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BIPOLAR NEUTROSOPHIC GRAPH STRUCTURES

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Abstract. In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

Key words and Phrases: Graph structure, bipolar single-valued neutrosophic graph structure, operations.

Abstrak. Pada penelitian ini, kami memperkenalkan konsep struktur graf neutrosofik bipolar bernilai tunggal. Kami mengkaji ide-ide tertentu dari struktur graf neutrosofik bipolar bernilai tunggal berserta contoh-contohnya. Kami menyajikan beberapa metode konstruksi struktur graf neutrosofik bipolar bernilai tunggal. Kami juga memeriksa beberapa sifat-sifat mereka.

Kata kunci: Striktur graf, struktur graf neutrosofik bipolar bernilai tunggal, operasi

1. Introduction

Fuzzy graph theory has a number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1973, Kauffmann [13] illustrated the notion of fuzzy graphs based on Zadeh's fuzzy relations [24]. Rosenfeld [16] discussed several basic graph-theoretic concepts, including bridges,

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cut-nodes, connectedness, trees and cycles. Later on, Bhattacharya [8] gave some remarks on fuzzy graphs. In 1994, Mordeson and Chang-Shyh [14] defined some operations on fuzzy graphs. The complement of fuzzy graph was defined in [14]. Further, this concept was discussed by Sunitha and Vijayakumar [20]. Akram described bipolar fuzzy graphs in 2011 [1]. Akram and Shahzadi [5] described the concept of neutrosophic soft graphs with applications. Dinesh and Ramakrishnan [12] introduced the concept of the fuzzy graph structure and investigated some related properties. Akram and Akmal [4] proposed the notion of bipolar fuzzy graph structures. On the other hand, Dhavaseelan et al. [10] defined strong neutrosophic graphs. Akram and Sarwar [3] portrayed bipolar neutrosophic graphs with applications. Akram and Shahzadi [5] introduced the notion of neutrosophic soft graphs with applications. Akram [2] introduced the notion of single-valued neutrosophic planar graphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [6]. Single-valued neutrosophic minimum spanning tree and its clustering method were studied by Ye [22]. In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

2. BIPOLAR SINGLE-VALUED NEUTROSOPHIC GRAPH STRUCTURES

Smarandache [19] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set has three constituents: truth-membership, indeterminacy-membership and falsity-membership, in which each membership value is a real standard or non-standard subset of the unit interval $]0^-, 1^+[$. In real-life problems, neutrosophic sets can be applied more appropriately by using the single-valued neutrosophic sets defined by Smarandache [19] and Wang et al [21].

Definition 2.1. [19] A *neutrosophic set* N on a non-empty set V is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where, $T_N, I_N, F_N : V \to]0^-, 1^+[$ and there is no restriction on the sum of $T_N(v)$, $I_N(v)$ and $F_N(v)$ for all $v \in V$.

Definition 2.2. [21]A *single-valued neutrosophic set* N on a non-empty set V is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where, $T_N, I_N, F_N : V \to [0, 1]$ and sum of $T_N(v)$, $I_N(v)$ and $F_N(v)$ is confined between 0 and 3 for all $v \in V$.

Deli et al. [9] defined bipolar neutrosophic sets a generalization of bipolar fuzzy sets. They also studied some operations and applications in decision making problems.

Definition 2.3. [9]A *bipolar single-valued neutrosophic set* on a non-empty set V is an object of the form

$$B = \{(v, T_B^P(v), I_B^P(v), F_B^P(v), T_B^N(v), I_B^N(v), F_B^N(v)) : v \in V\}$$

where, $T_B^P, I_B^P, F_B^P : V \to [0,1]$ and $T_B^N, I_B^N, F_B^N : V \to [-1,0]$. The positive values $T_B^P(v), I_B^P(v), F_B^P(v)$ denote the truth, indeterminacy and falsity membership values of an element $v \in V$, whereas negative values $T_B^N(v), I_B^N(v), F_B^N(v)$ indicates the implicit counter property of truth, indeterminacy and falsity membership values of an element $v \in V$.

Definition 2.4. A *bipolar single-valued neutrosophic graph* on a non-empty set V is a pair $G = (B, R)$, where B is a bipolar single-valued neutrosophic set on V and R is a bipolar single-valued neutrosophic relation in V such that

$$T_R^P(bd) \le T_B^P(b) \wedge T_B^P(d), \quad I_R^P(bd) \le I_B^P(b) \wedge I_B^P(d), \quad F_R^P(bd) \le F_B^P(b) \vee F_B^P(d),$$
$$T_R^N(bd) \ge T_B^N(b) \vee T_B^N(d), \quad I_R^N(bd) \ge I_B^N(b) \vee I_B^N(d), \quad F_R^N(bd) \ge F_B^N(b) \wedge F_B^N(d),$$

for all $b, d \in V$.

We now define bipolar single-valued neutrosophic graph structure.

Definition 2.5. $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is called *bipolar single-valued neutrosophic graph structure(BSVNGS)* of graph structure $\check{G}_s = (V, V_1, V_2, \ldots, V_m)$ if $B =< b, T^P(b), I^P(b), F^P(b), T^N(b), I^N(b), F^N(b) >$ and
$B_k =< (b,d), T_k^P(b,d), I_k^P(b,d), F_k^P(b,d), T_k^N(b,d), I_k^N(b,d), F_k^N(b,d) >$ are bipolar single-valued neutrosophic(BSVN) sets on V and V_k, respectively, such that

$$T_k^P(b,d) \le \min\{T^P(b), T^P(d)\},\ I_k^P(b,d) \le \min\{I^P(b), I^P(d)\},$$
$$F_k^P(b,d) \le \max\{F^P(b), F^P(d)\},\ T_k^N(b,d) \ge \max\{T^N(b), T^N(d)\},$$
$$I_k^N(b,d) \ge \max\{I^N(b), I^N(d)\},\ F_k^N(b,d) \ge \min\{F^N(b), F^N(d)\},$$

for all $b, d \in V$. Note that $0 \le T_k^P(b,d) + I_k^P(b,d) + F_k^P(b,d) \le 3$,
$-3 \le T_k^N(b,d) + I_k^N(b,d) + F_k^N(b,d) \le 0$ for all $(b,d) \in V_k$.

Example 2.6. Consider graph structure(GSR) $\check{G}_s = (V, V_1, V_2)$ such that $V = \{b_1, b_2, b_3, b_4\}$, $V_1 = \{b_1b_3, b_1b_2, b_3b_4\}$, $V_2 = \{b_1b_4, b_2b_3\}$. By defining bipolar single-valued neutrosophic sets B, B_1 and B_2 on V, V_1 and V_2, respectively, we can draw a bipolar SVNGS as depicted in Fig. 1.

Definition 2.7. Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS of GS $\check{G}_s$. If
$\check{H}_{bn} = (B', B'_1, B'_2, \ldots, B'_m)$ is a BSVNGS of $\check{G}_s$ such that

$$T'^P(b) \le T^P(k), I'^P(b) \le I^P(b), F'^P(b) \ge F^P(b),$$
$$T'^N(b) \ge T^P(k), I'^N(b) \ge I^P(b), F'^N(b) \le F^N(b)\ ,$$

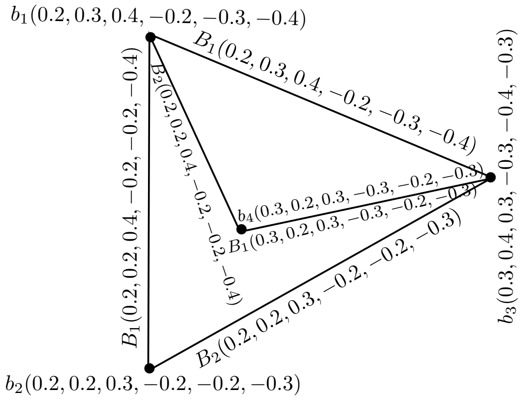


FIGURE 1. A bipolar single-valued neutrosophic graph structure

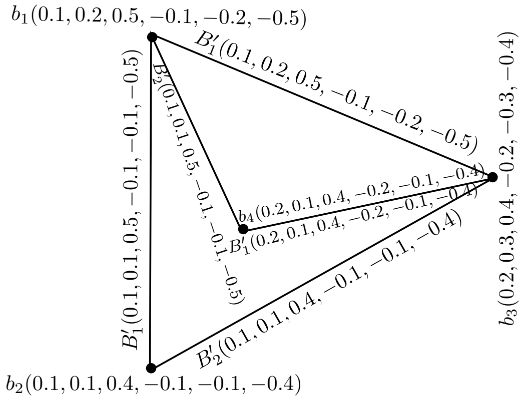


FIGURE 2. A BSVN subgraph structure

$$T_k'^P(b,d) \leq T_k^P(b,d),\ I_k'^P(b,d) \leq I_k^P(b,d),\ F_k'^P(b,d) \geq F_k^P(b,d),$$
$$T_k'^N(b,d) \geq T_k^N(b,d),\ I_k'^N(b,d) \geq I_k^N(b,d),\ F_k'^N(b,d) \leq F_k^N(b,d),$$
$$\forall\ b \in V \text{ and } (b,d) \in V_k,\ k = 1, 2, \ldots, m.$$

Then $\check{H}_{bn}$ is named as a *bipolar single-valued neutrosophic(BSVN) subgraph structure* of BSVNGS $\check{G}_{bn}$.

Example 2.8. Consider a BSVNGS $\check{H}_{bn} = (B', B_1', B_2')$ of GS $\check{G}_s = (V, V_1, V_2)$ as depicted in Fig. 2. Routine calculations indicate that $\check{H}_{bn}$ is BSVN subgraph-structure of BSVNGS $\check{G}_{bn}$.

Definition 2.9. A BSVNGS $\check{H}_{bn} = (B', B_1', B_2', \ldots, B_m')$ is called a *BSVN induced subgraph-structure* of BSVNGS $\check{G}_{bn}$ by $Q \subseteq V$ if

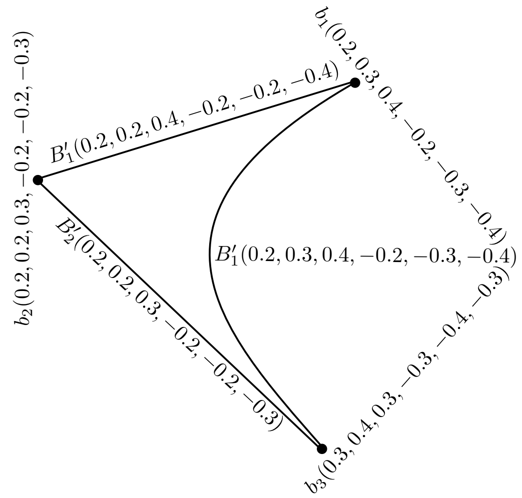


FIGURE 3. A BSVN induced subgraph-structure

$$\begin{gathered}
T'^P(b) = T^P(b), I'^P(b) = I^P(b), F'^P(b) = F^P(b),\\
T'^N(b) = T^N(b), I'^N(b) = I^N(b), F'^N(b) = F^N(b),\\
T_k'^P(b,d) = T_k^P(b,d),\ I_k'^P(b,d) = I_k^P(b,d),\ F_k'^P(b,d) = F_k^P(b,d),\\
T_k'^N(b,d) = T_k^N(b,d),\ I_k'^N(b,d) = I_k^N(b,d),\ F_k'^N(b,d) = F_k^N(b,d),\\
\forall\ b,d \in Q,\ k = 1,2,\ldots,m.
\end{gathered}$$

Example 2.10. A BSVNGS depicted in Fig.3 is a BSVN induced subgraph-structure of BSVNGS represented in Fig. 1.

Definition 2.11. A BSVNGS $\check{H}_{bn} = (B', B_1', B_2', \ldots, B_m')$ is called *BSVN spanning subgraph-structure* of BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ if $B' = B$ and

$$\begin{gathered}
T_k'^P(b,d) \le T_k^P(b,d),\ I_k'^P(b,d) \le I_k^P(b,d),\ F_k'^P(b,d) \ge F_k^P(b,d),\\
T_k'^N(b,d) \ge T_k^N(b,d),\ I_k'^N(b,d) \ge I_k^N(b,d),\ F_k'^N(b,d) \le F_k^N(b,d),\ k = 1,2,\ldots,m.
\end{gathered}$$

Example 2.12. A BSVNGS represented in Fig. 4 is a BSVN spanning subgraph-structure of BSVNGS represented in Fig. 1.

Definition 2.13. Let $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ be a BSVNGS. Then $bd \in B_k$ is called a *BSVN B_k-edge* or shortly *B_k-edge*, if
$T_k^P(b,d) > 0$ or $I_k^P(b,d) > 0$ or $F_k^P(b,d) > 0$ or $T_k^N(b,d) < 0$ or $I_k^N(b,d) < 0$ or $F_k^N(b,d) < 0$ or all these conditions are satisfied.
Consequently support of B_k is defined as:

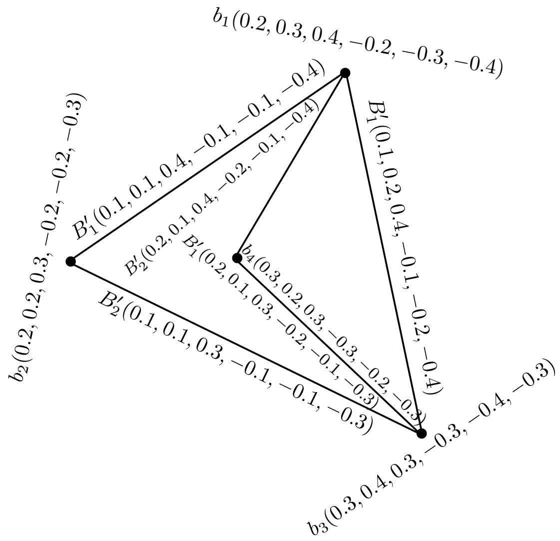


FIGURE 4. A BSVN spanning subgraph-structure

$supp(B_k) = \{bd \in B_k : T_k^P(b,d) > 0\} \cup \{bd \in B_k : I_k^P(b,d) > 0\} \cup \{bd \in B_k : F_k^P(b,d) > 0\} \cup \{bd \in B_k : T_k^N(b,d) < 0\} \cup \{bd \in B_k : I_k^N(b,d) < 0\} \cup \{bd \in B_k : F_k^N(b,d) < 0\}$, $k = 1, 2, \ldots, m$.

Definition 2.14. B_k-*path* in BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is a sequence $b_1, b_2, \ldots, b_m$ of distinct nodes(vertices) (except $b_m = b_1$) in V such that $b_{k-1}b_k$ is a BSVN B_k-edge for all $k = 2, \ldots, m$.

Definition 2.15. A BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is B_k-*strong* for any $k \in \{1, 2, \ldots, m\}$ if

$$\begin{gathered} T_k^P(b,d) = \min\{T^P(b), T^P(d)\},\ I_k^P(b,d) = \min\{I^P(b), I^P(d)\}, \\ F_k^P(b,d) = \max\{F^P(b), F^P(d)\},\ T_k^N(b,d) = \max\{T^N(b), T^N(d)\}, \\ I_k^N(b,d) = \max\{I^N(b), I^N(d)\},\ F_k^N(b,d) = \min\{F^N(b), F^N(d)\}, \end{gathered}$$

$\forall\ bd \in supp(B_k)$. If $\check{G}_{bn}$ is B_k-strong for all $k \in \{1, 2, \ldots, m\}$, then $\check{G}_{bn}$ is called *strong BSVNGS*.

Example 2.16. Consider BSVNGS $\check{G}_{bn} = (B, B_1, B_2, B_3)$ as depicted in Fig. 5. Then $\check{G}_{bn}$ is strong BSVNGS, since it is B_1-, B_2- and B_3- strong.

Definition 2.17. A BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ is called complete BSVNGS if

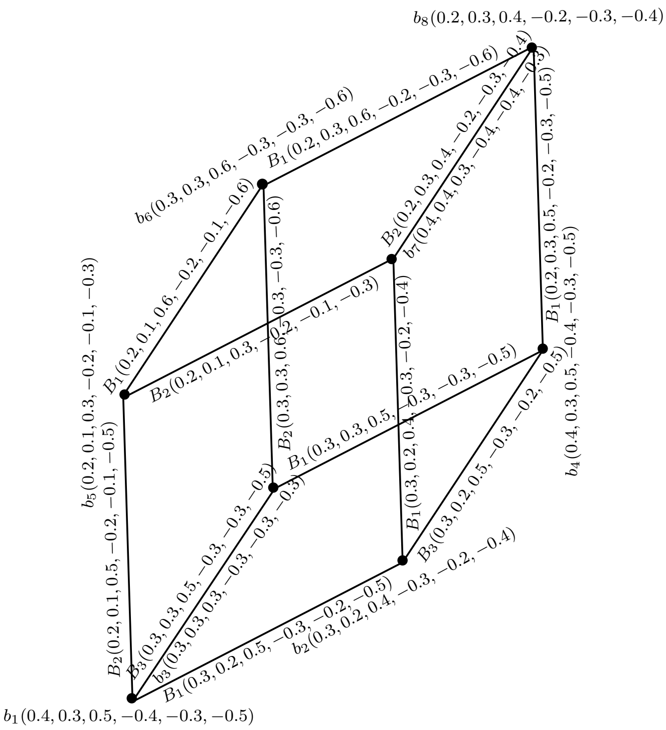


FIGURE 5. A Strong BSVNGS

(1) $\check{G}_{bn}$ is strong BSVNGS.
(2) $supp(B_k) \neq \emptyset$, for all k = 1, 2, ..., m.
(3) For all $b, d \in V$, bd is a $B_k - edge$ for some k.

Example 2.18. Let $\check{G}_{bn} = (B, B_1, B_2)$ be BSVNGS of GS $\check{G} = (V, V_1, V_2)$, such that $V = \{b_1, b_2, b_3, b_4\}$, $V_1 = \{b_1b_2, b_3b_4\}$, $V_2 = \{b_1b_3, b_2b_3, b_1b_4, b_2b_4\}$.
Through direct calculations it is easily shown that $\check{G}_{bn}$ is strong BSVNGS. Moreover, $supp(B_1) \neq \emptyset$, $supp(B_2) \neq \emptyset$ and each pair b_kb_l of nodes in V is either a B_1-edge or B_2-edge. Hence $\check{G}_{bn}$ is complete BSVNGS, that is, $B_1 - B_2-$complete BSVNGS.

Definition 2.19. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be two BSVNGSs. *Lexicographic product* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1m} \bullet B_{2m}),$$

is defined as:

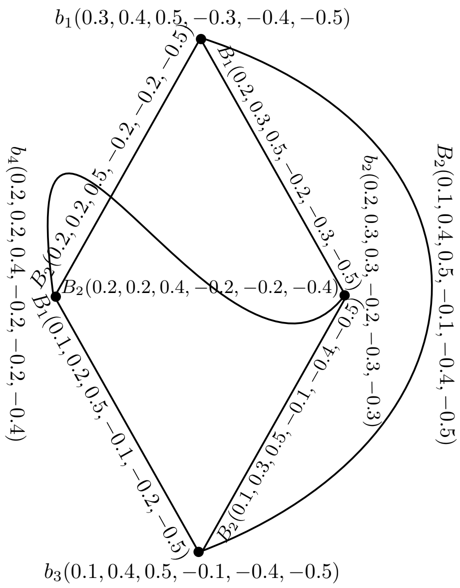


FIGURE 6. A complete BSVNGS

$$
\text{(i)} \begin{cases} T^P_{(B_1 \bullet B_2)}(bd) = (T^P_{B_1} \bullet T^P_{B_2})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_1 \bullet B_2)}(bd) = (I^P_{B_1} \bullet I^P_{B_2})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_1 \bullet B_2)}(bd) = (F^P_{B_1} \bullet F^P_{B_2})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}
$$

$$
\text{(ii)} \begin{cases} T^N_{(B_1 \bullet B_2)}(bd) = (T^N_{B_1} \bullet T^N_{B_2})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_1 \bullet B_2)}(bd) = (I^N_{B_1} \bullet I^N_{B_2})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_1 \bullet B_2)}(bd) = (F^N_{B_1} \bullet F^N_{B_2})(bd) = F^N_{B_1}(b) \wedge F^N_{B_2}(d) \end{cases}
$$

for all $(bd) \in V_1 \times V_2$,

$$
\text{(iii)} \begin{cases} T^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}} \bullet T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}} \bullet I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}} \bullet F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}
$$

$$
\text{(iv)} \begin{cases} T^N_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}} \bullet T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}} \bullet I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k} \bullet B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}} \bullet F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}
$$

for all $b \in V_1$, $(d_1d_2) \in V_{2k}$,

$$
\text{(v)} \begin{cases} T^P_{(B_{1k} \bullet B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} \bullet T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k} \bullet B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} \bullet I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k} \bullet B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} \bullet F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}
$$

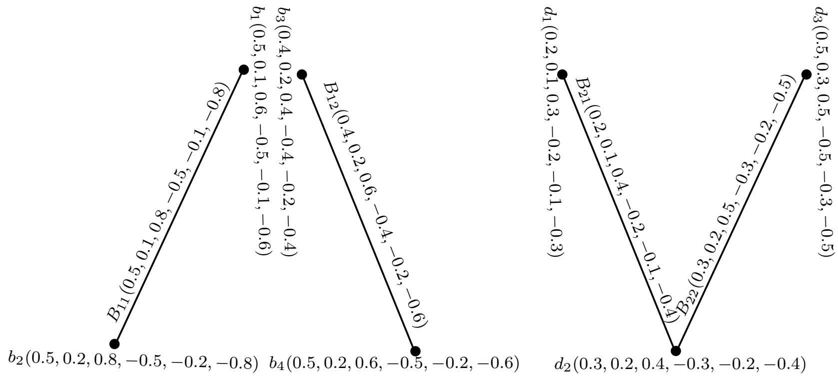
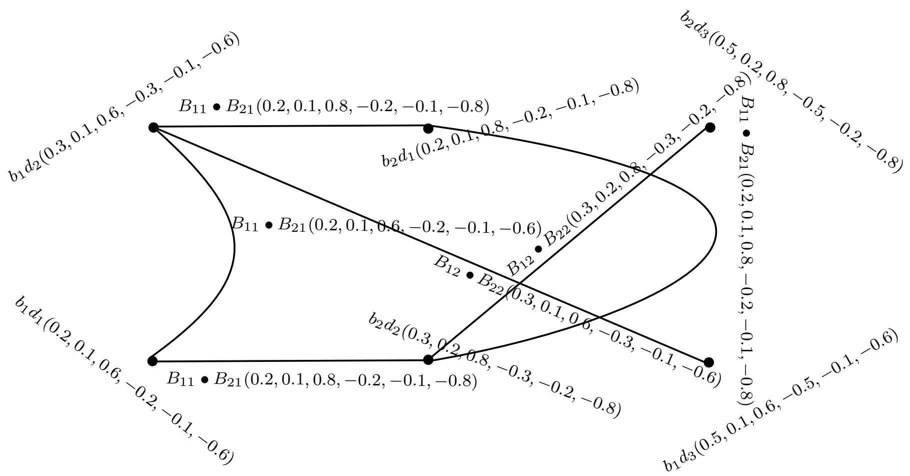


FIGURE 7. Two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$



$$\text{(vi)} \begin{cases} T^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} \bullet T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} \bullet I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k}\bullet B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} \bullet F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$$

for all $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$.

Example 2.20. Consider $\check{G}_{b1} = (B_1, B_{11}, B_{12})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22})$ are two BSVNGSs of GSs $\check{G}_{s1} = (V_1, V_{11}, V_{12})$ and $\check{G}_{s2} = (V_2, V_{21}, V_{22})$, respectively, as depicted in Fig. 7, where $V_{11} = \{b_1b_2\}$, $V_{12} = \{b_3b_4\}$, $V_{21} = \{d_1d_2\}$, $V_{22} = \{d_2d_3\}$.

Lexicographic product of BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 7 is defined as:
$\check{G}_{b1} \bullet \check{G}_{b2} = \{B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}\}$ and is depicted in Fig. 8.

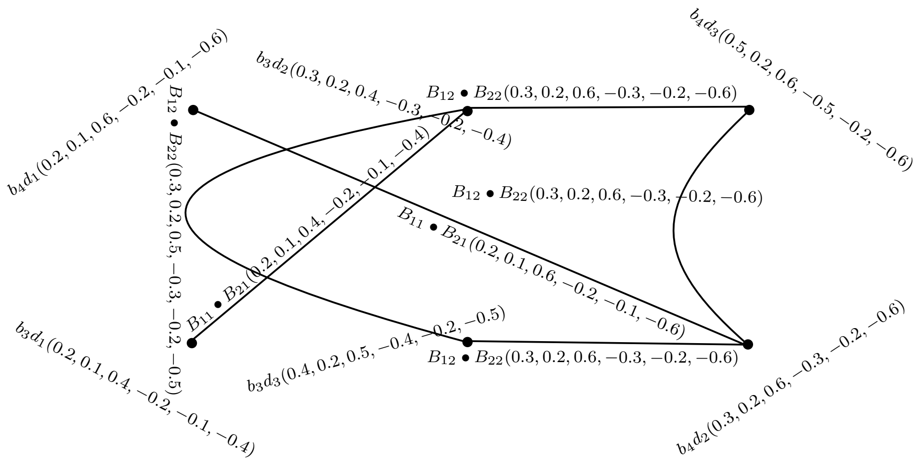


FIGURE 8. $\check{G}_{b1} \bullet \check{G}_{b2}$

Theorem 2.21. Lexicographic product $\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \ldots, B_{1m} \bullet B_{2m})$ of two BSVNSGSs of GSs $\check{G_{s1}}$ and $\check{G_{s2}}$ is a BSVNGS of $\check{G_{s1}} \bullet \check{G_{s2}}$.

PROOF. Consider two cases:

Case 1.: For $b \in V_1$, $d_1d_2 \in V_{2k}$

$$\begin{aligned}
T^P_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2)\\
&\leq T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)]\\
&= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)]\\
&= T^P_{(B_1\bullet B_2)}(bd_1) \wedge T^P_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2)\\
&\geq T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)]\\
&= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)]\\
&= T^N_{(B_1\bullet B_2)}(bd_1) \vee T^N_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2)\\
&\leq I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\bullet B_2)}(bd_1) \wedge I^P_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2)\\
&\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\bullet B_2)}(bd_1) \vee I^N_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2)\\
&\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\bullet B_2)}(bd_1) \vee F^P_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\bullet B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\bullet B_2)}(bd_1) \wedge F^N_{(B_1\bullet B_2)}(bd_2),
\end{aligned}$$

for $bd_1, bd_2 \in V_1 \bullet V_2$.

Case 2.: For $b_1b_2 \in V_{1k}$, $d_1d_2 \in V_{2k}$

$$\begin{aligned}
T^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2)\\
&\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2] \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)]\\
&= [T^P_{B_1}(b_1) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b_2) \wedge T^P_{B_2}(d_2)]\\
&= T^P_{(B_1\bullet B_2)}(b_1d_1) \wedge T^P_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2)\\
&\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2] \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)]\\
&= [T^N_{B_1}(b_1) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b_2) \vee T^N_{B_2}(d_2)]\\
&= T^N_{(B_1\bullet B_2)}(b_1d_1) \vee T^N_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2)\\
&\leq [I^P_{B_1}(b_1) \wedge I^P_{B_1}(b_2] \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b_1) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b_2) \wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\bullet B_2)}(b_1d_1) \wedge I^P_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}$$

$$
\begin{aligned}
I^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2)\vee I^N_{B_{2k}}(d_1d_2)\\
&\geq [I^N_{B_1}(b_1)\vee I^N_{B_1}(b_2]\vee[I^N_{B_2}(d_1)\vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b_1)\vee I^N_{B_2}(d_1)]\vee[I^N_{B_1}(b_2)\vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\bullet B_2)}(b_1d_1)\vee I^N_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
F^P_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2)\vee F^P_{B_{2k}}(d_1d_2)\\
&\leq [F^P_{B_1}(b_1)\vee F^P_{B_1}(b_2]\vee[F^P_{B_2}(d_1)\vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b_1)\vee F^P_{B_2}(d_1)]\vee[F^P_{B_1}(b_2)\vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\bullet B_2)}(b_1d_1)\vee F^P_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}
$$

$$
\begin{aligned}
F^N_{(B_{1k}\bullet B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2)\wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq [F^N_{B_1}(b_1)\wedge F^N_{B_1}(b_2]\wedge[F^N_{B_2}(d_1)\wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b_1)\wedge F^N_{B_2}(d_1)]\wedge[F^N_{B_1}(b_2)\wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\bullet B_2)}(b_1d_1)\wedge F^N_{(B_1\bullet B_2)}(b_2d_2),
\end{aligned}
$$

$b_1d_1, b_2d_2 \in V_1\bullet V_2$ and $h\in\{1,2,\ldots,m\}$. This completes the proof.

Definition 2.22. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be two BSVNGSs. *Strong product* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1}\boxtimes\check{G}_{b2} = (B_1\boxtimes B_2, B_{11}\boxtimes B_{21}, B_{12}\boxtimes B_{22}, \ldots, B_{1m}\boxtimes B_{2m}),$$

is defined as:

$$
\text{(i)}\left\{\begin{array}{l}
T^P_{(B_1\boxtimes B_2)}(bd) = (T^P_{B_1}\boxtimes T^P_{B_2})(bd) = T^P_{B_1}(b)\wedge T^P_{B_2}(d)\\
I^P_{(B_1\boxtimes B_2)}(bd) = (I^P_{B_1}\boxtimes I^P_{B_2})(bd) = I^P_{B_1}(b)\wedge I^P_{B_2}(d)\\
F^P_{(B_1\boxtimes B_2)}(bd) = (F^P_{B_1}\boxtimes F^P_{B_2})(bd) = F^P_{B_1}(b)\vee F^P_{B_2}(d)
\end{array}\right.
$$

$$
\text{(ii)}\left\{\begin{array}{l}
T^N_{(B_1\boxtimes B_2)}(bd) = (T^N_{B_1}\boxtimes T^N_{B_2})(bd) = T^N_{B_1}(b)\vee T^N_{B_2}(d)\\
I^N_{(B_1\boxtimes B_2)}(bd) = (I^N_{B_1}\boxtimes I^N_{B_2})(bd) = I^N_{B_1}(b)\vee I^N_{B_2}(d)\\
F^N_{(B_1\boxtimes B_2)}(bd) = (F^N_{B_1}\boxtimes F^N_{B_2})(bd) = F^N_{B_1}(b)\wedge F^P_{B_2}(d)
\end{array}\right.
$$

for all $(bd)\in V_1\times V_2$,

$$
\text{(iii)}\left\{\begin{array}{l}
T^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (T^P_{B_{1k}}\boxtimes T^P_{B_{2k}})(bd_1)(bd_2) = T^P_{B_1}(b)\wedge T^P_{B_{2k}}(d_1d_2)\\
I^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (I^P_{B_{1k}}\boxtimes I^P_{B_{2k}})(bd_1)(bd_2) = I^P_{B_1}(b)\wedge I^P_{B_{2k}}(d_1d_2)\\
F^P_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (F^P_{B_{1k}}\boxtimes F^P_{B_{2k}})(bd_1)(bd_2) = F^P_{B_1}(b)\vee F^P_{B_{2k}}(d_1d_2)
\end{array}\right.
$$

$$
\text{(iv)}\left\{\begin{array}{l}
T^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (T^N_{B_{1k}}\boxtimes T^N_{B_{2k}})(bd_1)(bd_2) = T^N_{B_1}(b)\vee T^N_{B_{2k}}(d_1d_2)\\
I^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (I^N_{B_{1k}}\boxtimes I^N_{B_{2k}})(bd_1)(bd_2) = I^N_{B_1}(b)\vee I^N_{B_{2k}}(d_1d_2)\\
F^N_{(B_{1k}\boxtimes B_{2k})}(bd_1)(bd_2) = (F^N_{B_{1k}}\boxtimes F^N_{B_{2k}})(bd_1)(bd_2) = F^N_{B_1}(b)\wedge F^N_{B_{2k}}(d_1d_2)
\end{array}\right.
$$

for all $b\in V_1$, $(d_1d_2)\in V_{2k}$,

$$
\text{(v)}\left\{\begin{array}{l}
T^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (T^P_{B_{1k}}\boxtimes T^P_{B_{2k}})(b_1d)(b_2d) = T^P_{B_2}(d)\wedge T^P_{B_{1k}}(b_1b_2)\\
I^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (I^P_{B_{1k}}\boxtimes I^P_{B_{2k}})(b_1d)(b_2d) = I^P_{B_2}(d)\wedge I^P_{B_{2k}}(b_1b_2)\\
F^P_{(B_{1k}\boxtimes B_{2k})}(b_1d)(b_2d) = (F^P_{B_{1k}}\boxtimes F^P_{B_{2k}})(b_1d)(b_2d) = F^P_{B_2}(d)\vee F^P_{B_{2k}}(b_1b_2)
\end{array}\right.
$$

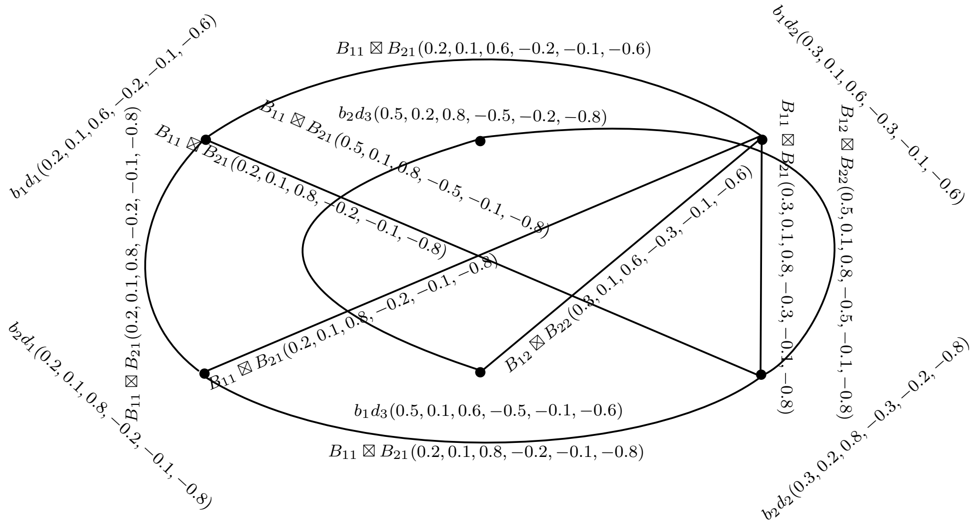


FIGURE 9. $\check{G}_{b1} \boxtimes \check{G}_{b2}$

$$\text{(vi)} \begin{cases} T^N_{(B_{1k} \boxtimes B_{2k})}(b_1d)(b_2d) = (T^N_{B_{1k}} \boxtimes T^N_{B_{2k}})(b_1d)(b_2d) = T^N_{B_2}(d) \vee T^N_{B_{1k}}(b_1b_2) \\ I^N_{(B_{1k} \boxtimes B_{2k})}(b_1d)(b_2d) = (I^N_{B_{1k}} \boxtimes I^N_{B_{2k}})(b_1d)(b_2d) = I^N_{B_2}(d) \vee I^N_{B_{2k}}(b_1b_2) \\ F^N_{(B_{1k} \boxtimes B_{2k})}(b_1d)(b_2d) = (F^N_{B_{1k}} \boxtimes F^N_{B_{2k}})(b_1d)(b_2d) = F^N_{B_2}(d) \wedge F^N_{B_{2k}}(b_1b_2) \end{cases}$$

for all $d \in V_2$, $(b_1b_2) \in V_{1k}$.

$$\text{(vii)} \begin{cases} T^P_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (T^P_{B_{1k}} \boxtimes T^P_{B_{2k}})(b_1d_1)(b_2d_2) = T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2) \\ I^P_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (I^P_{B_{1k}} \boxtimes I^P_{B_{2k}})(b_1d_1)(b_2d_2) = I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2) \\ F^P_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (F^P_{B_{1k}} \boxtimes F^P_{B_{2k}})(b_1d_1)(b_2d_2) = F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2) \end{cases}$$

$$\text{(viii)} \begin{cases} T^N_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (T^N_{B_{1k}} \boxtimes T^N_{B_{2k}})(b_1d_1)(b_2d_2) = T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2) \\ I^N_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (I^N_{B_{1k}} \boxtimes I^N_{B_{2k}})(b_1d_1)(b_2d_2) = I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2) \\ F^N_{(B_{1k} \boxtimes B_{2k})}(b_1d_1)(b_2d_2) = (F^N_{B_{1k}} \boxtimes F^N_{B_{2k}})(b_1d_1)(b_2d_2) = F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2) \end{cases}$$

for all $(b_1b_2) \in V_{1k}$, $(d_1d_2) \in V_{2k}$.

Example 2.23. Strong product of BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 7 is defined as $\check{G}_{b1} \boxtimes \check{G}_{b2} = \{B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}\}$ and is depicted in Fig. 9 and 10.

Theorem 2.24. Strong product $\check{G}_{b1} \boxtimes \check{G}_{b2} = (B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \ldots, B_{1m} \boxtimes B_{2m})$ of two BSVNGSs of GSs $\check{G_{s1}}$ and $\check{G_{s2}}$ is a BSVNGS of $\check{G_{s1}} \boxtimes \check{G_{s2}}$.

PROOF. Consider three cases:

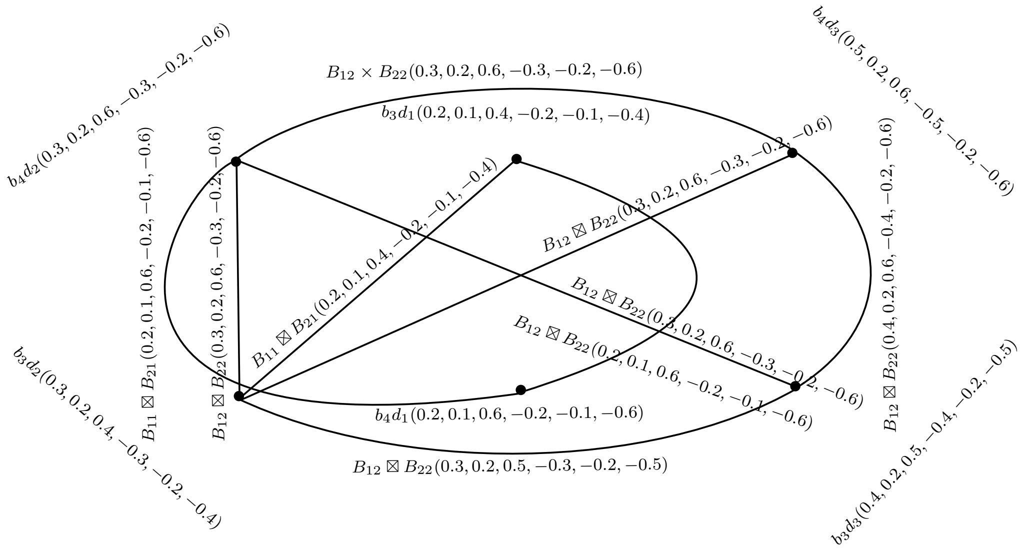


Figure 10. $\check{G}_{b1} \boxtimes \check{G}_{b2}$

Case 1.: For $b \in V_1$, $d_1d_2 \in V_{2k}$

$$\begin{aligned}
T^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= T^P_{B_1}(b) \wedge T^P_{B_{2k}}(d_1d_2)\\
&\leq T^P_{B_1}(b) \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)]\\
&= [T^P_{B_1}(b) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b) \wedge T^P_{B_2}(d_2)]\\
&= T^P_{(B_1\boxtimes B_2)}(bd_1) \wedge T^P_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= T^N_{B_1}(b) \vee T^N_{B_{2k}}(d_1d_2)\\
&\geq T^N_{B_1}(b) \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)]\\
&= [T^N_{B_1}(b) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b) \vee T^N_{B_2}(d_2)]\\
&= T^N_{(B_1\boxtimes B_2)}(bd_1) \vee T^N_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= I^P_{B_1}(b) \wedge I^P_{B_{2k}}(d_1d_2)\\
&\leq I^P_{B_1}(b) \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)]\\
&= [I^P_{B_1}(b) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b) \wedge I^P_{B_2}(d_2)]\\
&= I^P_{(B_1\boxtimes B_2)}(bd_1) \wedge I^P_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= I^N_{B_1}(b) \vee I^N_{B_{2k}}(d_1d_2)\\
&\geq I^N_{B_1}(b) \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b) \vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\boxtimes B_2)}(bd_1) \vee I^N_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= F^P_{B_1}(b) \vee F^P_{B_{2k}}(d_1d_2)\\
&\leq F^P_{B_1}(b) \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b) \vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\boxtimes B_2)}(bd_1) \vee F^P_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\boxtimes B_{2k})}((bd_1)(bd_2)) &= F^N_{B_1}(b) \wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq F^N_{B_1}(b) \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b) \wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\boxtimes B_2)}(bd_1) \wedge F^N_{(B_1\boxtimes B_2)}(bd_2),
\end{aligned}$$

for $bd_1, bd_2 \in V_1 \boxtimes V_2$.

Case 2.: For $b \in V_2$, $d_1d_2 \in V_{1k}$

$$\begin{aligned}
T^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= T^P_{B_2}(b) \wedge T^P_{B_{1k}}(d_1d_2)\\
&\leq T^P_{B_2}(b) \wedge [T^P_{B_1}(d_1) \wedge T^P_{B_1}(d_2)]\\
&= [T^P_{B_2}(b) \wedge T^P_{B_1}(d_1)] \wedge [T^P_{B_2}(b) \wedge T^P_{B_1}(d_2)]\\
&= T^P_{(B_1\boxtimes B_2)}(d_1b) \wedge T^P_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= T^N_{B_2}(b) \vee T^N_{B_{1k}}(d_1d_2)\\
&\geq T^N_{B_2}(b) \vee [T^N_{B_1}(d_1) \vee T^P_{B_1}(d_2)]\\
&= [T^N_{B_2}(b) \vee T^N_{B_1}(d_1)] \vee [T^N_{B_2}(b) \vee T^N_{B_1}(d_2)]\\
&= T^N_{(B_1\boxtimes B_2)}(d_1b) \vee T^N_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= I^P_{B_2}(b) \wedge I^P_{B_{1k}}(d_1d_2)\\
&\leq I^P_{B_2}(b) \wedge [I^P_{B_1}(d_1) \wedge I^P_{B_1}(d_2)]\\
&= [I^P_{B_2}(b) \wedge I^P_{B_1}(d_1)] \wedge [I^P_{B_2}(b) \wedge I^P_{B_1}(d_2)]\\
&= I^P_{(B_1\boxtimes B_2)}(d_1b) \wedge I^P_{(B_1\boxtimes B_2)}(d_2b),
\end{aligned}$$

$$\begin{aligned} I^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= I^N_{B_2}(b) \vee I^N_{B_{1k}}(d_1d_2)\\ &\geq I^N_{B_2}(b) \vee [I^N_{B_1}(d_1) \vee I^N_{B_1}(d_2)]\\ &= [I^N_{B_2}(b) \vee I^N_{B_1}(d_1)] \vee [I^N_{B_2}(b) \vee I^N_{B_1}(d_2)]\\ &= I^N_{(B_1\boxtimes B_2)}(d_1b) \vee I^N_{(B_1\boxtimes B_2)}(d_2b), \end{aligned}$$

$$\begin{aligned} F^P_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= F^P_{B_2}(b) \vee F^P_{B_{1k}}(d_1d_2)\\ &\leq F^P_{B_2}(b) \vee [F^P_{B_1}(d_1) \vee F^P_{B_1}(d_2)]\\ &= [F^P_{B_2}(b) \vee F^P_{B_1}(d_1)] \vee [F^P_{B_2}(b) \vee F^P_{B_1}(d_2)]\\ &= F^P_{(B_1\boxtimes B_2)}(d_1b) \vee F^P_{(B_1\boxtimes B_2)}(d_2b), \end{aligned}$$

$$\begin{aligned} F^N_{(B_{1k}\boxtimes B_{2k})}((d_1b)(d_2b)) &= F^N_{B_2}(b) \wedge F^N_{B_{1k}}(d_1d_2)\\ &\geq F^N_{B_2}(b) \wedge [F^N_{B_1}(d_1) \wedge F^N_{B_1}(d_2)]\\ &= [F^N_{B_2}(b) \wedge F^N_{B_1}(d_1)] \wedge [F^N_{B_2}(b) \wedge F^N_{B_1}(d_2)]\\ &= F^N_{(B_1\boxtimes B_2)}(d_1b) \wedge F^N_{(B_1\boxtimes B_2)}(d_2b), \end{aligned}$$

for $d_1b, d_2b \in V_1 \boxtimes V_2$.

Case 3.: For $b_1b_2 \in V_{1k},\ d_1d_2 \in V_{2k}$

$$\begin{aligned} T^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= T^P_{B_{1k}}(b_1b_2) \wedge T^P_{B_{2k}}(d_1d_2)\\ &\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2] \wedge [T^P_{B_2}(d_1) \wedge T^P_{B_2}(d_2)]\\ &= [T^P_{B_1}(b_1) \wedge T^P_{B_2}(d_1)] \wedge [T^P_{B_1}(b_2) \wedge T^P_{B_2}(d_2)]\\ &= T^P_{(B_1\boxtimes B_2)}(b_1d_1) \wedge T^P_{(B_1\boxtimes B_2)}(b_2d_2), \end{aligned}$$

$$\begin{aligned} T^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= T^N_{B_{1k}}(b_1b_2) \vee T^N_{B_{2k}}(d_1d_2)\\ &\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2] \vee [T^N_{B_2}(d_1) \vee T^N_{B_2}(d_2)]\\ &= [T^N_{B_1}(b_1) \vee T^N_{B_2}(d_1)] \vee [T^N_{B_1}(b_2) \vee T^N_{B_2}(d_2)]\\ &= T^N_{(B_1\boxtimes B_2)}(b_1d_1) \vee T^N_{(B_1\boxtimes B_2)}(b_2d_2), \end{aligned}$$

$$\begin{aligned} I^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= I^P_{B_{1k}}(b_1b_2) \wedge I^P_{B_{2k}}(d_1d_2)\\ &\leq [I^P_{B_1}(b_1) \wedge I^P_{B_1}(b_2] \wedge [I^P_{B_2}(d_1) \wedge I^P_{B_2}(d_2)]\\ &= [I^P_{B_1}(b_1) \wedge I^P_{B_2}(d_1)] \wedge [I^P_{B_1}(b_2) \wedge I^P_{B_2}(d_2)]\\ &= I^P_{(B_1\boxtimes B_2)}(b_1d_1) \wedge I^P_{(B_1\boxtimes B_2)}(b_2d_2), \end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= I^N_{B_{1k}}(b_1b_2) \vee I^N_{B_{2k}}(d_1d_2)\\
&\geq [I^N_{B_1}(b_1) \vee I^N_{B_1}(b_2] \vee [I^N_{B_2}(d_1) \vee I^N_{B_2}(d_2)]\\
&= [I^N_{B_1}(b_1) \vee I^N_{B_2}(d_1)] \vee [I^N_{B_1}(b_2) \vee I^N_{B_2}(d_2)]\\
&= I^N_{(B_1\boxtimes B_2)}(b_1d_1) \vee I^N_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= F^P_{B_{1k}}(b_1b_2) \vee F^P_{B_{2k}}(d_1d_2)\\
&\leq [F^P_{B_1}(b_1) \vee F^P_{B_1}(b_2] \vee [F^P_{B_2}(d_1) \vee F^P_{B_2}(d_2)]\\
&= [F^P_{B_1}(b_1) \vee F^P_{B_2}(d_1)] \vee [F^P_{B_1}(b_2) \vee F^P_{B_2}(d_2)]\\
&= F^P_{(B_1\boxtimes B_2)}(b_1d_1) \vee F^P_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\boxtimes B_{2k})}((b_1d_1)(b_2d_2)) &= F^N_{B_{1k}}(b_1b_2) \wedge F^N_{B_{2k}}(d_1d_2)\\
&\geq [F^N_{B_1}(b_1) \wedge F^N_{B_1}(b_2] \wedge [F^N_{B_2}(d_1) \wedge F^N_{B_2}(d_2)]\\
&= [F^N_{B_1}(b_1) \wedge F^N_{B_2}(d_1)] \wedge [F^N_{B_1}(b_2) \wedge F^N_{B_2}(d_2)]\\
&= F^N_{(B_1\boxtimes B_2)}(b_1d_1) \wedge F^N_{(B_1\boxtimes B_2)}(b_2d_2),
\end{aligned}$$

$b_1d_1, b_2d_2 \in V_1 \boxtimes V_2$.

All cases hold $\forall\ k \in \{1, 2, \ldots, m\}$.

Definition 2.25. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be BSVNGSs. *Union* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} \cup \check{G}_{b2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1m} \cup B_{2m}),$$

is defined as:

$$\text{(i)} \begin{cases} T^P_{(B_1\cup B_2)}(b) = (T^P_{B_1} \cup T^P_{B_2})(b) = T^P_{B_1}(b) \vee T^P_{B_2}(b)\\ I^P_{(B_1\cup B_2)}(b) = (I^P_{B_1} \cup I^P_{B_2})(b) = (I^P_{B_1}(b) + I^P_{B_2}(b))/2\\ F^P_{(B_1\cup B_2)}(b) = (F^P_{B_1} \cup F^P_{B_2})(b) = F^P_{B_1}(b) \wedge F^P_{B_2}(b) \end{cases}$$

$$\text{(ii)} \begin{cases} T^N_{(B_1\cup B_2)}(b) = (T^N_{B_1} \cup T^N_{B_2})(b) = T^N_{B_1}(b) \wedge T^N_{B_2}(b)\\ I^N_{(B_1\cup B_2)}(b) = (I^N_{B_1} \cup I^N_{B_2})(b) = (I^N_{B_1}(b) + I^N_{B_2}(b))/2\\ F^N_{(B_1\cup B_2)}(b) = (F^N_{B_1} \cup F^N_{B_2})(b) = F^N_{B_1}(b) \vee F^N_{B_2}(b) \end{cases}$$

for all $b \in V_1 \cup V_2$,

$$\text{(iii)} \begin{cases} T^P_{(B_{1k}\cup B_{2k})}(bd) = (T^P_{B_{1k}} \cup T^P_{B_{2k}})(bd) = T^P_{B_{1k}}(bd) \vee T^P_{B_{2k}}(bd)\\ I^P_{(B_{1k}\cup B_{2k})}(bd) = (I^P_{B_{1k}} \cup I^P_{B_{2k}})(bd) = (I^P_{B_{1k}}(bd) + I^P_{B_{2k}}(bd))/2\\ F^P_{(B_{1k}\cup B_{2k})}(bd) = (F^P_{B_{1k}} \cup F^P_{B_{2k}})(bd) = F^P_{B_{1k}}(bd) \wedge F^P_{B_{2k}}(bd) \end{cases}$$

$$\text{(iv)} \begin{cases} T^N_{(B_{1k}\cup B_{2k})}(bd) = (T^N_{B_{1k}} \cup T^N_{B_{2k}})(bd) = T^N_{B_{1k}}(bd) \wedge T^N_{B_{2k}}(bd)\\ I^N_{(B_{1k}\cup B_{2k})}(bd) = (I^N_{B_{1k}} \cup I^N_{B_{2k}})(bd) = (I^N_{B_{1k}}(bd) + I^N_{B_{2k}}(bd))/2\\ F^N_{(B_{1k}\cup B_{2k})}(bd) = (F^N_{B_{1k}} \cup F^N_{B_{2k}})(bd) = F^N_{B_{1k}}(bd) \vee F^N_{B_{2k}}(bd) \end{cases}$$

for all $(bd) \in V_{1k} \cup V_{2k}$.

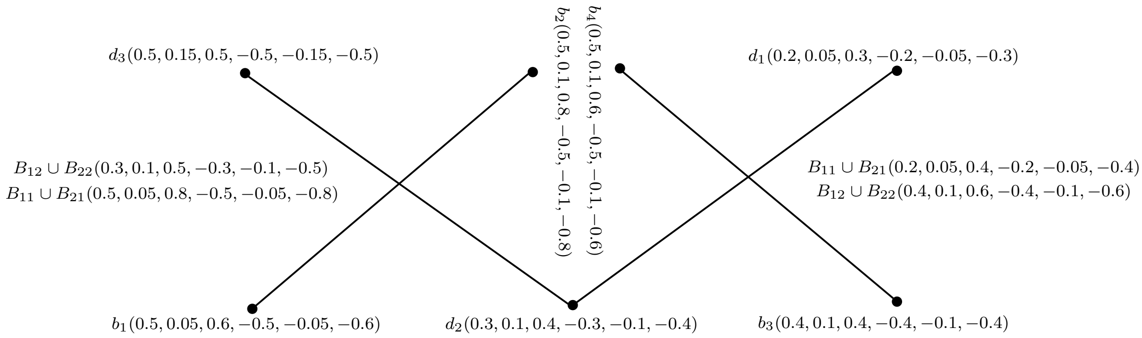


FIGURE 11. $\check{G}_{b1} \cup \check{G}_{b2}$

Example 2.26. Union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 7 is defined as $\check{G}_{b1} \cup \check{G}_{b2} = \{B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}\}$ and is depicted in Fig. 11.

Theorem 2.27. Union $\check{G}_{b1} \cup \check{G}_{b2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \ldots, B_{1m} \cup B_{2m})$ of two BSVNGSs of the GSs $\check{G}_1$ and $\check{G}_2$ is BSVNGS of $\check{G}_1 \cup \check{G}_2$.

PROOF.Let $b_1b_2 \in V_{1k} \cup V_{2k}$. Two cases arise:

Case 1.: For $b_1, b_2 \in V_1$, by definition 2.25
$T^P_{B_2}(b_1) = T^P_{B_2}(b_2) = T^P_{B_{2k}}(b_1b_2) = 0$, $I^P_{B_2}(b_1) = I^P_{B_2}(b_2) = I^P_{B_{2k}}(b_1b_2) = 0$,
$F^P_{B_2}(b_1) = F^P_{B_2}(b_2) = F^P_{B_{2k}}(b_1b_2) = 1$,
$T^N_{B_2}(b_1) = T^N_{B_2}(b_2) = T^N_{B_{2k}}(b_1b_2) = 0$, $I^N_{B_2}(b_1) = I^N_{B_2}(b_2) = I^N_{B_{2k}}(b_1b_2) = 0$,
$F^N_{B_2}(b_1) = F^N_{B_2}(b_2) = F^N_{B_{2k}}(b_1b_2) = -1$, so

$$\begin{aligned}
T^P_{(B_{1k} \cup B_{2k})}(b_1b_2) &= T^P_{B_{1k}}(b_1b_2) \vee T^P_{B_{2k}}(b_1b_2) \\
&= T^P_{B_{1k}}(b_1b_2) \vee 0 \\
&\leq [T^P_{B_1}(b_1) \wedge T^P_{B_1}(b_2)] \vee 0 \\
&= [T^P_{B_1}(b_1) \vee 0] \wedge [T^P_{B_1}(b_2) \vee 0] \\
&= [T^P_{B_1}(b_1) \vee T^P_{B_2}(b_1)] \wedge [T^P_{B_1}(b_2) \vee T^P_{B_2}(b_2)] \\
&= T^P_{(B_1 \cup B_2)}(b_1) \wedge T^P_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k} \cup B_{2k})}(b_1b_2) &= T^N_{B_{1k}}(b_1b_2) \wedge T^N_{B_{2k}}(b_1b_2) \\
&= T^N_{B_{1k}}(b_1b_2) \wedge 0 \\
&\geq [T^N_{B_1}(b_1) \vee T^N_{B_1}(b_2)] \wedge 0 \\
&= [T^N_{B_1}(b_1) \wedge 0] \vee [T^N_{B_1}(b_2) \wedge 0] \\
&= [T^N_{B_1}(b_1) \wedge T^N_{B_2}(b_1)] \vee [T^N_{B_1}(b_2) \wedge T^N_{B_2}(b_2)] \\
&= T^N_{(B_1 \cup B_2)}(b_1) \vee T^N_{(B_1 \cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^P_{B_{1k}}(b_1b_2) \wedge F^P_{B_{2k}}(b_1b_2)\\
&= F^P_{B_{1k}}(b_1b_2) \wedge 1\\
&\leq [F^P_{B_1}(b_1) \vee F^P_{B_1}(b_2)] \wedge 1\\
&= [F^P_{B_1}(b_1) \wedge 1] \vee [F^P_{B_1}(b_2) \wedge 1]\\
&= [F^P_{B_1}(b_1) \wedge F^P_{B_2}(b_1)] \vee [F^P_{B_1}(b_2) \wedge F^P_{B_2}(b_2)]\\
&= F^P_{(B_1\cup B_2)}(b_1) \vee F^P_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^N_{B_{1k}}(b_1b_2) \vee F^N_{B_{2k}}(b_1b_2)\\
&= F^N_{B_{1k}}(b_1b_2) \vee -1\\
&\geq [F^N_{B_1}(b_1) \wedge F^N_{B_1}(b_2)] \vee -1\\
&= [F^N_{B_1}(b_1) \vee -1] \wedge [F^N_{B_1}(b_2) \vee -1]\\
&= [F^N_{B_1}(b_1) \vee F^N_{B_2}(b_1)] \wedge [F^N_{B_1}(b_2) \vee F^N_{B_2}(b_2)]\\
&= F^N_{(B_1\cup B_2)}(b_1) \wedge F^N_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^P_{B_{1k}}(b_1b_2) + I^P_{B_{2k}}(b_1b_2)}{2}\\
&= \frac{I^P_{B_{1k}}(b_1b_2) + 0}{2}\\
&\leq \frac{[I^P_{B_1}(b_1) \wedge I^P_{B_1}(b_2)] + 0}{2}\\
&= [\frac{I^P_{B_1}(b_1)}{2} + 0] \wedge [\frac{I^P_{B_1}(b_2)}{2} + 0]\\
&= \frac{[I^P_{B_1}(b_1) + I^P_{B_2}(b_1)]}{2} \wedge \frac{[I^P_{B_1}(b_2) + I^P_{B_2}(b_2)]}{2}\\
&= I^P_{(B_1\cup B_2)}(b_1) \wedge I^P_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^N_{B_{1k}}(b_1b_2)+I^N_{B_{2k}}(b_1b_2)}{2}\\
&= \frac{I^N_{B_{1k}}(b_1b_2)+0}{2}\\
&\geq \frac{[I^N_{B_1}(b_1)\vee I^N_{B_1}(b_2)]+0}{2}\\
&= [\frac{I^N_{B_1}(b_1)}{2}+0]\vee[\frac{I^N_{B_1}(b_2)}{2}+0]\\
&= \frac{[I^N_{B_1}(b_1)+I^N_{B_2}(b_1)]}{2}\vee\frac{[I^N_{B_1}(b_2)+I^N_{B_2}(b_2)]}{2}\\
&= I^N_{(B_1\cup B_2)}(b_1)\vee I^N_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

for $b_1, b_2 \in V_1 \cup V_2$.

Case 2.: For $b_1, b_2 \in V_2$, by definition 2.25
$T^P_{B_1}(b_1) = T^P_{B_1}(b_2) = T^P_{B_{1k}}(b_1b_2) = 0$, $I^P_{B_1}(b_1) = I^P_{B_1}(b_2) = I^P_{B_{1k}}(b_1b_2) = 0$,
$F^P_{B_1}(b_1) = F^P_{B_2}(b_2) = F^P_{B_{1k}}(b_1b_2) = 1$,
$T^N_{B_1}(b_1) = T^N_{B_1}(b_2) = T^N_{B_{1k}}(b_1b_2) = 0$, $I^N_{B_1}(b_1) = I^N_{B_1}(b_2) = I^N_{B_{1k}}(b_1b_2) = 0$,
$F^N_{B_1}(b_1) = F^N_{B_2}(b_2) = F^N_{B_{1k}}(b_1b_2) = -1$, so

$$\begin{aligned}
T^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= T^P_{B_{1k}}(b_1b_2)\vee T^P_{B_{2k}}(b_1b_2)\\
&= T^P_{B_{2k}}(b_1b_2)\vee 0\\
&\leq [T^P_{B_2}(b_1)\wedge T^P_{B_2}(b_2)]\vee 0\\
&= [T^P_{B_2}(b_1)\vee 0]\wedge[T^P_{B_2}(b_2)\vee 0]\\
&= [T^P_{B_2}(b_1)\vee T^P_{B_1}(b_1)]\wedge[T^P_{B_2}(b_2)\vee T^P_{B_1}(b_2)]\\
&= T^P_{(B_1\cup B_2)}(b_1)\wedge T^P_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
T^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= T^N_{B_{1k}}(b_1b_2)\wedge T^N_{B_{2k}}(b_1b_2)\\
&= T^N_{B_{2k}}(b_1b_2)\wedge 0\\
&\geq [T^N_{B_2}(b_1)\vee T^N_{B_2}(b_2)]\wedge 0\\
&= [T^N_{B_2}(b_1)\wedge 0]\vee[T^N_{B_2}(b_2)\wedge 0]\\
&= [T^N_{B_2}(b_1)\wedge T^N_{B_1}(b_1)]\vee[T^N_{B_2}(b_2)\wedge T^N_{B_1}(b_2)]\\
&= T^N_{(B_1\cup B_2)}(b_1)\vee T^N_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^P_{B_{1k}}(b_1b_2) \wedge F^P_{B_{2k}}(b_1b_2) \\
&= F^P_{B_{2k}}(b_1b_2) \wedge (1) \\
&\le [F^P_{B_2}(b_1) \vee F^P_{B_2}(b_2)] \wedge (1) \\
&= [F^P_{B_2}(b_1) \wedge (1)] \vee [F^P_{B_2}(b_2) \wedge (1)] \\
&= [F^P_{B_2}(b_1) \wedge F^P_{B_1}(b_1)] \vee [F^P_{B_2}(b_2) \wedge F^P_{B_1}(b_2)] \\
&= F^P_{(B_1\cup B_2)}(b_1) \vee F^P_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
F^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= F^N_{B_{1k}}(b_1b_2) \vee F^N_{B_{2k}}(b_1b_2) \\
&= F^N_{B_{2k}}(b_1b_2) \vee (-1) \\
&\ge [F^N_{B_2}(b_1) \wedge F^N_{B_2}(b_2)] \vee (-1) \\
&= [F^N_{B_2}(b_1) \vee (-1)] \wedge [F^N_{B_2}(b_2) \vee (-1)] \\
&= [F^N_{B_2}(b_1) \vee F^N_{B_1}(b_1)] \wedge [F^N_{B_2}(b_2) \vee F^N_{B_1}(b_2)] \\
&= F^N_{(B_1\cup B_2)}(b_1) \wedge F^N_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^P_{B_{1k}}(b_1b_2) + I^P_{B_{2k}}(b_1b_2)}{2} \\
&= \frac{I^P_{B_{2k}}(b_1b_2) + 0}{2} \\
&\le \frac{[I^P_{B_2}(b_1) \wedge I^P_{B_2}(b_2)] + 0}{2} \\
&= [\frac{I^P_{B_2}(b_1)}{2} + 0] \wedge [\frac{I^P_{B_2}(b_2)}{2} + 0] \\
&= \frac{[I^P_{B_2}(b_1) + I^P_{B_1}(b_1)]}{2} \wedge \frac{[I^P_{B_2}(b_2) + I^P_{B_1}(b_2)]}{2} \\
&= I^P_{(B_1\cup B_2)}(b_1) \wedge I^P_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

$$\begin{aligned}
I^N_{(B_{1k}\cup B_{2k})}(b_1b_2) &= \frac{I^N_{B_{1k}}(b_1b_2)+I^N_{B_{2k}}(b_1b_2)}{2}\\
&= \frac{I^N_{B_{2k}}(b_1b_2)+0}{2}\\
&\geq \frac{[I^N_{B_2}(b_1)\vee I^N_{B_2}(b_2)]+0}{2}\\
&= [\frac{I^N_{B_2}(b_1)}{2}+0]\vee[\frac{I^N_{B_2}(b_2)}{2}+0]\\
&= \frac{[I^N_{B_2}(b_1)+I^N_{B_1}(b_1)]}{2}\vee\frac{[I^N_{B_2}(b_2)+I^N_{B_1}(b_2)]}{2}\\
&= I^N_{(B_1\cup B_2)}(b_1)\vee I^N_{(B_1\cup B_2)}(b_2),
\end{aligned}$$

for $b_1, b_2 \in V_1 \cup V_2$.

Both cases hold $\forall\ k \in \{1, 2, \ldots, m\}$. This completes the proof.

Theorem 2.28. Let $\check{G}_s = (V_1 \cup V_2, V_{11} \cup V_{21}, V_{12} \cup V_{22}, \ldots, V_{1m} \cup V_{2m})$ be union of GSs $\check{G_{s1}} = (V_1, V_{11}, V_{12}, \ldots, V_{1m})$ and $\check{G_{s2}} = (V_2, V_{21}, V_{22}, \ldots, V_{2m})$. Then every BSVNGS $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$ of $\check{G}_s$ is union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ of GSs $\check{G_{s1}}$ and $\check{G_{s2}}$, respectively.

PROOFFirstly, we define B_1, B_2, B_{1k} and B_{2k} for $k \in \{1, 2, \ldots, m\}$ as:
$T^P_{B_1}(b) = T^P_B(b), I^P_{B_1}(b) = I^P_B(b), F^P_{B_1}(b) = F^P_B(b)$,
$T^N_{B_1}(b) = T^N_B(b), I^N_{B_1}(b) = I^N_B(b), F^N_{B_1}(b) = F^N_B(b)$, if $b \in V_1$.

$T^P_{B_2}(b) = T^P_B(b), I^P_{B_2}(b) = I^P_B(b), F^P_{B_2}(b) = F^P_B(b)$,
$T^N_{B_2}(b) = T^N_B(b), I^N_{B_2}(b) = I^N_B(b), F^N_{B_2}(b) = F^N_B(b)$, if $b \in V_2$.

$T^P_{B_{1k}}(b_1b_2) = T^P_{B_k}(b_1b_2), I^P_{B_{1k}}(b_1b_2) = I^P_{B_k}(b_1b_2), F^P_{B_{1k}}(b_1b_2) = F^P_{B_k}(b_1b_2), T^N_{B_{1k}}(b_1b_2) = T^N_{B_k}(b_1b_2), I^N_{B_{1k}}(b_1b_2) = I^N_{B_k}(b_1b_2), F^N_{B_{1k}}(b_1b_2) = F^N_{B_k}(b_1b_2)$, if $b_1b_2 \in V_{1k}$.
$T^P_{B_{2k}}(b_1b_2) = T^P_{B_k}(b_1b_2), I^P_{B_{2k}}(b_1b_2) = I^P_{B_k}(b_1b_2), F^P_{B_{2k}}(b_1b_2) = F^P_{B_k}(b_1b_2), T^N_{B_{2k}}(b_1b_2) = T^N_{B_k}(b_1b_2), I^N_{B_{2k}}(b_1b_2) = I^N_{B_k}(b_1b_2), F^N_{B_{2k}}(b_1b_2) = F^N_{B_k}(b_1b_2)$, if $b_1b_2 \in V_{2k}$.

Then $B = B_1 \cup B_2$ and $B_k = B_{1k} \cup B_{2k}$, $k \in \{1, 2, \ldots, m\}$. Now for $b_1b_2 \in V_{tk}$, $t = 1, 2$, $k \in \{1, 2, \ldots, m\}$:
$T^P_{B_{tk}}(b_1b_2) = T^P_{B_k}(b_1b_2) \leq T^P_B(b_1) \wedge T^P_B(b_2) = T^P_{B_t}(b_1) \wedge T^P_{B_t}(b_2)$, $I^P_{B_{tk}}(b_1b_2) = I^P_{B_k}(b_1b_2) \leq I^P_B(b_1) \wedge I^P_B(b_2) = I^P_{B_t}(b_1) \wedge I^P_{B_t}(b_2)$, $F^P_{B_{tk}}(b_1b_2) = F^P_{B_k}(b_1b_2) \leq F^P_B(b_1) \vee F^P_B(b_2) = F^P_{B_t}(b_1) \vee F^P_{B_t}(b_2)$,

$T^N_{B_{tk}}(b_1b_2) = T^N_{B_k}(b_1b_2) \geq T^N_B(b_1) \vee T^N_B(b_2) = T^N_{B_t}(b_1) \vee T^N_{B_t}(b_2)$, $I^N_{B_{tk}}(b_1b_2) = I^N_{B_k}(b_1b_2) \geq I^N_B(b_1) \vee I^N_B(b_2) = I^N_{B_t}(b_1) \vee I^N_{B_t}(b_2)$, $F^N_{B_{tk}}(b_1b_2) = F^N_{B_k}(b_1b_2) \geq F^N_B(b_1) \wedge F^N_B(b_2) = F^N_{B_t}(b_1) \wedge F^N_{B_t}(b_2)$, i.e.,
$\check{G}_{bl} = (B_l, B_{l1}, B_{l2}, \ldots, B_{lm})$ is a BSVNGS of $\check{G}_t$, $t = 1, 2$. Thus $\check{G}_{bn} = (B, B_1, B_2, \ldots, B_m)$, a BSVNGS of $\check{G}_s = \check{G}_{s1} \cup \check{G}_{s2}$, is the union of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$.

Definition 2.29. Let $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \ldots, B_{1m})$ and $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \ldots, B_{2m})$ be BSVNGSs and let $V_1 \cap V_2 = \emptyset$. *Join* of $\check{G}_{b1}$ and $\check{G}_{b2}$, denoted by

$$\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1m} + B_{2m}),$$

is defined as:

$$\text{(i)} \begin{cases} T^P_{(B_1+B_2)}(b) = T^P_{(B_1\cup B_2)}(b) \\ I^P_{(B_1+B_2)}(b) = I^P_{(B_1\cup B_2)}(b) \\ F^P_{(B_1+B_2)}(b) = F^P_{(B_1\cup B_2)}(b) \end{cases}$$

$$\text{(ii)} \begin{cases} T^N_{(B_1+B_2)}(b) = T^N_{(B_1\cup B_2)}(b) \\ I^N_{(B_1+B_2)}(b) = I^N_{(B_1\cup B_2)}(b) \\ F^N_{(B_1+B_2)}(b) = F^N_{(B_1\cup B_2)}(b) \end{cases}$$

for all $b \in V_1 \cup V_2$,

$$\text{(iii)} \begin{cases} T^P_{(B_{1k}+B_{2k})}(bd) = T^P_{(B_{1k}\cup B_{2k})}(bd) \\ I^P_{(B_{1k}+B_{2k})}(bd) = I^P_{(B_{1k}\cup B_{2k})}(bd) \\ F^P_{(B_{1k}+B_{2k})}(bd) = F^P_{(B_{1k}\cup B_{2k})}(bd) \end{cases}$$

$$\text{(iv)} \begin{cases} T^N_{(B_{1k}+B_{2k})}(bd) = T^N_{(B_{1k}\cup B_{2k})}(bd) \\ I^N_{(B_{1k}+B_{2k})}(bd) = I^N_{(B_{1k}\cup B_{2k})}(bd) \\ F^N_{(B_{1k}+B_{2k})}(bd) = F^N_{(B_{1k}\cup B_{2k})}(bd) \end{cases}$$

for all $(bd) \in V_{1k} \cup V_{2k}$,

$$\text{(v)} \begin{cases} T^P_{(B_{1k}+B_{2k})}(bd) = (T^P_{B_{1k}} + T^P_{B_{2k}})(bd) = T^P_{B_1}(b) \wedge T^P_{B_2}(d) \\ I^P_{(B_{1k}+B_{2k})}(bd) = (I^P_{B_{1k}} + I^P_{B_{2k}})(bd) = I^P_{B_1}(b) \wedge I^P_{B_2}(d) \\ F^P_{(B_{1k}+B_{2k})}(bd) = (F^P_{B_{1k}} + F^P_{B_{2k}})(bd) = F^P_{B_1}(b) \vee F^P_{B_2}(d) \end{cases}$$

$$\text{(vi)} \begin{cases} T^N_{(B_{1k}+B_{2k})}(bd) = (T^N_{B_{1k}} + T^N_{B_{2k}})(bd) = T^N_{B_1}(b) \vee T^N_{B_2}(d) \\ I^N_{(B_{1k}+B_{2k})}(bd) = (I^N_{B_{1k}} + I^N_{B_{2k}})(bd) = I^N_{B_1}(b) \vee I^N_{B_2}(d) \\ F^N_{(B_{1k}+B_{2k})}(bd) = (F^N_{B_{1k}} + F^N_{B_{2k}})(bd) = F^N_{B_1}(b) \wedge F^N_{B_2}(d) \end{cases}$$

for all $b \in V_1$, $d \in V_2$.

Example 2.30. Join of two BSVNGSs $\check{G}_{b1}$ and $\check{G}_{b2}$ shown in Fig. 7 is defined as $\check{G}_{b1} + \check{G}_{b2} = \{B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}\}$ and is depicted in Fig. 12.

Theorem 2.31. Join $\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \ldots, B_{1m} + B_{2m})$ of two BSVNGSs of the GSs $\check{G}_1$ and $\check{G}_2$ is BSVNGS of $\check{G}_1 + \check{G}_2$.

3. CONCLUDING REMARKS

Bipolar fuzzy graph theory has numerous applications in various fields of science and technology including, artificial intelligence, operations research and decision making. A bipolar neutrosophic graph constitutes a generalization of the notion bipolar fuzzy graph. In this research paper, We have introduced the idea of bipolar single-valued neutrosophic graph structure and discussed many relevant notions. We also discussed a worthwhile application of bipolar single-valued neutrosophic graph structure in decision-making. In future, we aim to generalize our notions to (1) BSVN hypergraph structures, (2) BSVN vague hypergraph structures,

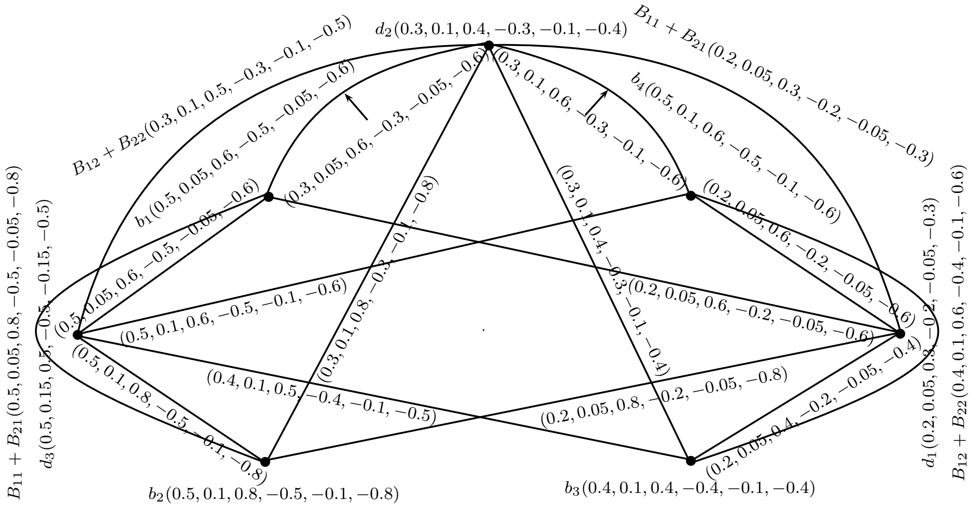


Figure 12. $\check{G}_{b1} + \check{G}_{b2}$

(3) BSVN interval-valued hypergraph structures, and(4) BSVN rough hypergraph structures.

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