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ON A ROUGH CAYLEY GRAPH RELATED TO CONJUGACY CLASSES

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Abstract. In this paper, we study concepts of lower and upper approximations edge Cayley graphs and vertex pseudo-Cayley graphs of Cayley graphs with respect to conjugacy classes. In the following, we discuss the properties of automorphisms in the Cayley rough graphs.

Key words and Phrases: Rough sets, Cayley rough graph, Pseudo-Cayley graph, Conjugacy classes.

1. INTRODUCTION

Pawlak [21, 22, 23] proposed the concept of rough set to handle uncertainty in data analysis. This concept has been applied in a variety of problems [24, 25, 29].

The theory of rough sets deals with the approximation of an arbitrary subset of a universe by two definable or visible subsets called lower and upper approximations. Many mathematicians are attached in studying the relation between rough sets and algebraic systems. Biswas and Nanda [4] studied the notion of rough subgroups. Kuroki and Wang [20] studied the lower and upper approximations with respect to the normal subgroups. Kuroki [18] defined the concepts of a rough ideal in a subgroup. Xiao and Zhang [28] introduced the notion of rough prime ideals in a semigroup. Kuroki and Mordeson [19] studied the structure of rough sets and rough groups. Davvaz [5] introduced the notion of rough subring and rough ideal of a ring. Davvaz and Mahdavipour [6] defined the notion of rough submodule with respect to a submodule of an R-module. Jun [15, 16] studied the roughness of ideals in BCK-algebras, roughness of Γ-subsemigroups and ideals in Γ-semigroups. Davvaz and Kazansi [17] introduced and studied the rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Arthur Cayley in 1878 introduced the definition of Cayley graph. In [26] was introduced the concepts of rough approximations of Cayley graphs and rough edge Cayley graphs and studied a new definition called pseudo-Cayley graphs containing Cayley graphs was

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proposed. Some studies have been done in the fields of fuzzy graph, rough fuzzy set and fuzzy rough set which can be found In [7, 8, 10, 11, 12, 13, 14].

2. preliminaries

We first summarize some basic definitions and theorems which can be found in [9, 26].

A graph Γ is a mathematical structure used to model pairwise relations between objects from a certain collection. A graph in this context refers to a nonempty set of vertices and a collection of edges that connect pairs of vertices. The set of vertices is usually denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$. The edges can be directed or undirected. A graph with all directed edges is called directed graph, otherwise it is called undirected.
Let Γ_1 and Γ_2 be graphs. The union $\Gamma_1 \cup \Gamma_2$ of Γ_1 and Γ_2 is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. The intersection $\Gamma_1 \cap \Gamma_2$ of Γ_1 and Γ_2 is graph with vertex set $V(\Gamma_1) \cap V(\Gamma_2)$ and edge set $E(\Gamma_1 \cap \Gamma_2) = E(\Gamma_1) \cap E(\Gamma_2)$.
Let G be a finite group and S a subset of G not containing the identity element 1. We define the Cayley graph $\Gamma = Cay(G, S)$ of G by $V(\Gamma) = G$,

$$E(\Gamma) = \{\{g, sg\} : g \in G, s \in S\}.$$

Let R be a subset of G, if R contains S and $SR \subseteq R$ where $SR = \{sr : s \in S, r \in R\}$, then the pseudo-Cayley graph $PCay(R, S)$ is a graph whose vertices are labelled with the elements of R, in which there is an edge between two vertices r and rs if and only if $s \in S$.

Theorem 2.1. [9] *A Cayley graph (G, S) is connected if and only if S generates G.*

A subset S of G is called minimal Cayley set if it generates G, and $S - \{s, s^{-1}\}$ generates a proper subgroup of G for all $s \in S$.

Theorem 2.2. [26] *If $\Gamma_1 = Cay(G, S_1)$ and $\Gamma_2 = Cay(G, S_2)$ are Cayley graphs, then*
(1) $\Gamma_1 \cup \Gamma_2 = Cay(G, S_1 \cup S_2)$,
(2) $\Gamma_1 \cap \Gamma_2 = Cay(G, S_1 \cap S_2)$.

Theorem 2.3. [26] *If $\Gamma_1 = Cay(H_1, S)$ and $\Gamma_2 = Cay(H_2, S)$ $(H_1, H_2 \leq G)$ which means H_1 and H_2 are subgroups of G are Cayley graphs, then*
(1) $\Gamma_1 \cup \Gamma_2 = Cay(H_1 \cup H_2, S)$,
(2) $\Gamma_1 \cap \Gamma_2 = Cay(H_1 \cap H_2, S)$.

Theorem 2.4. [26] *If $\Gamma_1 = Cay(H_1, S_1)$ and $\Gamma_2 = Cay(H_2, S_2)$ $(H_1, H_2 \leq G)$ are Cayley graphs, then $\Gamma_1 \cap \Gamma_2 = Cay(H_1 \cap H_2, S_1 \cap S_2)$.*

Theorem 2.5. [26] *If $\Gamma_1 = Cay(G, S_1)$, $\Gamma_2 = Cay(G, S_2)$, $\Omega_1 = Cay(G_1, S)$ and $\Omega_1 = Cay(G_2, S)$ are Cayley graphs, then*
(1) $\Gamma_1 \subseteq \Gamma_2$ if and only if $S_1 \subseteq S_2$,
(2) $\Omega_1 \subseteq \Omega_2$ if and only if $G_1 \subseteq G_2$.

3. Rough sets with conjugacy class

In this section, we introduce the concept of rough set with respect conjugacy class in a group and prove some preliminary properties.

In a group G, two elements g and h are called conjugate when $h = xgx^{-1}$ for some $x \in G$. The relation is symmetric, since $g = yhy^{-1}$ with $y = x^{-1}$. When $xgx^{-1} = h$, we say x conjugates g to h.

It can be easily shown that conjugacy is an equivalence relation and therefore partitions G into equivalence classes. (This means that every element of the group belongs to precisely one conjugacy class, and the classes $Cl(a)$ and $Cl(b)$ are equal if and only if a and b are conjugate, and disjoint otherwise.) The equivalence class that contains the element a in G is,

$$Cl(a) = \{b \in G; \text{there exist } g \in G \text{ with } b = gag^{-1}\}.$$

Let G be a group. If S is nonempty subset of G, then the sets

$$\underline{Cl} = \{x \in G; x^G \subseteq S\}, \qquad \overline{Cl} = \{x \in G; x^G \cap S \neq \varnothing\}$$

are called, respectively, lower and upper approximations of a set S with respect to conjugacy classes.

Theorem 3.1. *Let G be a group. Let X and Y be any nonempty subsets of G. Then,*
(1) $\underline{Cl}(X) \subseteq X \subseteq \overline{Cl}(X)$,

(2) $X \subseteq Y$ implies $\underline{Cl}(X) \subseteq \underline{Cl}(Y)$,

(3) $X \subseteq Y$ implies $\overline{Cl}(X) \subseteq \overline{Cl}(Y)$,

(4) $\underline{Cl}(X \cap Y) = \underline{Cl}(X) \cap \underline{Cl}(Y)$,

(5) $\overline{Cl}(X \cup Y) = \overline{Cl}(X) \cup \overline{Cl}(Y)$,

(6) $\overline{Cl}(X \cap Y) \subseteq \overline{Cl}(X) \cap \overline{Cl}(Y)$,

(7) $\underline{Cl}(X \cup Y) \supseteq \underline{Cl}(X) \cup \underline{Cl}(Y)$.

Proof. (1) Suppose that $x \in \underline{Cl}(X)$. Then $x^G \subseteq X$, hence $x \in x^G \subseteq X$. Therefore, $\underline{Cl}(X) \subseteq X$.
Now suppose that $x \in X$. Then $x \in x^G \cap X$, So $x \in \overline{Cl}(X)$. Therefore, $X \subseteq \overline{Cl}(X)$.

(2) Suppose that $x \in \underline{Cl}(X)$. Then $x^G \subseteq X$. Since $X \subseteq Y$, $x^G \subseteq Y$. Therefore $x \in \underline{Cl}(Y)$.

(3) The proof is similar to (2).

(4) Since $(X\cap Y)\subseteq X, Y$, according to relation (2), we conclude $\underline{Cl}(X\cap Y)\subseteq \underline{Cl}(X)$ and $\underline{Cl}(X\cap Y)\subseteq \underline{Cl}(Y)$. Hence $\underline{Cl}(X\cap Y)\subseteq \underline{Cl}(X)\cap \underline{Cl}(Y)$.

Conversely, let $x\in \underline{Cl}(X)\cap \underline{Cl}(Y)$. Then $x\in \underline{Cl}(X)$ and $x\in \underline{Cl}(Y)$. So $x^G\subseteq X$ and $x^G\subseteq Y$. Thus $x^G\subseteq (X\cap Y)$ and so $x\in \underline{Cl}(X\cap Y)$, which implies that $\underline{Cl}(X)\cap \underline{Cl}(Y)\subseteq \underline{Cl}(X\cap Y)$.

(5) The proof is similar to (4).

(6) Since $(X\cap Y)\subseteq X, Y$, $\overline{Cl}(X\cap Y)\subseteq \overline{Cl}(X)$ and $\overline{Cl}(X\cap Y)\subseteq \overline{Cl}(Y)$, we have $\overline{Cl}(X\cap Y)\subseteq \overline{Cl}(X)\cap \overline{Cl}(Y)$.

(7) The proof is similar to (6). □

4. Rough edge Cayley graphs

In this section, we give concept of lower and upper approximations edge Cayley graphs of a Cayley graph with respect to conjugacy classes and in the following, we bring some necessary properties of them.

Definition 4.1. *Let G be a finite group with identity 1, S be a subset of G and $\Gamma = Cay(G,S)$ be a Cayley graph. Then the graphs*

$$\overline{\Gamma} = Cay(G,\overline{Cl}(S)), \qquad \underline{\Gamma} = Cay(G,\underline{Cl}(S))$$

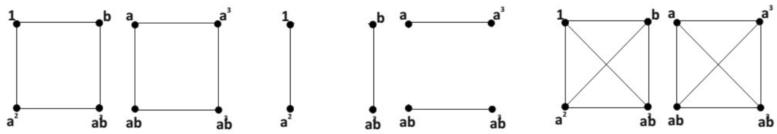
are called, respectively, lower and upper approximations edge Cayley graphs of the Cayley graph $\Gamma = Cay(G,S)$ with respect to the conjugacy classes.

Theorem 4.2. *The graphs $\underline{\Gamma}$ and $\overline{\Gamma}$ are Cayley graphs.*

Proof. Suppose that $x\in \underline{Cl}(S)$. Then $x^G\subseteq S$ so we have $(g^{-1}xg)^{-1} = gx^{-1}g^{-1}\in S^{-1} = S$, for all $g\in G$ and thus $(x^{-1})^G\subseteq S$. Therefore $x^{-1}\in \underline{Cl}(S)$.

Now, let $x\in \overline{Cl}(S)$. Then $x^G\cap S\neq \varnothing$ so we have $g\in G$ such that $t = gxg^{-1}\in S$, so $t^{-1} = g^{-1}x^{-1}g\in S^{-1} = S$. On the other hand, we have $t^{-1} = g^{-1}x^{-1}g\in (x^{-1})^G$. So $t^{-1}\in (x^{-1})^G\cap S$, which implies that $(x^{-1})^G\cap S\neq\varnothing$ and Therefore $x^{-1}\in \overline{Cl}(S)$. □

Example 4.3. *Let $G =< a, b\ ;\ b^2 = a^4 = 1, ab = ba^{-1} >$ and $\Gamma = Cay(G,S)$ be Cayley graph such that $S = \{a^2, b\}$. We have $\overline{\Gamma} = (G,\{a^2, b, a^2b\})$ and $\underline{\Gamma} = Cay(G,\{a^2\})$.(See the figure below)*



Γ $\underline{\Gamma}$ $\overline{\Gamma}$

Theorem 4.4. *Let S, S_1, S_2 be subsets of a group G. Let $\Gamma = Cay(G, S)$, $\Gamma_1 = Cay(G, S_1)$ and $\Gamma_2 = Cay(G, S_2)$ be Cayley graphs. Then we have*
(1) $\underline{\Gamma} \subseteq \Gamma \subseteq \overline{\Gamma}$,

(2) $\overline{\Gamma_1 \cup \Gamma_2} = \overline{\Gamma_1} \cup \overline{\Gamma_2}$,

(3) $\underline{\Gamma_1 \cap \Gamma_2} = \underline{\Gamma_1} \cap \underline{\Gamma_2}$,

(4) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow \underline{\Gamma_1} \subseteq \underline{\Gamma_2}$,

(5) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow \overline{\Gamma_1} \subseteq \overline{\Gamma_2}$,

(6) $\underline{\Gamma_1 \cup \Gamma_2} \supseteq \underline{\Gamma_1} \cup \underline{\Gamma_2}$,

(7) $\overline{\Gamma_1 \cap \Gamma_2} \subseteq \overline{\Gamma_1} \cap \overline{\Gamma_2}$.

Proof. We will prove $(1), (2), (4)$ and (6). The proof of cases $(3), (5)$ and (7) is similar.

(1) Since $\underline{Cl}(S) \subseteq S \subseteq \overline{Cl}(S)$ then by Theorem 2.5 leads $\underline{\Gamma} \subseteq \Gamma \subseteq \overline{\Gamma}$.

(2) By Theorem 2.3, $\overline{\Gamma_1} \cup \overline{\Gamma_2} = (G, \overline{Cl}(S_1) \cup \overline{Cl}(S_2))$. Then by Theorem 3.1, we have $\overline{Cl}(S_1), \overline{Cl}(S_2) \subseteq \overline{Cl}(S_1 \cup S_2)$. So $\overline{\Gamma_1} \cup \overline{\Gamma_2} \subseteq \overline{\Gamma_1 \cup \Gamma_2}$.

Conversely, by Theorem 3.1, we have $\overline{Cl}(S_1) \cup \overline{Cl}(S_2) = \overline{Cl}(S_1 \cup S_2)$. Let (g, gs) is an arbitrary edge of $E(\overline{\Gamma_1 \cup \Gamma_2})$. Then $s \in \overline{Cl}(S_1) \cup \overline{Cl}(S_2)$, hence $s \in \overline{Cl}(S_1)$ or $s \in \overline{Cl}(S_2)$. So (g, gs) is an edge in $\overline{\Gamma_1}$ or $\overline{\Gamma_2}$. Therefore, $\overline{\Gamma_1 \cup \Gamma_2} \subseteq \overline{\Gamma_1} \cup \overline{\Gamma_2}$.

(4) If $\Gamma_1 \subseteq \Gamma_2$, then, we have $S_1 \subseteq S_2$ and so $\underline{Cl}(S_1) \subseteq \underline{Cl}(S_2)$. Therefore, $\underline{\Gamma_1} \subseteq \underline{\Gamma_2}$.

(6)According to the Theorem 3.1 (7), $\underline{Cl}(S_1 \cup S_2) \supseteq \underline{Cl}(S_1) \cup \underline{Cl}(S_2)$, thus $\underline{Cl}(S_1 \cup S_2) \supseteq \underline{Cl}(S_1)$ and $\underline{Cl}(S_1 \cup S_2) \supseteq \underline{Cl}(S_2)$. Therefore, $\underline{\Gamma_1 \cup \Gamma_2} \supseteq \underline{\Gamma_1} \cup \underline{\Gamma_2}$. □

5. Rough vertex pseudo-Cayley graphs

In this section, we give concept of lower and upper approximations edge Cayley graphs of a Cayley graph with respect to conjugacy classes and in the following, we bring some necessary properties of them.

Definition 5.1. *Let G be a finite group with identity 1, S a subset of G such that $S^G = S$. Let R be a subset of G and $\Gamma = PCay(R,S)$ be a pseudo-Cayley graph. Then*

$$\overline{\Gamma}^{'} = PCay(\overline{Cl}(R), S)$$

is called, upper approximations vertex pseudo-Cayley graph of the pseudo-Cayley graphs $\Gamma = PCay(R,S)$.

Theorem 5.2. *The graph $\overline{\Gamma}^{'}$ is pseudo-Cayley graph.*

Proof. Since $S \subseteq R$, then $S \subseteq \overline{Cl}(R)$. Suppose $sx \in S\overline{Cl}(R)$ for all $s \in S$ and $x \in \overline{Cl}(R)$, then $x^G \cap R \neq \varnothing$, therefore there exists $g \in G$ such that $gxg^{-1} \in R$. Since Γ is a pseudo-Cayley graph, $SR \subseteq R$, so $s(gxg^{-1}) \subseteq R$. From $S = S^G$, it follows $(sx)^G \subseteq R$ which implies that $(sx) \in \overline{Cl}(R)$. Hence $\overline{\Gamma}^{'}$ is a pseudo-Cayley graph. □

The following example shows that it is possible that graph $\underline{\Gamma}^{'}$ isn't a pseudo-Cayley graph.

Example 5.3. *Let $G =< a, b\ ;\ b^2 = a^4 = 1, ab = ba^{-1} >$ and $R = \{1, a, b, a^2, a^2, a^2b, a^3, a^3b\}$ be a subset of G and $S = \{b\}$. Then $\underline{\Gamma}^{'} = (\underline{Cl}(R), \underline{Cl}(R) \cap S)$ isn't pseudo-Cayley graph.*

Theorem 5.4. *Let R, R_1 and R_2 be subsets of a group G and $S \subseteq R, R_1, R_2$. Let $\Gamma = (R,S), \Gamma_1 = (R_1, S)$ and $\Gamma_2 = (R_2, S)$ be pseudo-Cayley graphs. Then we have*

(1) $\Gamma \subseteq \overline{\Gamma}^{'}$,

(2) $\overline{\Gamma_1 \cup \Gamma_2}^{'} = \overline{\Gamma_1}^{'} \cup \overline{\Gamma_2}^{'}$,

(3) $\Gamma_1 \subseteq \Gamma_2 \Rightarrow \overline{\Gamma_1}^{'} \subseteq \overline{\Gamma_2}^{'}$,

(4) $\overline{\Gamma_1 \cap \Gamma_2}^{'} \subseteq \overline{\Gamma_1}^{'} \cap \overline{\Gamma_2}^{'}$.

Proof. (1) According to the Theorem 3.1, we have $R \subseteq \overline{Cl}(R)$. So $\Gamma = (R,S) \subseteq \Gamma^{'} = (\overline{Cl}(R), S)$.

(2) We have,

$$\begin{aligned}
\overline{\Gamma_1 \cup \Gamma_2}^{'} &= (\overline{Cl}(R_1 \cup R_2), S) \\
&= (\overline{Cl}(R_1) \cup \overline{Cl}(R_2), S) \\
&= (\overline{Cl}(R_1), S) \cup (\overline{Cl}(R_2, S)) \\
&= \overline{\Gamma_1}^{'} \cup \overline{\Gamma_2}^{'}.
\end{aligned}$$

.

(3) Since $\Gamma_1 \subseteq \Gamma_2$,we have $R_1 \subseteq R_2$. Theorem 3.1 shows, $\overline{Cl}(R_1) \subseteq \overline{Cl}(R_2)$, so $\overline{\Gamma_1}' \subseteq \overline{\Gamma_2}'$.

(4) We have,

$$\begin{aligned}\overline{\Gamma_1 \cap \Gamma_2}' &= (\overline{Cl}(R_1 \cap R_2), S)\\ &\subseteq (\overline{Cl}(R_1), S) \cap (\overline{Cl}(R_2, S))\\ &= \overline{\Gamma_1}' \cap \overline{\Gamma_2}'.\end{aligned}$$

. $\square$

Theorem 5.5. *Let R be a subset of a group G and $S_1, S_2 \subseteq R$. Then*
(1) $PCay(\overline{Cl(R)}, S_1) \cup PCay(\overline{Cl(R)}, S_2) = PCay(\overline{Cl(R)}, S_1 \cup S_2)$,
(2) $PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2) = PCay(\overline{Cl(R)}, S_1 \cap S_2)$.

Proof. (1) It is easy to see that

$$V(PCay(\overline{Cl(R)}, S_1) \cup PCay(\overline{Cl(R)}, S_2)) = V(PCay(\overline{Cl(R)}, S_1 \cup S_2)) = \overline{Cl(R)}.$$

We have

$$\begin{aligned}&e \in E(PCay(\overline{Cl(R)}, S_1) \cup PCay(\overline{Cl(R)}, S_2))\\ &\qquad \Leftrightarrow e \in E(PCay(\overline{Cl(R)}, S_1)) \text{ or } e \in E(PCay(\overline{Cl(R)}, S_2))\\ &\qquad \Leftrightarrow e = \{r, sr\} \;\; r \in \overline{Cl(R)}, s \in S_1 \text{ or } S_2\\ &\qquad \Leftrightarrow e = \{r, sr\} \;\; r \in \overline{Cl(R)}, s \in S_1 \cup S_2\\ &\qquad \Leftrightarrow e \in E(PCay(\overline{Cl(R)}, S_1 \cup S_2)).\end{aligned}$$

(2) We have

$$V(PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2)) = V(PCay(\overline{Cl(R)}, S_1 \cap S_2)) = \overline{Cl(R)}.$$

Now, suppose that $e \in E(PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2))$, So there exists $r_1, r_2 \in \overline{Cl(R)}$, $s_1 \in S_1$ and $s_2 \in S_2$, such that

$$e = \{r_1, s_1r_1\} = \{r_2, s_2r_2\}.$$

We consider the following cases:

Case (1): If $r_1 = r_2$ and $r_1s_1 = r_2s_2$, then $s_1 = s_2$. So $s_1, s_2 \in S_1 \cap S_2$. which implies that $e \in E(PCay(\overline{Cl(R)}, S_1 \cap S_2))$.

Case (2): If $r_1 = s_2r_2$ and $r_2 = s_1s_2r_2$, then $s_1 = s_2^{-1}$. Thus $s_1, s_2 \in S_1 \cap S_2$. So $s_1r_1, s_2r_2 \in (S_1 \cap S_2)\overline{Cl(R)}$. It follows that $e \in E(PCay(\overline{Cl(R)}, S_1 \cap S_2))$. Hence, we have

$$E(PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2)) \subseteq E(PCay(\overline{Cl(R)}, S_1 \cap S_2)).$$

Conversely, let $e \in E(PCay(\overline{Cl(R)}, S_1 \cap S_2))$, then there exists $r \in \overline{Cl(R)}$ and $s \in S_1 \cap S_2$, such that $e = \{r, sr\}$. Since $s \in S_1 \cap S_2$, then $s \in S_1$ and $s \in S_2$.

Hence we have $e \in E(PCay(\overline{Cl(R)}, S_1))$ and $e \in E(PCay(\overline{Cl(R)}, S_2))$. So

$$E(PCay(\overline{Cl(R)}, S_1 \cap S_2)) \subseteq E(PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2)).$$

Finally, we have

$$PCay(\overline{Cl(R)}, S_1) \cap PCay(\overline{Cl(R)}, S_2) = PCay(\overline{Cl(R)}, S_1 \cap S_2).$$

□

Theorem 5.6. *Let R_1 and R_2 be subgroups of a group G. Then*
(1) $PCay(\overline{Cl(R_1)}, S) \cup PCay(\overline{Cl(R_2)}, S) = PCay(\overline{Cl(R_1)} \cup \overline{Cl(R_2)}, S)$,
(2) $PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}, S) = PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S)$.

Proof. (1) We have
$V(PCay(\overline{Cl(R_1)}, S) \cup PCay(\overline{Cl(R_2)}, S)) = V(PCay(\overline{Cl(R_1)} \cup \overline{Cl(R_2)}, S)) = \overline{Cl(R_1)} \cup \overline{Cl(R_2)}$. Also,

$$\begin{aligned} & e \in E(PCay(\overline{Cl(R_1)}, S) \cup PCay(\overline{Cl(R_2)}, S)) \\ &\Leftrightarrow e \in E(PCay(\overline{Cl(R_1)}, S)) \text{ or } e \in E(PCay(\overline{Cl(R_2)}, S)) \\ &\Leftrightarrow e = \{r, sr\} \;\; r \in \overline{Cl(R_1)} \text{ or } \overline{Cl(R_2)}, s \in S \\ &\Leftrightarrow e = \{r, sr\} \;\; r \in \overline{Cl(R_1)} \cup \overline{Cl(R_2)}, s \in S \\ &\Leftrightarrow e \in E(PCay(\overline{Cl(R_1)} \cup \overline{Cl(R_2)}, S)). \end{aligned}$$

(2) It is easy to see that

$$V(PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}, S)) = V(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S)) = (\overline{Cl(R_1)} \cap \overline{Cl(R_2)}).$$

If $e \in E(PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}))$, then $e \in E(PCay(\overline{Cl(R_1)}, S))$ and $e \in E(PCay(\overline{Cl(R_2)}, S))$. So there exists $r_1 \in \overline{Cl(R_1)}, r_2 \in \overline{Cl(R_2)}$ and $s_1, s_2 \in S$, such that

$$e = \{r_1, s_1 r_1\} = \{r_2, s_2 r_2\}.$$

We consider the following cases:

Case (1): If $r_1 = r_2$, then $r_1 = r_2 \in \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$, which implies that $e \in E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S))$

Case (2): If $r_1 = s_2 r_2$, then $r_1 = s_2 r_2 \in S\overline{Cl(R_2)} \subseteq \overline{Cl(R_2)}$, which implies that $e \in E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S))$. Hence, we have

$$E(PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}, S)) \subseteq E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S)).$$

Conversely, if $e \in E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S))$, then there exists $r \in \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$ and $s \in S$, such that $e = \{r, sr\}$, hence $e \in E(PCay(\overline{Cl(R_1)}, S))$, $e \in E(PCay(\overline{Cl(R_2)}, S))$. Thus

$$E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S)) \subseteq E(PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}, S)).$$

Finally, we have

$$PCay(\overline{Cl(R_1)}, S) \cap PCay(\overline{Cl(R_2)}, S) = PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S).$$

□

Theorem 5.7. *Let R_1 and R_2 be subsets of a group G, $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ Then*

$$PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2) = PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2).$$

Proof. We have
$V(PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2)) = V(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2)) = \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$.
Now suppose that $e \in E(PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2))$, then $e \in E(PCay(\overline{Cl(R_1)}, S_1))$ and $e \in E(PCay(\overline{Cl(R_2)}, S_2))$. So there exist $r_1 \in \overline{Cl(R_1)}, r_2 \in \overline{Cl(R_2)}, s_1 \in S_1, s_2 \in S_2$ such that

$$e = \{r_1, s_1 r_1\} = \{r_2, s_2 r_2\}.$$

We consider the following cases:

Case (1): If $r_1 = r_2$, then $r_1 = r_2 \in \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$. On the other hand, $s_1 r_1 = s_2 r_2$, thus $s_1 = s_2 \in S_1 \cap S_2$. Hence $e \in E(PCay(\overline{Cl(R_1)}) \cap \overline{Cl(R_2)}, S_1 \cap S_2)$.

Case (2): If $r_1 = s_2 r_2$, then $r_1 = s_2 r_2 \in S\overline{Cl(R_2)} \subseteq \overline{Cl(R_2)}$. Hence $r_1 \in \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$. From $s_1 r_1 = r_2$, we have $s_1 s_2 = 1$, so $s_1 = s_2^{-1}$, then $s_1 \in S_1 \cap S_2$. Similarly, $s_2 \in S_1 \cap S_2$.
It follows $e \in E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2))$. Hence

$$E(PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2)) \subseteq E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2)).$$

Conversely, if $e \in E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2))$, then there exists $r \in \overline{Cl(R_1)} \cap \overline{Cl(R_2)}$ and $s \in S_1 \cap S_2$, such that $e = \{r, sr\}$. Therefore, $r \in \overline{Cl(R_1)}, \overline{Cl(R_2)}$ and $s \in S_1, S_2$. Hence, $e \in E(PCay(\overline{Cl(R_1)}, S_1))$ and $e \in E(PCay(\overline{Cl(R_2)}, S_2))$, and so

$$E(PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2)) \subseteq E(PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2)).$$

Finally, we have

$$PCay(\overline{Cl(R_1)}, S_1) \cap PCay(\overline{Cl(R_2)}, S_2) = PCay(\overline{Cl(R_1)} \cap \overline{Cl(R_2)}, S_1 \cap S_2).$$

□

Theorem 5.8. *Let f be a automorphism of group G. let $\underline{\Gamma} = Cay(G, \underline{Cl}(S))$ and $\overline{\Gamma} = Cay(G, \overline{Cl}(S))$, then*
(1) $Cay(G, \underline{Cl}(S)) \cong Cay(G, \underline{Cl}(f(S)))$

(2) $Cay(G, \overline{Cl}(S)) \cong Cay(G, \overline{Cl}(f(S)))$.

Proof. (1) We show that f is a automorphism between groups. Let $e \in E(Cay(G, \underline{Cl}(S)))$. Then there exist $g \in G$ and $s \in \underline{Cl}(S)$, such that $e = \{g, gs\}$. Thus $s^G \subseteq S$, and there exist $g \in G$ such that $s^g \in S$. It follows $f(s)^{f(g)} \in f(S)$. Hence $f(s) \in \underline{Cl}(f(S))$ which implies that $f(e) = \{f(g), f(g)f(s)\} \in E(Cay(G, \underline{Cl}(f(S))))$.

(2) Let $e \in E(Cay(G, \overline{Cl}(S)))$. Then there exist $g \in G$ and $s \in \overline{Cl}(S)$, such that $e = \{g, gs\}$. Hence $s^G \cap S \neq \varnothing$, and there exist $t \in s^G \cap S$ and $g \in G$ such that $s^g = t$. Then $f(s)^{f(g)} = f(t) \in f(s)^G$ and $f(t) \in f(S)$, therefore, $f(s)^G \cap f(S) \neq \varnothing$ and $f(s) \in \overline{Cl}(f(S))$. Thus $f(e) = \{f(g), f(g)f(s)\}$. Finally $f(e) \in E(Cay(G, \overline{Cl}(f(S))))$. $\square$

Remark 5.9. *According to the following example, we show* $Cay(G, \underline{Cl}(S)) \cong Cay(G, \underline{Cl}(S^{'}))$*, but* $Cay(G, S) \ncong Cay(G, S^{'})$.

Example 5.10. *Let* $G =< a, b\ ;\ b^2 = a^4 = 1, ab = ba^{-1} >$ *and* $S = \{a^2\}$ *and* $S^{'} = \{a^2, b\}$ *be subsets of* G*. Then* $\underline{Cl}(S) = \underline{Cl}(S^{'}) = \{a^2\}$.

6. CONCLUDING

The theory of rough set is an extension of set theory. Many mathematicians are interested in studying the relationship between rough sets and algebraic systems. In this paper, we studied the properties of the rough approximations of Cayley and pseudo-Cayley graphs. Also, we gave rough vertex pseudo-Cayley graphs and prove some theorems.

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