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ON THE SPHERICAL INDICATRIES OF CURVES IN GALILEAN 4-SPACE

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Abstract. Euclidean and non-Euclidean geometries can be considered as spaces that are invariant under a given group of transformations [8]. The geometry established by this approach is called Cayley-Klein geometry. Galilean 4-space is simply defined as a Cayley-Klein geometry of the product space $\mathbb{R} \times \mathbb{E}^3$ whose symmetry group is Galilean transformation group which has an important place in classical and modern physics.

Key words and Phrases: Galilean 4D space, Spherical indicatrices, $ccr-$curve, Involute-evolute curves, Bertrand curve, Helix, Spherical curve.

Abstrak. Geometri Euclidean dan non-Euclidean dapat dipandang sebagai ruang yang invarian terhadap suatu grup transformasi [8]. Geometri yang terbentuk dari pendekatan ini disebut Geometri Cayley-Klein. Galilean 4-space didefinisikan sebagai suatu geometri Cayley-Klein dari hasil kali dari ruang $\mathbb{R} \times \mathbb{E}^3$ yang grup simetrinya adalah grup transformasi Galilean yang memiliki aplikasi penting di fisika klasik dan modern.

Kata kunci: Ruang Galilean 4D, *indicatrice* speris, kurva *ccr*, kurva *involute-evolute*, kurva Bertrand, Heliks, kurva speris.

1. INTRODUCTION

Euclidean and non-Euclidean geometries can be considered as spaces that are invariant under a given group of transformations [8]. The geometry established by this approach is called Cayley-Klein geometry. Galilean 4-space is simply defined

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as a Cayley-Klein geometry of the product space $\mathbb{R} \times \mathbb{E}^3$ whose symmetry group is Galilean transformation group which has an important place in classical and modern physics.

Discovering Galilean space-time is probably one of the major achievements of non relativistic physics. One may consider Galilean space as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. Nowadays Galilean space is becoming increasingly popular as evidenced from the connection of the fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles as indicated in [17].

A curve in 3D Galilean space is a graph of a plane motion. Note that such a curve is called a world line in 3-dimensional Galilean space. It is well known that the idea of world lines originates in physics and was pioneered by Einstein. The term is now often used in relativity theories, that is, general relativity and special relativity [18].

There is a vast of literature about researches on differential geometry of curves in both 3D and 4D Galilean space as explained below: firstly, Some special curves such as helices, involute-evolute curves were studied in [2, 5, 13, 14, 16]. Spherical curves and spherical indicatrix curves were examined in different spaces such as Galilean space, Heisenberg group, and de Sitter-space [1, 7, 9, 10, 11, 19].

Yılmaz constructed Frenet-Serret frame of a curve in 4D Galilean space and gave some characterizations of spherical curves in the same space [20]. Bektaş et al. characterized Mannheim curves in 4D Galilean space [4]. Yoon gave some characterizations of inclined curves by using their curvatures in 4D Galilean space [21]. Aydın et al. established equiform differential geometry of curves in 4D Galilean space [3]. Yoon et al. gave some characterizations of osculating curves by using their curvatures in 4D Galilean space [22]. Öztekin introduced special Bertrand curves and characterized them in 4D Galilean space [15].

In this paper, firstly we obtain the tangent, principal normal, binormal, and trinormal spherical indicatrices of a regular curve in 4D Galilean space. Then we get their Frenet elements of these four special curves in terms of the Frenet elements of the original curve at the same space. Moreover, using these, we give some theorems characterizing these special curves as ccr-curves, general helix, involute-evolute curve pairs, and Bertrand mates.

2. Preliminaries

The 3-dimensional Galilean space $\mathbb{G}_3$ is the Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean space consists of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f.

The 4-dimensional Galilean geometry $\mathbb{G}_4$ can be described as the study of properties of four-dimensional space with coordinates that are invariant under general Galilean transformations

$$\begin{aligned}
x' &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x + (\sin\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)y \\
&\quad +(\sin\gamma\sin\alpha)z + (v\cos\delta_1)t + a, \\
y' &= -(\cos\beta\sin\alpha - \cos\gamma\sin\beta\cos\alpha)x + (-\sin\beta\sin\alpha - \cos\gamma\cos\beta\cos\alpha)y \\
&\quad +(\sin\gamma\cos\alpha)z + (v\cos\delta_2)t + b, \\
z' &= (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y + (\cos\gamma)z + (v\cos\delta_3)t + c, \\
t' &= t + d,
\end{aligned}$$

where $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$ [20]. Along the paper, the four dimensional Galilean space will be denoted by $\mathbb{G}_4$.

Given two vectors $a = (a_1, a_2, a_3, a_4)$and $b = (b_1, b_2, b_3, b_4)$, then the Galilean scalar product between these vectors is, as in [20], defined by

$$\langle a, b\rangle_{\mathbb{G}_4} = \begin{cases} a_1 b_1, & \text{if } a_1 \neq 0 \vee b_1 \neq 0, \\ a_2 b_2 + a_3 b_3 + a_4 b_4, & \text{if } a_1 = 0 \wedge b_1 = 0. \end{cases}$$

For the vectors $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$, and $c = (c_1, c_2, c_3, c_4)$, the cross product in the $\mathbb{G}_4$ is, as in [20], given by

$$(a \wedge b \wedge c)_{\mathbb{G}_4} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \text{ if } a_1 \neq 0, b_1 \neq 0 \vee c_1 \neq 0, \\[2ex] \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \text{ if } a_1 = 0, b_1 = 0 \wedge c_1 = 0. \end{cases}$$

In [20], the Galilean sphere of radius r and center m of the space $\mathbb{G}_4$ is defined by

$$S^3_G(m, r) = \{\varphi - m \in \mathbb{G}_4 (\varphi - m, \varphi - m)_{\mathbb{G}_4} = \mp r^2\}.$$

A curve $\alpha : I \to \mathbb{G}_4$ of the class C^r $(r \geq 4)$ in the Galilean space $\mathbb{G}_4$ is defined by the parametrization:

$$\alpha(s) = (s, y(s), z(s), w(s)),$$

where s is arclength parameter [20]. The orthonormal frame in $\mathbb{G}_4$ is defined by

$$\begin{cases} t(s) = \alpha'(s) = (1, y'(s), z'(s), w'(s)), \\ n(s) = \dfrac{\alpha''(s)}{\|\alpha''(s)\|} = \dfrac{(0, y''(s), z''(s), w''(s)}{\kappa}, \\ b(s) = \dfrac{1}{\tau(s)}(0, (\dfrac{1}{\kappa(s)}y''(s))', (\dfrac{1}{\kappa(s)}z''(s))', (\dfrac{1}{\kappa(s)}w''(s))'), \\ e(s) = \mu t \wedge n \wedge b, \end{cases} \tag{1}$$

where the coefficient μ is taken ± 1 to make $+1$ the matrix $[t, n, b, e]$ [20].

The curvature $\kappa(s)$, the torsion (the second curvature) $\tau(s)$, and the third curvature $\sigma(s)$ are defined by

$$\begin{aligned}
&\kappa(s) = \|\alpha''(s)\|_G = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2}, \\
&\tau(s) = \|n'(s)\|_G, \\
&\sigma(s) = \langle b', e\rangle_G.
\end{aligned} \tag{2}$$

It is well known that the set $\{t, n, b, e, \kappa, \tau, \sigma\}$ is called the Frenet-Serret apparatus of the curve α. Here, we know that the vectors $\{t, n, b, e\}$ are mutually orthogonal vectors satisfying

$$\begin{aligned}
&\langle t, t\rangle_G = \langle n, n\rangle_G = \langle b, b\rangle_G = \langle e, e\rangle_G = 1, \\
&\langle t, n\rangle_G = \langle t, b\rangle_G = \langle t, e\rangle_G = \langle n, b\rangle_G = \langle n, e\rangle_G = \langle b, e\rangle_G = 0.
\end{aligned}$$

The vectors t, n, b, and e in represent the tangent, the principal normal, the binormal, and the trinormal vectors of α, respectively. The Frenet derivative formulas can be given as in [20] in $\mathbb{G}_4$

$$\begin{cases}
t' = \kappa(s)n(s), \\
n' = \tau(s)b(s), \\
b' = -\tau(s)n(s) + \sigma(s)e(s), \\
e' = -\sigma(s)b(s).
\end{cases}$$

Definition 2.1. [12] *A curve $\alpha : I \to \mathbb{G}_4$ is said to have constant curvature ratios (that is to say, it is a ccr−curve) if the quotients $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constant.*

Definition 2.2. [21] *Let $\alpha : I \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. The curve α is called helix if its tangent vector t makes a constant angle with a fixed direction U.*

Definition 2.3. [2] *Let α and α^* be two curves in $\mathbb{G}_4$. The curve α^* is called involute of the curve α if the tangent vector of the curve α at the point $\alpha(s)$ passes through the tangent vector of the curve α^* at the point $\alpha^*(s)$ and $\langle t, t^*\rangle = 0$, where $\{t, n, b, e\}$ and $\{t^*, n^*, b^*, e^*\}$ are Frenet frames of α and α^*, respectively. Also, the curve α is called the evolute of the curve α^*.*

Definition 2.4. [15] *A $C^\infty-$special Frenet-Serret curve α in $\mathbb{G}_4$ is called a Bertrand curve if there exist a $C^\infty-$special Frenet-Serret curve $\widetilde{\alpha}$, distinct from α, and a regular $C^\infty-$map $\varphi : I \to \widetilde{I}$ ($\widetilde{s} = \varphi(s)$, $\frac{d\varphi(s)}{ds} \neq 0$ for all $s \in I$) such that curves α and $\widetilde{\alpha}$ have the same principal normal line at each pair of corresponding points $\alpha(s)$ and $\widetilde{\alpha}(\widetilde{s}) = \widetilde{\alpha}(\varphi(s))$ under φ. Here s and $\widetilde{s}$ are arclength parameters of α and $\widetilde{\alpha}$, respectively. In this case, $\widetilde{\alpha}$ is called a Bertrand mate of α and the mate of curves $(\alpha, \widetilde{\alpha})$ is said to be a Bertrand mate in $\mathbb{G}_4$.*

3. Spherical indicatrices of curves in $\mathbb{G}_4$

In this section, the tangent, principal normal, binormal, and trinormal spherical indicatrices of the curve α in $\mathbb{G}_4$ are expressed in terms of the Frenet apparatus of the curve α.

3.1. **Tangent spherical indicatrix of a curve in $\mathbb{G}_4$.**

Definition 3.1. *Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc-length s in $\mathbb{G}_4$. If we translate the tangent vector field of Frenet frame to the center O of the unit sphere S^3_G, we obtain a spherical indicatrix which is called tangent spherical indicatrix α_t the curve $\alpha = \alpha(s)$.*

Let $\varphi = \varphi(s_\varphi)$ be α_t Galilean spherical indicatrix of a regular curve $\alpha = \alpha(s)$ in $\mathbb{G}_4$. First, we differentiate it, we have

$$\varphi' = \frac{d\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = (0, y''(s), z''(s), w''(s)), \tag{3}$$

where we denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Taking the norm of both sides of (3), we have $\frac{ds_\varphi}{ds} = \kappa(s)$ and

$$t_\varphi = n(s) = \frac{1}{\kappa}(0, y''(s), z''(s), w''(s)). \tag{4}$$

Differentiating (4), then we get

$$\frac{dt_\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = (0, (\tfrac{1}{\kappa(s)}y''(s))', (\tfrac{1}{\kappa(s)}z''(s))', (\tfrac{1}{\kappa(s)}w''(s))'),$$

and $\dot{t}_\varphi = \dfrac{\tau(s)b(s)}{\kappa(s)}$.

Thus we have the first curvature and the principal normal vector of φ as

$$\kappa_\varphi(s) = \left\|\dot{t}_\varphi\right\| = \frac{\tau(s)}{\kappa(s)}, \tag{5}$$

and $n_\varphi = b(s)$. Differentiating (5) with respect to s, we have

$$n'_\varphi = \frac{dn_\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = -\tau n + \sigma e,$$
$$n'_\varphi = \tau_\varphi b_\varphi = -\tau n + \frac{\sigma e}{\kappa}.$$

Hence, we find the second curvature of binormal vector of φ as follow:

$$\tau_\varphi = \sqrt{(\frac{\tau}{\kappa})^2 + (\frac{\sigma}{\kappa})^2}, \tag{6}$$

and

$$b_\varphi = \frac{1}{\tau_\varphi}(-\frac{\tau}{\kappa})n + (\frac{\sigma}{\kappa})e.$$

The cross (exterior) product of $\mu t_\varphi \wedge n_\varphi \wedge b_\varphi$ gives the trinormal vector field of the tangent spherical indicatrix α_t of $\alpha = \alpha(s)$ in $\mathbb{G}_4$. Since $e_\varphi = \mu t_\varphi \wedge n_\varphi \wedge b_\varphi$, we have

$$e_\varphi = \mu\frac{\kappa}{\sqrt{\tau^2 + \sigma^2}}t.$$

Since $\sigma_\varphi = -e'_\varphi b$, we find

$$\sigma_\varphi = \frac{\kappa^2 \tau}{\tau^2 + \sigma^2}. \tag{7}$$

3.2. **The principal normal spherical indicatrix of a curve in $\mathbb{G}_4$.**

Definition 3.2. *Let $\alpha = (s, y(s), z(s), w(s))$ be a curve parametrized by arc-length s in $\mathbb{G}_4$. If we translate the principal normal vector field of Frenet frame to the center O of the unit sphere S^3_G, we obtain a spherical indicatrix $\beta = \beta(s_\beta)$ which is called the principal normal spherical indicatrix α_n of a regular curve $\alpha = \alpha(s)$.*

Let $\beta = \beta(s_\beta)$ be the principal normal spherical indicatrix α_n of a regular curve $\alpha = \alpha(s)$. We can write that

$$\beta' = \frac{d\beta}{ds_\beta} . \frac{ds_\beta}{ds} = \tau(s) b(s),$$

similar to the tangent spherical indicatrix α_t, one can have

$$t_\beta = b(s), \tag{8}$$

and

$$\frac{ds_\beta}{ds} = \tau(s). \tag{9}$$

Differentiating of the formula (8), we obtain

$$t'_\beta = \dot{t}_\beta \frac{ds_\beta}{ds} = -\tau(s) n(s) + \sigma(s) e(s),$$

or

$$t'_\beta = -n(s) + \frac{\sigma(s)}{\tau(s)} e(s).$$

Therefore, we have the first curvature and the principal normal vector of β

$$\kappa_\beta(s) = \left\| t'_\beta \right\| = \sqrt{1 + (\frac{\sigma}{\tau})^2}, \tag{10}$$

and

$$n_\beta = \frac{1}{\kappa_\beta} [-n + \frac{\sigma}{\tau}]. \tag{11}$$

Differentiating (11) gives

$$n'_\beta = \frac{dn_\beta}{ds_\beta} \frac{ds_\beta}{ds} = n'_\beta \frac{ds_\beta}{ds}, \quad \text{or} \quad n'_\beta = \frac{ds_\beta}{ds} (\tau_\beta b_\beta) = (\frac{1}{\kappa_\beta \tau} [-n + \frac{\sigma}{\tau} e])'.$$

Thus, the torsion and the binormal vector of β are expressed as

$$\tau_\beta = ([\frac{1}{\kappa_\beta \tau} (1 + (\frac{\sigma}{\tau})^2)]')^2, \tag{12}$$

and

$$b_\beta = -(\frac{1}{\kappa_\beta})' n - b + (\frac{\sigma}{\tau}) [(\frac{1}{\kappa_\beta})' + \frac{1}{\kappa_\beta}] e.$$

Calculating $\mu(t_\beta \wedge n_\beta \wedge b_\beta)$, we get

$$e_\beta = \mu(\frac{1}{\kappa_\beta})' t,$$

and also using the third curvature formula, we find

$$\sigma_\beta = (\frac{1}{\kappa_\beta})'^2 \kappa. \tag{13}$$

3.3. **The binormal spherical indicatrix of a curve in $\mathbb{G}_4$.**

Definition 3.3. *Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc-length s in $\mathbb{G}_4$. If we translate the binormal vector field of Frenet frame to the center O of the unit sphere S^3_G, we obtain a spherical indicatrix $\phi = \phi(s_\phi)$ which is called the binormal spherical indicatrix α_b of the curve $\alpha = \alpha(s)$.*

Let $\phi = \phi(s_\phi)$ be the binormal spherical indicatrix α_b of a regular curve $\alpha = \alpha(s)$. In terms of the Frenet frame vector fields in $\mathbb{G}_4$, one can differentiate of ϕ respect to s

$$\phi' = \frac{d\phi}{ds_\phi}.\frac{ds_\phi}{ds} = -\tau(s)n(s) + \sigma(s)e(s).$$

In terms of the Frenet frame vector fields in $\mathbb{G}_4$, we have the binormal vector of the spherical indicatrix as follows:

$$t_\phi = \frac{-\tau n + \sigma e}{\sqrt{\tau^2 + \sigma^2}}, \tag{14}$$

where

$$\frac{ds_\phi}{ds} = \sqrt{\tau^2 + \sigma^2}.$$

In order to determine the first curvature of ϕ , we write

$$\begin{aligned} t'_\phi &= \dot{t}_\phi \frac{ds_\phi}{ds} \\ &= \frac{1}{\tau^2 + \sigma^2} \left[-(\frac{\tau}{\sqrt{\tau^2 + \sigma^2}})' n + (\sqrt{\tau^2 + \sigma^2}) b + (\frac{\sigma}{\sqrt{\tau^2 + \sigma^2}})' e \right], \end{aligned} \tag{15}$$

hence we arrive at

$$\kappa_\phi = \left\| \dot{t}_\phi \right\| = \frac{1}{\tau^2 + \sigma^2} \sqrt{((\frac{\tau}{\sqrt{\tau^2 + \sigma^2}})')^2 + \tau^2 + \sigma^2 + (\frac{\sigma}{\sqrt{\tau^2 + \sigma^2}})^2}, \tag{16}$$

and the normal vector field of the spherical indicatrix α_b is obtained as follows:

$$n_\phi = \frac{1}{\kappa_\phi(\tau^2 + \sigma^2)} [-(\frac{\tau}{\sqrt{\tau^2 + \sigma^2}})' n + \sqrt{\tau^2 + \sigma^2} b + (\frac{\sigma}{\sqrt{\tau^2 + \sigma^2}})' e]. \tag{17}$$

Differentiating (17) gives

$$n'_\phi = \frac{dn_\phi}{ds_\phi} \frac{ds_\phi}{ds} = \tau_\phi b_\phi \frac{ds_\phi}{ds},$$

or

$$\begin{aligned}\tau_\phi b_\phi = \frac{1}{\sqrt{\tau^2+\sigma^2}}(\{-&\frac{1}{\kappa_\phi(\tau^2+\sigma^2)}[(\frac{1}{\sqrt{\tau^2+\sigma^2}})''+\tau\sqrt{\tau^2+\sigma^2}]\\ &-(\frac{1}{\kappa_\phi(\tau^2+\sigma^2)})'(\frac{\tau}{\sqrt{\tau^2+\sigma^2}})'\}n+\frac{1}{\kappa_\phi(\tau^2+\sigma^2)}\{(\sqrt{\tau^2+\sigma^2})'\\ &-(\frac{\tau}{\sqrt{\tau^2+\sigma^2}})'\tau+(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}})'\sigma-(\frac{1}{\kappa_\phi(\tau^2+\sigma^2)})'\sqrt{\tau^2+\sigma^2}\}b\\ &+(\{\frac{1}{\kappa_\phi(\tau^2+\sigma^2)}[(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}})''+\sigma\sqrt{\tau^2+\sigma^2}]\\ &+(\frac{1}{\kappa_\phi(\tau^2+\sigma^2)})'(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}})'\}e).\end{aligned} \tag{18}$$

For (18), using the expression $\tau_\phi b_\phi = \eta(s)$, then we obtain

$$\tau_\phi = \|\eta(s)\|,$$

and also

$$b_\phi = \frac{\eta(s)}{\|\eta(s)\|}, \tag{19}$$

where η_i are the components of the $\eta(s)$ $(i = 1, 2, 3, 4)$,

$$\eta_\varphi(s) = \eta_1(s) + \eta_2(s) + \eta_3(s) + \eta_4(s).$$

By the cross product $\mu t_\phi \wedge n_\phi \wedge b_\phi$ we find

$$\begin{aligned}e_\phi = \frac{1}{\kappa_\phi^2(\tau^2+\sigma^2)^2}\{&\tau[(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}})'\eta_3-\eta_4\sqrt{\tau^2+\sigma^2}]\\ &-\sigma[\eta_3(\frac{\tau}{\sqrt{\tau^2+\sigma^2}})'+\eta_2\sqrt{\tau^2+\sigma^2}]\},\end{aligned}$$

we express derivative of e_ϕ so that

$$e'_\phi = \xi(s). \tag{20}$$

Using the third curvature formula, we have

$$\sigma_\phi = -e'_\phi b_\phi, \tag{21}$$

and substituting (19) and (20) into (21), we obtain

$$\sigma_\phi = -\frac{\xi'(s)\eta(s)}{\|\eta(s)\|}.$$

3.4. **The trinormal spherical indicatrix of a curve in $\mathbb{G}_4$.**

Definition 3.4. *Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc-length s in $\mathbb{G}_4$. If we translate the trinormal vector field of Frenet frame to the center O of the unit sphere S^3_G, we obtain a spherical indicatrix $\psi = \psi(s_\psi)$ which is called the trinormal spherical indicatrix α_e of the curve $\alpha = \alpha(s)$.*

Let $\psi = \psi(s_\psi)$ be the trinormal spherical indicatrix α_e of the curve $\alpha = \alpha(s)$. Differentiating $\psi = \psi(s_\psi)$ respect to s gives

$$\psi' = \frac{d\psi}{ds_\psi}\frac{ds_\psi}{ds} = -\sigma(s)b(s).$$

The tangent vector of the trinormal spherical indicatrix is obtained as

$$t_\psi = -b, \tag{22}$$

and $\|\psi'\| = \dfrac{ds_\psi}{ds} = \sigma.$

Differentiating (22) we obtain

$$\dot{t}_\psi = \kappa_\psi n_\psi \sigma = \tau n - \sigma e,$$

then one can easily have the principal normal vector and the first curvature

$$n_\psi = \tau n - \frac{\sigma}{\sqrt{\tau^2+\sigma^2}} e, \tag{23}$$

and

$$\kappa_\psi = \sqrt{1 + (\frac{\tau}{\sigma})^2}, \tag{24}$$

respectively. Differentiating (23) we get

$$\begin{aligned} n'_\psi = \frac{dn_\psi}{ds_\psi}.\frac{ds_\psi}{ds} = [(&\frac{1}{\sqrt{\tau^2+\sigma^2}})'\tau + \frac{\tau'}{\sqrt{\tau^2+\sigma^2}}]n + \sqrt{\tau^2+\sigma^2} b \\ &-[(\frac{1}{\sqrt{\tau^2+\sigma^2}})'\sigma + \frac{\sigma'}{\sqrt{\tau^2+\sigma^2}}]e. \end{aligned} \tag{25}$$

Taking the norm of both sides of (25), we obtain the second curvature as follows:

$$\begin{aligned} \tau_\psi = &\frac{1}{\sigma}\sqrt{\tau^2(\frac{1}{\sqrt{\tau^2+\sigma^2}})'^2 + \frac{\tau'^2}{\tau^2} + \sigma^2 + \tau^2 + \sigma^2 + (\frac{1}{\sqrt{\tau^2+\sigma^2}})'^2\sigma^2 + \frac{\sigma'^2}{\tau^2+\sigma^2}} \\ =&\Gamma(s). \end{aligned} \tag{26}$$

To determine the binormal vector field, we express

$$\begin{aligned} b_\psi = \frac{1}{\Gamma(s)}\{[(&\frac{1}{\sqrt{\tau^2+\sigma^2}})'\tau + \frac{\tau'}{\sqrt{\tau^2+\sigma^2}}]n \\ &+(\sqrt{\tau^2+\sigma^2})b - [(\frac{1}{\sqrt{\tau^2+\sigma^2}})'\sigma + \frac{\sigma'}{\sqrt{\tau^2+\sigma^2}}]e\}. \end{aligned}$$

Finally, let us form the vector $\mu t_\psi \wedge n_\psi \wedge b_\psi$ as

$$e_\psi = \mu[\sigma(\frac{\tau}{\sqrt{\tau^2+\sigma^2}})' - \tau(\frac{\sigma}{\sqrt{\tau^2+\sigma^2}})']t.$$

4. CHARACTERIZATIONS OF THE SPHERICAL INDICATRICES OF CURVES IN $\mathbb{G}_4$

Theorem 4.1. *Let $\alpha = \alpha(s)$ be a unit speed curve and $\varphi(s_\varphi)$ be its tangent spherical indicatrix. If α is a $ccr-$curve or a helix (i.e. W curve), then φ is also a helix in $\mathbb{G}_4$.*

Proof. Let $\alpha = \alpha(s)$ be a unit speed $ccr-$curve in $\mathbb{G}_4$. Then, we know that

$$l_1 = \frac{\tau}{\kappa} = const. \quad \text{and} \quad l_2 = \frac{\sigma}{\kappa} = const. \tag{27}$$

From (5), (6), and (7), we have

$$\kappa_\varphi = l_1, \quad \tau_\varphi = \sqrt{l_1^2(1+l_2^2)}, \quad \sigma_\varphi = \frac{1}{l_1^2(1+l_2^2)}, \tag{28}$$

respectively. Therefore $\varphi(s_\varphi)$ is also helix. The condition for α to be helix can be immediately seen by (27). $\square$

By this theorem , it is presented that a characterization of the tangent spherical indicatrix is associated to the Frenet-Serret curvature ratios to be constant (or helices). It is also observed that the mentioned indicatrix can be a helix, so one can ask whether this tangent spherical indicatrix is a general helix or not? Therefore it is characterized by the following statements.

Theorem 4.2. *Let $\alpha = \alpha(s)$ be a unit speed $ccr-$curve and $\varphi = \varphi(s_\varphi)$ be its tangent spherical indicatrix. If φ is a general helix, then there exists a relation among the Frenet-Serret curvatures of α as*

$$\varphi = c_1 t_\varphi + c_2 n_\varphi + c_3 b_\varphi - c_3 \textstyle\int_0^s \sigma ds c_\varphi.$$

Proof. Let $\alpha = \alpha(s)$ be a unit speed curve and $\varphi = \varphi(s_\varphi)$ be its tangent spherical indicatrix in $\mathbb{G}_4$. If $\varphi = \varphi(s_\varphi)$ is a general helix, then for a constant vector U, we can express

$$\langle t_\varphi, U\rangle_G = const., \tag{29}$$

differentiating (29), we obtain $\kappa \langle n, U\rangle = 0$. For $\kappa \neq 0$, we have

$$\langle n, U\rangle = const. \tag{30}$$

One can write linear combination of $\{t, n, b, e\}$ of the constant vector U as

$$U = \varepsilon_1 t + \varepsilon_2 n + \varepsilon_3 b + \varepsilon_4 e. \tag{31}$$

Differentiating (31) respect to s, we have the following system of ordinary differential equations

$$\begin{cases} \varepsilon_1' = 0, \\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_3\tau = 0, \\ \varepsilon_3' + \varepsilon_2\tau - \varepsilon_4\sigma = 0, \\ \varepsilon_3\sigma + \varepsilon_4' = 0. \end{cases} \tag{32}$$

We know that $\varepsilon_2 = c_2 \neq 0$ is a constant. Moreover, from (32_1), we get $\varepsilon_1 = c_1$ constant. Using this system, we have two differential equations due to ε_3 as

$$\begin{cases} \varepsilon_3'(\frac{\tau}{\kappa}) + \varepsilon_3(\frac{\tau}{\kappa})' = 0, \\ (\frac{\varepsilon_3'}{\sigma})' + c(\frac{\tau}{\sigma})' + \varepsilon_3\sigma = 0. \end{cases} \tag{33}$$

Suppose $\alpha = \alpha(s)$is a unit speed $ccr-$curve and $\varphi = \varphi(s_\varphi)$ is its tangent spherical indicatrix, then the relations (33) are satisfied. We know that $ccr-$curve has constant curvature ratios such that $\frac{\kappa}{\tau} = l_1 = const.$, and $\frac{\tau}{\sigma} = const.$ Therefore the differential equations (33) turn into the following form:

$$\begin{cases} \frac{1}{l_1}\varepsilon_3' = 0, \\ (\frac{\varepsilon_3'}{\sigma})' + \varepsilon_3\sigma = 0, \end{cases} \tag{34}$$

from (34_1), we get $\varepsilon_3 = c_3 = const.$ □

Corollary 4.3. *The fixed direction (constant vector U) can be composed by the components*

$$\begin{cases} \varepsilon_1 = c_1 = const., \\ \varepsilon_2 = c_2 = const., \\ \varepsilon_3 = c_3 = const., \\ \varepsilon_4 = -c_3 \int_0^s \sigma ds. \end{cases}$$

Theorem 4.4. *Given that $X = X(s), Y = Y(s)$ are curves in $\mathbb{G}_4$, and let the curvature $\sqrt{(\tau_X)^2 + (\sigma_X)^2}$ of $X = X(s)$ be constant. If trinormal of $X = X(s)$ is tangent indicatrix of $Y = Y(s)$, then σ_X is constant.*

Proof. Lets calculate Frenet apparatus of $Y = Y(s)$. Consider the Frenet apparatus of curve $X = X(s)$, and $Y = Y(s)$, respectively, $\{t_X, n_X, b_X, e_X, \kappa_X, \tau_X, \sigma_X\}$ and $\{t_Y, n_Y, b_Y, e_Y, \kappa_Y, \tau_Y, \sigma_Y\}$. Suppose that s_Y is the parameterized arc-length of $Y = Y(s)$. Then, we can write

$$Y = \int_0^s e_X(s)ds. \tag{35}$$

Differentiating both sides of (35) respect to s, we get

$$\frac{dY}{ds} = \frac{dY}{ds_Y}.\frac{ds_Y}{ds} = e_X. \tag{36}$$

Since

$$\frac{dY}{ds_Y} = t_Y, \tag{37}$$

from (36), we find

$$t_Y.\frac{ds_Y}{ds} = e_X. \tag{38}$$

Using (37) and (38) in (36), we obtain

$$t_Y = e_X, \tag{39}$$

and

$$\frac{ds_Y}{ds} = 1.$$

Taking the derivative of (39) respect to s, we have

$$\kappa_Y n_Y = -\sigma_X b_X. \tag{40}$$

From (40), we get

$$n_Y = -b_X, \tag{41}$$

and $\kappa_Y = \sigma_X$. Differentiating (41) gives

$$\tau_Y b_Y = \tau_X n_X - \sigma_X e_X,$$

and using the expression $\tau_Y = \sqrt{\tau_X^2 + \sigma_X^2}$, we obtain

$$b_Y = \frac{\tau_X n_X - \sigma_X e_X}{\sqrt{\tau_X^2 + \sigma_X^2}}. \tag{42}$$

If exterior product $t_Y \wedge n_Y \wedge b_Y$ is calculated, we get

$$e_Y = \frac{t_X}{\sqrt{\tau_X^2 + \sigma_X^2}}, \tag{43}$$

and differentiating (43) respect to s , we find

$$-\sigma_Y b_Y = \frac{\kappa_X n_X}{\sqrt{\tau_X^2 + \sigma_X^2}} + \left(\frac{1}{\sqrt{\tau_X^2 + \sigma_X^2}}\right)' t_X.$$

Since $r = \sqrt{\tau_X^2 + \sigma_X^2}$ is constant, then

$$\sigma_Y = \sigma \frac{\kappa_X}{r}, \quad \text{and} \quad b_Y = -n_X.$$

Therefore, we arrive at

$$\sigma_Y = \frac{\kappa_X}{\sqrt{\tau_X^2 + \sigma_X^2}}, \quad \text{and} \quad \sigma_X = \frac{1}{\sqrt{\left(\frac{\tau}{\kappa_X}\right)^2 + \left(\frac{\sigma_X}{\kappa_X}\right)^2}}.$$

So, the third curvature of $X = X(s)$ is as follows:

$$\sigma_X = \frac{1}{\sqrt{l_1^2 + l_1.l_2}}.$$

Since l_1 and l_2 are constant, then $\sigma_X = const.$ □

Theorem 4.5. *Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a unit speed curve and $\beta = \beta(s_\beta)$ be its principal normal spherical indicatrix. If α is a $ccr-$curve helix (i.e. W-curve), then β is also a plane curve.*

Proof. Let $\alpha = \alpha(s)$ be unit speed $ccr-$curve. We know that

$$\frac{\kappa}{\tau} = \frac{1}{l_1} = const. \text{ and } \frac{\tau}{\sigma} = \frac{1}{l_2} = const.$$

From (10) and (12), we get the first and second curvatures of $\beta = \beta(s_\beta)$ as follows

$$\kappa_\beta = \sqrt{1 + l_1^2} = const., \text{ and } \tau_\beta = 0 = const.$$

Substituting κ_β and τ_β to the first part of formula a spherical curve has to satisfy, it is seen that $\beta(s_\beta)$ is a spherical curve. By the way, we get

$$\sigma_\beta = \left(\left(\frac{1}{\sqrt{1 + l_1^2}}\right)'\right)^2 \kappa = 0 = const.$$

Therefore $\beta(s_\beta)$ is also a flat curve. □

Theorem 4.6. *Let ψ and δ be unit speed curves such that ψ is trinormal spherical indicatrix of δ, and δ be an evolute of ψ. The Frenet apparatus of ψ $\{t_\psi, n_\psi, b_\psi, e_\psi, \kappa_\psi, \tau_\psi, \sigma_\psi\}$ can be formed according to the Frenet apparatus of δ, $\{t_\delta, n_\delta, b_\delta, e_\delta, \kappa_\delta, \tau_\delta, \sigma_\delta\}$.*

Proof. From definition of involute-evolute curve pair we write

$$\delta_\psi = \psi + \lambda t_\psi, \tag{44}$$

and differentiating (44) respect to s we obtain

$$\frac{d\delta}{ds_\delta}.\frac{ds_\delta}{ds} = t + \frac{d\lambda}{ds}t_\psi + \lambda t_\psi^{'}.$$

Recalling definition of involute and evolute we can say

$$t_\psi \perp t_\delta,$$

hence we have

$$1 + \frac{d\lambda}{ds} = 0,$$

and we find

$$\lambda = c - s,$$

where c is constant. Rewriting (44), we have

$$\delta = \psi + (c - s)t_\psi, \tag{45}$$

and differentiating (45) respect to s, we have

$$t_\delta \frac{ds_\delta}{ds} = -(c - s)b, \tag{46}$$

and

$$t_\delta = -b.$$

Taking the norm of both sides of (46) we have

$$\frac{ds_\delta}{ds} = (c - s),$$

and

$$\dot{t}_\delta = \kappa_\delta n_\delta = -\tau n + \sigma e, \tag{47}$$

then one can easily have the principal vector and the first curvature

$$n_\delta = \frac{-\tau n + \sigma e}{\sqrt{\tau^2 + \sigma^2}}, \quad \text{and} \quad \kappa_\delta = \left\|\dot{t}_\delta\right\| = \sqrt{\tau^2 + \sigma^2}. \tag{48}$$

Differentiating (48) gives

$$\begin{aligned} n'_\delta = \frac{1}{c - s}\{(&\frac{-\tau}{(\sqrt{\tau^2 + \sigma^2})'} - \tau')n + (\frac{-\tau^2 - \sigma^2}{\sqrt{\tau^2 + \sigma^2}})b \\ &+(\frac{\sigma}{(\sqrt{\tau^2 + \sigma^2})'} + \sigma')e\}. \end{aligned}$$

Let us denote the derivative of the principal normal vector field as

$$n_\delta^{'} = P(s), \tag{49}$$

where $P(s) = (p_1(s), p_2(s), p_3(s), p_4(s))$. Taking the norm of both sides of (49), we have the second curvature and binormal vector as follows:

$$\tau_\delta b_\delta = P'(s),$$

and

$$\tau_\delta = P'(s) \text{ and } b_\delta = \frac{P'(s)}{\tau_\delta}.$$

Calculating $t_\delta \wedge n_\delta \wedge b_\delta$ gives

$$e_\delta = \frac{\tau e - \sigma n}{\tau_\delta \sqrt{\tau^2 + \sigma^2}} p_1^{'}(s),$$

and using the third curvature formula, we obtain

$$\sigma_\delta = -e_\delta^{'} b_\delta,$$

and

$$\sigma_\delta = -(\frac{\tau e - \sigma n}{\tau_\delta . \sqrt{\tau^2 + \sigma^2}} p_1'(s))' \frac{p'(s)}{\tau_\delta}.$$

□

Theorem 4.7. *Let $\alpha = \alpha(s)$ be regular curve in $\mathbb{G}_4$. The curves β and ψ which are spherical indicatrices of α are Bertrand mates. Also the curve β is the principal normal spherical indicatrix, and ψ the trinormal spherical indicatrix of α.*

Proof. Denote the principal normal vectors of ϕ and ψ, n_ϕ and n_ψ, respectively, and the principal normal vectors are given

$$n_\beta = \frac{-\tau n + \sigma e}{\sqrt{\tau^2 + \sigma^2}}, \text{ and } n_\phi = \frac{\tau n - \sigma e}{\sqrt{\tau^2 + \sigma^2}}.$$

It can be seen that $n_\beta = -n_\phi$. So the principal normal vectors of spherical indicatrices β and ψ are linear dependent. Hence these curves are Bertrand mates. □

Theorem 4.8. *Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{G}_4$. Let us denote the principal normal, binormal, and trinormal spherical indicatrices of α by β, ϕ and ψ, respectively. Both of β and ψ which are spherical indicatrices of α are spherical involutes for the binormal spherical indicatrices ϕ of α.*

Proof. Let us denote the tangent vectors of the spherical indicatrices β, ϕ, and ψ as t_β, t_ϕ, and t_ψ, respectively. By (8), (14), and (22), these tangent vectors are as follows

$$t_\beta = b, t_\phi = \frac{-\tau n + \sigma e}{\sqrt{\tau^2 + \sigma^2}}, \text{ and } t_\psi = -b,$$

If the inner products of these vectors are calculated, then we get

$$\langle t_\beta, t_\phi \rangle = 0, \text{ and } \langle t_\psi, t_\phi \rangle = 0.$$

The tangent vectors of the spherical indicatrices β, ψ are orthogonal to tangent vector of binormal spherical indicatrix ϕ. So the proof is completed. □

5. EXAMPLE

Let's consider the following parameterized curve

$$\alpha(s) = (s, \cos s, \sqrt{2}\sin s, \cos s) \tag{50}$$

in $\mathbb{G}_4$. Differentiating (50), we have $\alpha'(s) = t(s) = (1, -\sin s, \sqrt{2}\cos s, -\sin s)$. The principal normal vector $n(s)$ becomes $n(s) = \frac{1}{\sqrt{2}}(0, -\cos s, -\sqrt{2}\sin s, -\cos s)$. The second curvature is obtained as $\tau(s) = 1$. The binormal vector is found as $b(s) = \frac{1}{\sqrt{2}}(0, \sin s, -\sqrt{2}\cos s, \sin s)$. Using $e(s) = \mu t(s) \wedge n(s) \wedge b(s)$, the trinormal vector is $e(s) = \frac{\mu}{\sqrt{2}}(0, -1, 0, 1)$, hence the third curvature is calculated as $\sigma(s) = \langle b'(s), e(s)\rangle_G = 0$.

Let us calculate the spherical indicatrices of the curve α, first let us express the Frenet elements of the tangent indicatrix $\varphi = \varphi(s_\varphi)$ in terms of the Frenet elements of the curve α. Using the equations (4), (5), (6), the tangent vector of the tangent indicatrix is calculated as $t_\varphi = n(s) = \frac{1}{\sqrt{2}}(0, -\cos s, -\sqrt{2}\sin s, -\cos s)$, and the first and second curvatures of the tangent indicatrix are found as $\kappa_\varphi = \frac{1}{\sqrt{2}}, \tau_\varphi = \frac{1}{\sqrt{2}}$,respectively. The binormal and trinormal vectors of the tangent indicatrix are obtained as $b_\varphi = \frac{1}{\sqrt{2}}(0, \cos s, \sqrt{2}\sin s, \cos s)$, and $e_\varphi = \frac{\mu}{\sqrt{2}}(1, -\sin s, \sqrt{2}\cos s, -\sin s)$, respectively.

Let us express the Frenet elements of the principal normal indicatrix $\beta = \beta(s_\beta)$ in terms of the Frenet elements of the curve α. Using the equations (8), (10), (11), and (13), the tangent vector of the principal normal indicatrix is $t_\beta = n(s) = \frac{1}{\sqrt{2}}(0, \sin s, -\sqrt{2}\cos s, \sin s)$, and the first and second curvatures of the principal normal indicatrix are obtained as $\kappa_\beta = 1$, $\tau_\beta = 0$ respectively. The binormal and trinormal vectors of the principal normal indicatrix are found as $b_\beta = \frac{1}{\sqrt{2}}(0, -\sin s, \sqrt{2}\cos s, -\sin s)$, and $e_\beta = (0, 0, 0, 0)$, respectively. Hence the third curvature is calculated as $\sigma_\beta(s) = 0$.

Let us express the Frenet elements of the binormal indicatrix $\phi = \phi(s_\phi)$ in terms of the Frenet elements of the curve α. By the equations (9), (14), (16), (17) and (20), the tangent vector of the binormal indicatrix is $t_\phi = \frac{1}{\sqrt{2}}(0, \cos s, \sqrt{2}\sin s, \cos s)$, and the first curvature of the binormal indicatrix is obtained as $\kappa_\phi = 1$. The principal normal vector of the binormal indicatrix is as $n_\phi = b = \frac{1}{\sqrt{2}}(0,\text{sin}s, -\sqrt{2}\cos s, \sin s)$, and also the second curvature is $\tau_\phi = 1$. The binormal and trinormal vectors of the binormal indicatrix are found as $b_\phi = -n = \frac{1}{\sqrt{2}}(0, \cos s, \sqrt{2}\sin s, \cos s)$, and

$e_\phi = \dfrac{1}{\sqrt{2}}(0,0,0,-\cos s)$, respectively. Hence the third curvature is calculated as $\sigma_\phi(s) = \dfrac{1}{4}\sin 2s$.

Finally, Let us express the Frenet elements of the trinormal indicatrix $\psi = \psi(s_\psi)$ in terms of the Frenet elements of the curve α. By equations (22), (23), (24), and (26), the tangent and principal normal vectors of the trinormal indicatrix are calculated $t_\psi = -b = \dfrac{1}{\sqrt{2}}(0,-\sin s,\sqrt{2}\cos s,-\sin s)$, and $n_\psi = n = \dfrac{1}{\sqrt{2}}(0,-\cos s,-\sqrt{2}\sin s,-\cos s)$, respectively. Also the first curvature of the trinormal indicatrix is obtained as $\kappa_\psi = 1$, and the second curvature τ_ψ is undefined since $\dfrac{1}{\sigma}$ $(\sigma = 0)$. The binormal and trinormal vectors of the trinormal indicatrix is found as $b_\psi = (0,0,0,0)$, and $e_\psi = (0,0,0,0)$, respectively. Hence the third curvature is $\sigma_\psi(s) = 0$.

6. CONCLUDING REMARKS

In this study, the tangent, principal normal, binormal, and trinormal spherical indicatrices of a regular curve were obtained in 4D Galilean space. Then their Frenet elements of these four special curves were found in terms of the Frenet elements of the original curve at the same space. Moreover, using Frenet elements of spherical indicatrix curves, some theorems were given characterizing these special curves as $ccr-$curves, general helix, involute-evolute curve pairs, and Bertrand mates. As an open problem, all the results in this study can be studied in pseudo-Galilean spaces.

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