right) \tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c) \right) d\lambda \right) dt$$

$$= \tfrac{1}{\eta_2(d,c)} \int_0^1 \left(t^\tau - (1-t)^\tau \right)$$

$$\times \left(\tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \eta_2(d,c) \right) + \tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c \right) \right) dt$$

$$- \tfrac{\beta}{\eta_2(d,c)} \int_0^1 \left(\lambda^{\beta-1} + (1-\lambda)^{\beta-1} \right)$$

$$\times \left(\int_0^1 \left(t^\tau - (1-t)^\tau \right) \tfrac{\partial f}{\partial t} \left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c) \right) dt \right) d\lambda$$

$$= \tfrac{f(a+\eta_1(b,a), c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a, c+\eta_2(d,c)) + f(a,c)}{\eta_1(b,a)\eta_2(d,c)}$$

$$- \tfrac{\tau}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 \left(t^{\tau-1} + (1-t)^{\tau-1} \right)$$

$$\times \left(f\left(a + t\eta_1(b,a), c + \eta_2(d,c) \right) + f\left(a + t\eta_1(b,a), c \right) \right) dt$$

$$- \tfrac{\beta}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 \left(\lambda^{\beta-1} + (1-\lambda)^{\beta-1} \right)$$

$$\times \left(f\left(a + \eta_1(b,a), c + \lambda \eta_2(d,c) \right) + f\left(a, c + \lambda \eta_2(d,c) \right) \right) d\lambda$$

$$+ \tfrac{\beta\tau}{\eta_1(b,a)\eta_2(d,c)} \int_0^1 \int_0^1 \left(\lambda^{\beta-1} + (1-\lambda)^{\beta-1} \right) \left(t^\tau + (1-t)^\tau \right)$$

$$\times f\left(a + t\eta_1(b,a), c + \lambda \eta_2(d,c) \right) dt d\lambda. \tag{19}$$

Substituting (18) and (19) in (15), then using the change of variable $u = a + t\eta_1(b,a)$ and $v = c + \lambda \eta_2(d,c)$, and (2)-(9), we obtain the desired result. □

Theorem 2.4. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|$ is co-ordinated prequasiinvex function on*

K, then

$$\begin{aligned}
&\left|O\left(x,y,\tau,\beta,a,a+\eta_1\left(b,a\right),c,c+\eta_2\left(d,c\right),A\right)\right| \\
\leq&\eta_1\left(b,a\right)\eta_2\left(d,c\right)\left(\tfrac{1}{4}+\tfrac{1}{(\tau+1)(\beta+1)}\left(1-\tfrac{1}{2^{\tau}}\right)\left(1-\tfrac{1}{2^{\beta}}\right)\right) \\
&\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}, \qquad (20)
\end{aligned}$$

where O is defined as in (11).

Proof. From Lemma 2.3, properties of modulus, and prequasiinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we have

$$\begin{aligned}
&\left|O\left(x,y,\tau,\beta,a,a+\eta_1\left(b,a\right),c,c+\eta_2\left(d,c\right),A\right)\right| \\
\leq&\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1\left(b,a\right),c+\lambda\eta_2\left(d,c\right)\right)\right|d\lambda dt\right. \\
&+\int_0^1\int_0^1\left|t^{\tau}-(1-t)^{\tau}\right|\left|\lambda^{\beta}-(1-\lambda)^{\beta}\right| \\
&\left.\times\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1\left(b,a\right),c+\lambda\eta_2\left(d,c\right)\right)\right|d\lambda dt\right) \\
\leq&\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\} \\
&\times\left(1+\int_0^1\int_0^1\left|t^{\tau}-(1-t)^{\tau}\right|\left|\lambda^{\beta}-(1-\lambda)^{\beta}\right|d\lambda dt\right) \\
=&\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\} \\
&\times\left(1+\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)\left(\int_0^1\left|\lambda^{\beta}-(1-\lambda)^{\beta}\right|d\lambda\right)\right) \\
=&\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\} \\
&\times\left(1+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\tau}-t^{\tau}\right)dt+\int_{\frac{1}{2}}^1\left(t^{\tau}-(1-t)^{\tau}\right)dt\right)\right. \\
&\left.\times\left(\int_0^{\frac{1}{2}}\left((1-\lambda)^{\beta}-\lambda^{\beta}\right)d\lambda+\int_{\frac{1}{2}}^1\left(\lambda^{\beta}-(1-\lambda)^{\beta}\right)d\lambda\right)\right) \\
=&\eta_1\left(b,a\right)\eta_2\left(d,c\right)\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\} \\
&\times\left(\tfrac{1}{4}+\tfrac{1}{(\tau+1)(\beta+1)}\left(1-\tfrac{1}{2^{\tau}}\right)\left(1-\tfrac{1}{2^{\beta}}\right)\right),
\end{aligned}$$

which is the desired result. □

Theorem 2.5. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is co-ordinated prequasiinvex function on K, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following fractional inequality holds*

$$
\begin{aligned}
&|O(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A)| \\
\leq & \eta_1(b,a)\,\eta_2(d,c)\left(\tfrac{1}{4} + \left(\tfrac{1}{(p\tau+1)(p\beta+1)}\right)^{\frac{1}{p}}\right) \\
& \times \left\{\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q\right\}\right\}^{\frac{1}{q}},
\end{aligned}
$$

where O is defined as in (11).

Proof. By Lemma 2.3, properties of modulus, Hölder inequality, and prequasiinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$, we have

$$
\begin{aligned}
&|O(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A)| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}\right. \\
& + \left(\left(\int_0^1\int_0^1 t^{p\tau}\lambda^{p\beta} d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 t^{\tau p}(1-\lambda)^{\beta p} d\lambda dt\right)^{\frac{1}{p}}\right. \\
& + \left.\left(\int_0^1\int_0^1 (1-t)^{\tau p}\lambda^{\beta p} d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{\tau p}(1-\lambda)^{\beta p} d\lambda dt\right)^{\frac{1}{p}}\right) \\
& \left.\times \left(\int_0^1\int_0^1 \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}\right) \\
\leq & \eta_1(b,a)\,\eta_2(d,c)\left(\tfrac{1}{4} + \left(\tfrac{1}{(p\tau+1)(p\beta+1)}\right)^{\frac{1}{p}}\right) \\
& \times \left\{\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q, \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q\right\}\right\}^{\frac{1}{q}},
\end{aligned}
$$

which is the desired result. □

Theorem 2.6. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is co-ordinated α-prequasiinvex function on*

K, then we have

$$
\begin{aligned}
& \left|O\left(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A\right)\right| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\tfrac{\alpha}{\alpha+1}+\tfrac{2^\beta-1}{(\beta+1)2^{\beta-1}}\right.\right. \\
& \times\left(\tfrac{2^\tau-1}{(\tau+1)2^{\tau-1}}+\tfrac{1-2^{\tau+\alpha}}{(\tau+\alpha+1)2^{\tau+\alpha}}+B(\alpha+1,\tau+1)-2B_{\frac{1}{2}}(\alpha+1,\tau+1)\right)\Big) \\
& \times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\} \\
& +\left(\tfrac{1}{\alpha+1}+\tfrac{2^\beta-1}{(\beta+1)2^{\beta-1}}\left(2B_{\frac{1}{2}}(\alpha+1,\tau+1)+\tfrac{2^\alpha-1}{(\alpha+1)2^\alpha}-B(\alpha+1,\tau+1)\right)\right) \\
& \times\ \max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}\Big)\Big),
\end{aligned}
$$

where O is defined as in (11), *$B(.,.)$ and $B_{\frac{1}{2}}(.,.)$ are the beta and incomplete beta functions.*

Proof. By Lemma 2.3, properties of modulus, and α-prequasiinvexity on the coordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we have

$$
\begin{aligned}
& \left|O\left(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A\right)\right| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|d\lambda dt\right. \\
& +\int_0^1\int_0^1\left|t^\tau-(1-t)^\tau\right|\left|\lambda^\beta-(1-\lambda)^\beta\right| \\
& \times\left.\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|d\lambda dt\right) \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1(1-t^\alpha)\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}d\lambda dt\right. \\
& +\int_0^1\int_0^1 t^\alpha\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}d\lambda dt \\
& +\int_0^1\int_0^1\left|t^\tau-(1-t)^\tau\right|\left|\lambda^\beta-(1-\lambda)^\beta\right|(1-t^\alpha) \\
& \times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}d\lambda dt \\
& +\int_0^1\int_0^1\left|t^\tau-(1-t)^\tau\right|\left|\lambda^\beta-(1-\lambda)^\beta\right|t^\alpha
\end{aligned}
$$

$$\times \max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}d\lambda dt\Bigg)$$

$$=\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\frac{\alpha}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}\right.$$

$$+\frac{1}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}$$

$$+\left(\int_0^{\frac{1}{2}}\left((1-\lambda)^\beta-\lambda^\beta\right)d\lambda+\int_{\frac{1}{2}}^1\left(\lambda^\beta-(1-\lambda)^\beta\right)d\lambda\right)$$

$$\times\left(\left(\int_0^{\frac{1}{2}}\left((1-t)^\tau-t^\tau\right)(1-t^\alpha)\,dt+\int_{\frac{1}{2}}^1\left(t^\tau-(1-t)^\tau\right)(1-t^\alpha)\,dt\right)\right.$$

$$\times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}$$

$$+\left(\int_0^{\frac{1}{2}}\left((1-t)^\tau-t^\tau\right)t^\alpha\ dt+\int_{\frac{1}{2}}^1\left(t^\tau-(1-t)^\tau\right)t^\alpha\ dt\right)$$

$$\left.\left.\times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}\right)\right)$$

$$=\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\tfrac{\alpha}{\alpha+1}+\tfrac{2^\beta-1}{(\beta+1)2^{\beta-1}}\right.\right.$$

$$\left.\times\left(\tfrac{2^\tau-1}{(\tau+1)2^{\tau-1}}+\tfrac{1-2^{\tau+\alpha}}{(\tau+\alpha+1)2^{\tau+\alpha}}+B(\alpha+1,\tau+1)-2B_{\frac{1}{2}}(\alpha+1,\tau+1)\right)\right)$$

$$\times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right\}$$

$$+\left(\tfrac{1}{\alpha+1}+\tfrac{2^\beta-1}{(\beta+1)2^{\beta-1}}\left(2B_{\frac{1}{2}}(\alpha+1,\tau+1)+\tfrac{2^\alpha-1}{(\alpha+1)2^\alpha}-B(\alpha+1,\tau+1)\right)\right)$$

$$\left.\times\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}\right)\right),$$

which is the desired result. □

Theorem 2.7. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K with $\eta_1(b,a)>0$ and $\eta_2(d,c)>0$. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated α-prequasiinvex function on K where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$|O(x,y,\tau,\beta,a,a+\eta_1(b,a),c,c+\eta_2(d,c),A)|$$

$$\leq\eta_1(b,a)\eta_2(d,c)\left(\frac{1}{4}+\left(\tfrac{1}{(p\tau+1)(p\beta+1)}\right)^{\frac{1}{p}}\right)$$

$$\times\left(\frac{\alpha}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right\}\right.$$

$$\left.+\frac{1}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where O is defined as in (11).

Proof. By Lemma 2.3, properties of modulus, Hölder inequality, and α-prequasiinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$, we have

$$
\begin{aligned}
&|O(x, y, \tau, \beta, a, a+\eta_1(b, a), c, c+\eta_2(d, c), A)| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b, a), c+\lambda\eta_2(d, c))\right|^q d\lambda dt\right)^{\frac{1}{q}}\right. \\
& +\left(\left(\int_0^1\int_0^1 t^{p\tau}\lambda^{p\beta} d\lambda dt\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1 t^{\tau p}(1-\lambda)^{\beta p} d\lambda dt\right)^{\frac{1}{p}}\right. \\
& \left.+\left(\int_0^1\int_0^1(1-t)^{\tau p}\lambda^{\beta p} d\lambda dt\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1(1-t)^{\tau p}(1-\lambda)^{\beta p} d\lambda dt\right)^{\frac{1}{p}}\right) \\
& \left.\times\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\eta_1(b, a), c+\lambda\eta_2(d, c))\right|^q d\lambda dt\right)^{\frac{1}{q}}\right) \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(1+4\left(\tfrac{1}{(p\tau+1)(p\beta+1)}\right)^{\frac{1}{p}}\right) \\
& \times\left(\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a, c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a, d)\right|^q\right\}\int_0^1\int_0^1(1-t^\alpha) d\lambda dt\right. \\
& \left.+\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b, c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b, d)\right|^q\right\}\int_0^1\int_0^1 t^\alpha d\lambda dt\right)^{\frac{1}{q}} \\
= & \eta_1(b, a)\eta_2(d, c)\left(\frac{1}{4}+\left(\tfrac{1}{(p\tau+1)(p\beta+1)}\right)^{\frac{1}{p}}\right) \\
& \times\left(\frac{\alpha}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a, c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a, d)\right|^q\right\}\right. \\
& r+\left.\frac{1}{\alpha+1}\max\left\{\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b, c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(b, d)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}
$$

which is the desired result. □

REFERENCES

[1] W. Alshanti, and A. Qayyum, A Note On New Ostrowski Type Inequalities Using A Generalized Kernel, *Bulletin of Mathematical Analysis and Applications.* **9** (2017), no.1, 1-18.

[2] N. S. Barnett and S. S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae. *Soochow J. Math.* **27** (2001), no. 1, 1–10.

[3] A. Ben-Israel and B. Mond, What is invexity? *J. Austral. Math. Soc. Ser. B* **8** (1986), no. 1, 1–9.

[4] G. Farid, Some new Ostrowski type inequalities via fractional integrals. *Int. J. Anal. App.* **14** (2017), no. 1, 64-68.

[5] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **80** (1981), no. 2, 545–550.

[6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. *Elsevier Science B.V., Amsterdam*, 2006.

[7] M. A. Latif, S. Hussain and S. S. Dragomir, Refinements of Hermite-Hadamard-type inequalities for co-ordinated quasi-convex functions. *International Journal of Mathematical Archive.* **3** (2012), no 1, 161-171.

[8] M. A. Latif and S. Hussain, New inequalities of Ostrowski type for co-ordineted convex functions via fractional integrals. *J. Fract. Calc. Appl.* **2** (2012), no. 9, 1-15.

[9] M. A. Latif and S. S. Dragomir, Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absolute value are preinvex on the co-ordinates. *Facta Univ. Ser. Math. Inform.* **28** (2013), no. 3, 257–270.

[10] M. Matloka, On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates. *J. Inequal. Appl.* 2013, 2013:227, 12 pp.

[11] B. Meftah, Some New Ostrwoski's Inequalities for Functions Whose nth Derivatives are r-Convex. *International Journal of Analysis*, 2016, 7 pages.

[12] B. Meftah, Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex. *Chin. J. Math.* (N.Y.) 2016, Art. ID 5292603, 10 pp.

[13] B. Meftah, New Ostrowski's inequalities. *Rev. Colombiana Mat.* **51** (2017), no. 1, 57–69.

[14] B. Meftah, Ostrowski inequality for functions whose first derivatives are s-preinvex in the second sense. *Khayyam J. Math.* **3** (2017), no. 1, 61–80.

[15] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense. *International Journal of Analysis and Applications.* **15** (2017), no. 2, 146–154.

[16] B. Meftah, Some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex. *Facta Univ. Ser. Math. Inform.* **32** (2017), no. 3, 319–327.

[17] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are φ-preinvex. *J. Adv. Math. Stud.* **10** (2017), no. 3, 335-347.

[18] B. Meftah , Some new Ostrowski inequalities for functions whose n^{th} derivatives are logarithmically convex. *Ann. Math. Sil. De Gruyter Open.*

[19] M. A. Noor, Variational-like inequalities. *Optimization.* **30** (1994), no. 4, 323–330.

[20] M. A. Noor, Invex equilibrium problems. *J. Math. Anal. Appl.* **302** (2005), no. 2, 463–475.

[21] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions. *J. Math. Anal. Approx. Theory.* **2** (2007), no. 2, 126–131.

[22] M. A. Noor, Hadamard integral inequalities for product of two preinvex functions. *Nonlinear Anal. Forum.* **14** (2009), 167–173.

[23] M. A. Noor, K. I. Noor and M. U. Awan, Fractional Ostrowski inequalities for s-Godunova-Levin functions. *International Journal of Analysis and Applications.* **5** (2014) no.2, 167-173.

[24] M. A. Noor, K. I. Noor and M. U. Awan, Fractional Ostrowski inequalities for (s, m)-Godunova-Levin functions. *Facta Univ. Ser. Math. Inform.* **30** (2015), no. 4, 489–499.

[25] M. A. Noor, K. I. Noor and M. U. Awan, Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **269** (2015), 242–251.

[26] M. A. Noor, M. U. Awan and K. I. Noor, Quantum Ostrowski inequalities for q-differentiable convex functions. *J. Math. Inequal.* **10** (2016), no. 4, 1013–1018.

[27] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. (German) *Comment. Math. Helv.* **10** (1937), no. 1, 226–227.

[28] M. E. Özdemir, H. Kavurmaci and E. Set, Ostrowski's type inequalities for (α, m)-convex functions. *Kyungpook Math. J.* **50** (2010), no. 3, 371–378.

[29] B. G. Pachpatte, A new Ostrowski type inequality for double integrals. *Soochow J. Math.* **32** (2006), no. 2, 317–322.
[30] R. Pini, Invexity and generalized convexity. *Optimization.* **22** (1991), no. 4, 513–525.
[31] M. Z. Sarikaya, On the Ostrowski type integral inequality. *Acta Math. Univ. Comenian. (N.S.)* **79** (2010), no. 1, 129–134.
[32] M. Z. Sarikaya and H. Ogunmez, On the weighted Ostrowski-type integral inequality for double integrals. *Arab. J. Sci. Eng.* **36** (2011), no. 6, 1153–1160.
[33] M. Z. Sarıkaya, On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals. *Integral Transforms Spec. Funct.* **25** (2014), no. 2, 134–147.
[34] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **63** (2012), no. 7, 1147–1154.
[35] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization. *J. Math. Anal. Appl.* **136** (1988), no. 1, 29–38.
[36] B. -Y. Xi, J. Sun, and S. -P. Bai, On some Hermite-Hadamard-type integral inequalities for co-ordinated (α, QC)- and (α, CJ)-convex functions. *Tbilisi Math. J.* **8** (2015), no. 2, 75–86.
[37] Q. Xue, J. Zhu and W. Liu, A new generalization of Ostrowski-type inequality involving functions of two independent variables. *Comput. Math. Appl.* **60** (2010), no. 8, 2219–2224.
[38] X. -M. Yang and D. Li, On properties of preinvex functions. *J. Math. Anal. Appl.* **256** (2001), no. 1, 229–241.