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ON MINIMUM AND MAXIMUM OF FUNCTIONS OF SMALL BAIRE CLASSES

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Abstract. A real-valued function on a Polish space X is said to of Baire class one (or simply, a Baire-1 function) if it is the pointwise limit of a sequence of continuous functions. Let $\mathcal{B}_1(X)$ be the set of all real-valued Baire-1 functions on X. Kechris and Louveau defined the set of functions of *small Baire class* ξ for each countable ordinal ξ as $\mathcal{B}_1^\xi(X) = \{f \in \mathcal{B}_1(X) : \beta(f) \le \omega^\xi\}$, where $\beta(f)$ denotes the oscillation index of f. In this paper we prove that the minimum and maximum of two functions of small Baire class ξ are also functions of small Baire class ξ. This extends a result of Chaatit, Mascioni, and Rosenthal [1] who obtained the result for $\xi = 1$.

1. INTRODUCTION

Let X be a metrizable space. A function $f : X \to \mathbf{R}$ is said to be of Baire class one (or simply, a Baire-1 function) if it is the pointwise limit of a sequence of continuous functions on X. The Baire Characterization Theorem states that if X is a Polish space, that is, a separable completely metrizable space, then $f : X \to \mathbf{R}$ is of Baire class one if and only if $f|_F$ has a point of continuity for every nonempty closed subset F of X. This leads naturally to the oscillation index for Baire-1 functions. This ordinal index was used by Kechris and Louvaeu [2] to give a finer gradation of Baire-1 functions into *small Baire classes*. Let $\mathcal{B}_1(X)$ be the set of all Baire-1 functions on X. For every ordinal $\xi < \omega_1$, the set of functions of small Baire class ξ is defined as

$$\mathcal{B}_1^\xi(X) = \{f \in \mathcal{B}_1(X) : \beta(f) \le \omega^\xi\}.$$

This study was continued by various authors. (See, e.g., [3],[4],and [5]).

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In this paper, we prove that if f and g belong to a small Baire class ξ for some $\xi < \omega_1$, then the minimum and maximum of f and g also belong to that class. This extends a result of Chaatit, Mascioni and Rosenthal [1] who obtained the result for $\xi = 1$.

We begin by recalling the definition of oscillation index β. The oscillation index β is associated with a family of derivations. Let X be a metrizable space and $\mathcal{C}$ denote the collection of all closed subsets of X. A derivation is a map $\mathcal{D} : \mathcal{C} \to \mathcal{C}$ such that $\mathcal{D}(H) \subseteq H$ for all $H \in \mathcal{C}$. Let $\varepsilon > 0$ and a function $f : X \to \mathbf{R}$ be given. For any closed subset H of X set $\mathcal{D}^0(f, \varepsilon, H) = H$ and $\mathcal{D}^1(f, \varepsilon, H)$ be the set of all $x \in H$ such that for every open set U containing x there are two points x_1 and x_2 in $U \cap H$ with $|f(x_1) - f(x_2)| \geq \varepsilon$. For $\alpha < \omega_1$, let

$$\mathcal{D}^{\alpha+1}(f, \varepsilon, H) = \mathcal{D}^1(f, \varepsilon, \mathcal{D}^\alpha(f, \varepsilon, H)).$$

If α is a countable limit ordinal,

$$\mathcal{D}^\alpha(f, \varepsilon, H) = \bigcap_{\alpha' < \alpha} \mathcal{D}^{\alpha'}(f, \varepsilon, H).$$

The *ε-oscillation index* of f on H is defined by

$$\beta_H(f, \varepsilon) = \begin{cases} \text{the smallest ordinal } \alpha < \omega_1 \text{ such that } \mathcal{D}^\alpha(f, \varepsilon, H) = \emptyset \\ \text{if such an } \alpha \text{ exists,} \\ \\ \omega_1, \quad \text{otherwise.} \end{cases}$$

The *oscillation index* of f on the set H is defined by

$$\beta_H(f) = \sup\{\beta_H(f, \varepsilon) \, : \varepsilon > 0\}.$$

We shall write $\beta(f, \varepsilon)$ and $\beta(f)$ for $\beta_X(f, \varepsilon)$ and $\beta_X(f)$ respectively.

2. MAIN RESULTS

Throughout, let X be a Polish space. For $f, g : X \to \mathbf{R}$, we denote their minimum and their maximum by $f \wedge g$ and $f \vee g$ respectively. A result in [1] is that if the oscillation indices of f and g are finite then the oscillation indices of $f \wedge g$ and $f \vee g$ are also finite. We extend this result into the classes of small Baire functions. We get the following result.

Theorem 2.1. *Let $f, g : X \to \mathbf{R}$. If $\beta(f) \leq \omega^\xi$ and $\beta(g) \leq \omega^\xi$ for some $\xi < \omega_1$, then $\beta(f \wedge g) \leq \omega^\xi$ and $\beta(f \vee g) \leq \omega^\xi$.*

Theorem 2.1 is proved by the method used in [2]. Following [5], we define a derivation $\mathcal{G}$ which closely related to $\mathcal{D}$. Given a real-valued function f on X,

$\varepsilon > 0$, and a closed subset H of X. Define $G(f, \varepsilon, H)$ to be the set of all $x \in H$ such that for any open neighborhood U of x, there exists $x' \in H \cap U$ such that $|f(x) - f(x')| \geq \varepsilon$. Let

$$\mathcal{G}^1(f, \varepsilon, H) = \overline{G(f, \varepsilon, H)}$$

where the closure is taken in X. If $\alpha < \omega_1$, let

$$\mathcal{G}^{\alpha+1}(f, \varepsilon, H) = \mathcal{G}^1(f, \varepsilon, \mathcal{G}^\alpha(f, \varepsilon, H)).$$

If $\alpha < \omega_1$ is a limit ordinal, let

$$\mathcal{G}^\alpha(f, \varepsilon, H) = \bigcap_{\alpha' < \alpha} \mathcal{G}^{\alpha'}(f, \varepsilon, H).$$

The relationship between derivations $\mathcal{D}$ and $\mathcal{G}$ is given in the following lemma that can be seen in [5, Lemma 4].

Lemma 2.2. *If f be real-valued function on X, $\varepsilon > 0$ and H is a closed subset of X, then*

$$\mathcal{D}^\alpha(f, 2\varepsilon, H) \subseteq \mathcal{G}^\alpha(f, \varepsilon, H) \subseteq \mathcal{D}^\alpha(f, \varepsilon, H),$$

for all $\alpha < \omega_1$.

Before we prove the main result, we show the following results first.

Lemma 2.3. *If f_1 and f_2 are real-valued functions on X, $\varepsilon > 0$, H is a closed subset of X and $f = f_1 \wedge f_2$ then*

$$\mathcal{G}^1(f, \varepsilon, H) \subseteq \mathcal{G}^1(f_1, \varepsilon, H) \cup \mathcal{G}^1(f_2, \varepsilon, H).$$

Proof. Let $x \in G(f, \varepsilon, H)$. If U is an open neighborhood of x in X then there exists $x' \in U \cap H$ such that $|f(x) - f(x')| \geq \varepsilon$. If $|f(x) - f(x')| = f(x) - f(x')$, then

$$\begin{aligned} |f(x) - f(x')| &= f_i(x) - f_j(x'), \quad i, j \in \{1, 2\} \\ &\leq f_j(x) - f_j(x'), \quad j \in \{1, 2\} \\ &= |f_j(x) - f_j(x')|, \quad j \in \{1, 2\}. \end{aligned}$$

Therefore $|f_j(x) - f_j(x')| \geq \varepsilon$, $j \in \{1, 2\}$. This shows $x \in G(f_1, \varepsilon, H) \cup G(f_2, \varepsilon, H)$. Similarly, whenever $|f(x) - f(x')| = f(x') - f(x)$.

It follows that

$$\mathcal{G}^1(f, \varepsilon, H) = \mathcal{G}^1(f_1, \varepsilon, H) \cup \mathcal{G}^1(f_2, \varepsilon, H).$$

Similarly, we obtain the following lemma.

Lemma 2.4. *If f_1 and f_2 are real-valued functions on X, $\varepsilon > 0$, H is a closed subset of X and $f = f_1 \vee f_2$ then*

$$\mathcal{G}^1(f, \varepsilon, H) \subseteq \mathcal{G}^1(f_1, \varepsilon, H) \cup \mathcal{G}^1(f_2, \varepsilon, H).$$

Now, we are ready to prove the main result.

Proof of Theorem 2.1. We prove for the minimum of f and g, for the $f \vee g$ we can prove in the similar way, by using Lemma 2.4 instead of Lemma 2.3. Let $\varepsilon > 0$. First, we prove that

$$\mathcal{G}^{\omega^\xi}(f \wedge g, \varepsilon, H) \subseteq \mathcal{G}^{\omega^\xi}(f, \varepsilon, H) \cup \mathcal{G}^{\omega^\xi}(g, \varepsilon, H). \tag{1}$$

for all closed subset H of X and $\xi < \omega_1$.

We prove (1) by transfinite induction on ξ. For $\xi = 0$, i.e., $\omega^\xi = 1$, this just Lemma 2.3. Since $(\mathcal{G}^\alpha(f, \varepsilon, H))_\alpha$ and $(\mathcal{G}^\alpha(g, \varepsilon, H))_\alpha$ are non-increasing, then (1) is immediate for a limit ordinal $\xi < \omega_1$.

Suppose that (1) is true for some ordinal $\xi < \omega_1$, we have to prove that (1) is also true for $\xi + 1$. For this, we need to prove that

$$\mathcal{G}^{\omega^\xi \cdot 2n}(f \wedge g, \varepsilon, H) \subseteq \mathcal{G}^{\omega^\xi \cdot n}(f, \varepsilon, H) \cup \mathcal{G}^{\omega^\xi \cdot n}(g, \varepsilon, H) \tag{2}$$

for all $n \in \mathbf{N}$.

For this, let for $s \in 2^k = \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \ : \epsilon_i = 0 \text{ or } 1\}$ $, k \in \mathbf{N},$ we define H_s as follows

$$H_0 = \mathcal{G}^{\omega^\xi}(f, \varepsilon, H),$$

$$H_1 = \mathcal{G}^{\omega^\xi}(g, \varepsilon, H),$$

and

$$H_{s^\wedge 0} = \mathcal{G}^{\omega^\xi}(f, \varepsilon, H_s),$$

$$H_{s^\wedge 1} = \mathcal{G}^{\omega^\xi}(g, \varepsilon, H_s).$$

In order to prove (2), we need to show that

$$\mathcal{G}^{\omega^\xi \cdot k}(f \wedge g, \varepsilon, H) \subseteq \bigcup_{s \in 2^k} H_s \tag{3}$$

for all $k \in \mathbf{N}$. By the assumption induction, statement (3) is true for $k = 1$.

Suppose that (3) is true fo some $k \in \mathbf{N}$. We obtain

$$\begin{aligned}
\mathcal{G}^{\omega^\xi \cdot (k+1)}(f \wedge g, \varepsilon, H) &= \mathcal{G}^{\omega^\xi \cdot k + \omega^\xi}(f \wedge g, \varepsilon, H) \\
&= \mathcal{G}^{\omega^\xi}(f \wedge g, \varepsilon, \mathcal{G}^{\omega^\xi \cdot k}(f \wedge g, \varepsilon, H)) \\
&\subseteq \mathcal{G}^{\omega^\xi}(f \wedge g, \varepsilon, \bigcup_{s \in 2^k} H_s) \\
&\subseteq \bigcup_{s \in 2^k} \mathcal{G}^{\omega^\xi}(f \wedge g, \varepsilon, H_s) \ \text{ by [5, Lemma 4]} \\
&\subseteq \bigcup_{s \in 2^k} (\mathcal{G}^{\omega^\xi}(f, \varepsilon H_s) \cup \mathcal{G}^{\omega^\xi}(g, \varepsilon, H_s) \\
&= (\bigcup_{s \in 2^k} \mathcal{G}^{\omega^\xi}(f, \varepsilon, H_s) \cup (\bigcup_{s \in 2^k} \mathcal{G}^{\omega^\xi}(g, \varepsilon, H_s)) \\
&= (\bigcup_{s \in 2^k} H_{s^\wedge 0}) \cup (\bigcup_{s \in 2^k} H_{s^\wedge 1}) \\
&= \bigcup_{s \in 2^{k+1}} H_s.
\end{aligned}$$

By (3), for all $n \in \mathbf{N}$, we have

$$\begin{aligned}
\mathcal{G}^{\omega^\xi \cdot 2n}(f \wedge g, \varepsilon, H) &\subseteq \bigcup_{s \in 2^{2n}} H_s \\
&\subseteq \bigcup \{H_s \ : s \in 2^{2n} \text{ dan card } (\{k : s(k) = 0\}) \geq n\} \\
&\quad \cup \bigcup \{H_s \ : s \in 2^{2n} \text{ dan card } (\{k : s(k) = 1\}) \geq n\}.
\end{aligned}$$

If s takes at least n values 0, then $H_s \subseteq \mathcal{G}^{\omega^\xi \cdot n}(f, \varepsilon, H)$. Similarly, if s takes at least n values 1, then $H_s \subseteq \mathcal{G}^{\omega^\xi \cdot n}(g, \varepsilon, H)$. Therefore, the proof of (2) is finished.

Since $(\mathcal{G}^\alpha(f, \varepsilon, H))_\alpha$ and $(\mathcal{G}^\alpha(g, \varepsilon, H))_\alpha$ are non-increasing, then by taking the intersection over n in (2) gives

$$\mathcal{G}^{\omega^{\xi+1}}(f \wedge g, \varepsilon, H) \subseteq \mathcal{G}^{\omega^{\xi+1}}(f, \varepsilon, H) \cup \mathcal{G}^{\omega^{\xi+1}}(g, \varepsilon, H).$$

Using (1) and Lemma 2.1, since $\beta(f) \leq \omega^\xi$ and $\beta(g) \leq \omega^\xi$, then

$$\mathcal{G}^{\omega^\xi}(f, \varepsilon, H) = \emptyset \text{ dan } \mathcal{G}^{\omega^\xi}(g, \varepsilon, H) = \emptyset.$$

Therefore,

$$\mathcal{D}^{\omega^\xi}(f \wedge g, 2\varepsilon, H) \subseteq \mathcal{G}^{\omega^\xi}(f \wedge g, \varepsilon, H) = \emptyset.$$

It follows that $\beta(f \wedge g) \leq \omega^\xi$.

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