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ON STRONGLY ASSOCIATIVE (SEMI)HYPERGROUPS

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Abstract. In this paper, we introduce the notion of a strongly associative hyperoperation that we call SASS and obtain a new class of (semi)hypergroups. Moreover, we study this concept in the context of K_H-semihypergroups, complete hypergroups, polygroups and Rosenberg hypergroups.

Key words and Phrases: Strongly associative hyperoperation, polygroups, complementable semihypergroup, Rosenberg hypergroup.

Abstrak. Pada makalah ini kami memperkenalkan ide suatu hiperoperasi asosiatif kuat yang kami nyatakan dengan SASS dan mendapatkan suatu kelas baru (semi)hipergrup. Lebih jauh, kami mempelajari konsep di atas dalam konteks K_H-semihipergrup, hipergrup lengkap, poligrup dan hipergrup Rosenberg.

Kata kunci: Hiperoperasi asosiatif kuat, poligrup, semihipergrup dapat dikomplemen, hipergrup Rosenberg.

1. Introduction

Hyperstructures state a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician Marty [11]. A set H endowed with a mapping $\circ : H \times H \longrightarrow P^*(H)$, named hyperoperation, is called hypergroupoid, where $P^*(H)$ denotes the set of all non-empty subsets of H. The image of a pair (x, y) is denoted by $x \circ y$ or xy. If $x \in H$ and A, B are non-empty subsets of H, then by $A \circ B$, $A \circ x$ and $x \circ B$ we mean $A \circ B = \cup\{ab|a \in A, b \in$

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$B\}$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively. A semihypergroup is a hypergroupoid $(H, \circ)$ such that for all x, y and z in H we have $(x \circ y) \circ z = x \circ (y \circ z)$ and a hypergroup is a semihypergroup which satisfies the reproductive axiom. i.e., for all $x \in H$, $H \circ x = x \circ H = H$. Resent decades numerous papers and books on algebraic hyperstructures have been published that surveys of the researches can be found in the books of Corsini [4], Corsini and Leoreanu [3], Vougiouklis [13], Davvaz and Leoreanu [6] and Davvaz [8, 7]. We know that the quotient of a group with respect to an invariant subgroup is a group. Marty states that the quotient of a group with respect to any subgroup is a hypergroup. Generally the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [14]. H_v-structures for the first time introduced by Vougiouklis at the Fourth AHA congress (1990) [14]. The concept of H_v-structures constitutes a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning H_v-structures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Some basic definitions and theorems about H_v-structures can be found in [13]. Let $(H, \circ)$ be a hypergroupoid. The hyperoperation "$\circ$ "on H is called *weak associative* (WASS) if for all $x, y, z \in H$, $(x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset$. The hyperstructure $(H, \circ)$ is called an H_v-semigroup if "$\circ$ "is weak associative. An H_v-semigroup is called an H_v-group if for all $a \in H$, $a \circ H = H \circ a = H$. For more definitions and applications on H_v-structures, see the books [3, 4, 6] and papers as [2, 9, 15, 16, 17]. A semihypergroup H is *regular* if it has at least one identity and each element has at least one inverse. A regular hypergroup $(H, \circ)$ is called *reversible* if it satisfies the following conditions, for all $(x, y, a) \in H^3$:

(1) If $y \in a \circ x$, an inverse a' of a exists such that $x \in a' \circ y$,
(2) If $y \in x \circ a$, an inverse a'' of a exists such that $x \in y \circ a''$.

If H is regular and $a \in H$, we denote by E the set of bilateral identities and by $i(a)$ the set of inverses of a. A semihypergroup H is *complete* if it satisfies one of the following equivalent conditions:

(1) $\forall (x, y) \in H^2, \forall a \in x \circ y, \mathcal{C}(a) = x \circ y$, where $\mathcal{C}(a)$ denotes the complete closure of a,
(2) $\forall (x, y) \in H^2, \mathcal{C}(x \circ y) = x \circ y$,
(3) $\forall (n, m) \in \mathbf{N}^2$, $m, n \geq 2$, $\forall (x_1, ..., x_n) \in H^n, \forall (y_1, ..., y_m) \in H^m$, the following implication is valid:

$$\Pi_{i=1}^n x_i \cap \Pi_{j=1}^m y_j \neq \emptyset \Rightarrow \Pi_{i=1}^n x_i = \Pi_{j=1}^m y_j.$$

In this paper, we introduce the notion of a strongly associative hyperoperation called SASS hyperoperation and obtain a new class of (semi)hypergroups which we call strongly associative (semi)hypergroups. Our first aim is to investigate this concept for some semihypergroups such K_H-semihypergroups, complete hypergroups, polygroups, complement hypergroups and Rosenberg hypergroups. Moreover, we introduce a class of non-strongly associative hypergroups which are derived from groups.

Theorem 1.1. *([3], 138) A semihypergroup $(H, \circ)$ is complete if it can be written as a union $H = \cup_{s \in S} A_s$ of its subsets, where S and A_s satisfy the conditions:*

(1) *$(S, \cdot)$ is a semigroup;*
(2) *for all $(s, t) \in S^2$, where $s \neq t$, we have $A_s \cap A_t = \emptyset$;*
(3) *if $(a, b) \in A_s \times A_t$, then $a \circ b = A_{st}$.*

Theorem 1.2. *([3], 141) If H is a complete hypergroup, then H is regular and reversible.*

2. On Strongly Associative Hyperoperations

In this section, we present some definitions and examples on strongly associative hyperstructures and we introduce strongly associative semihypergroups.

Definition 2.1. *Let H be a non-empty set and $\circ\colon H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. Then*

(1) *$\circ$ is called left strongly associative if for all $x, y, z \in H$ and for all $t \in y \circ z$, there exists $s \in x \circ y$ such that $x \circ t = s \circ z$,*
(2) *$\circ$ is called right strongly associative if for all $x, y, z \in H$ and for all $t \in x \circ y$, there exists $s \in y \circ z$ such that $t \circ z = x \circ s$,*
(3) *$\circ$ is strongly associative or for simplicity SASS if it is left and right strongly associative.*

Definition 2.2. *A hypergroupoid $(H, \circ)$ is called left (right) strongly associative if the hyperoperation $\circ$ is left (right) strongly associative. $(H, \circ)$ is called strongly associative if $\circ$ is strongly associative.*

Remark 2.1. *The group operation is strongly associative.*

Example 2.3. *Let H be a hypergroup, defined as follows:*

$\cdot$	e	a	b
e	$\{e, a\}$	$\{e, b\}$	$\{e, b\}$
a	$\{e, b\}$	$\{e, a\}$	$\{e, a\}$
b	$\{e, b\}$	$\{e, a\}$	$\{e, a\}$

The hyperoperation "$\cdot$"is strongly associative and H is a strongly associative hypergroup.

Example 2.4. *Let H be a hypergroup defined by the following table:*

$\cdot$	x	y	z
x	x	$\{x, y, z\}$	$\{x, y, z\}$
y	$\{x, y, z\}$	$\{y, z\}$	$\{y, z\}$
z	$\{x, y, z\}$	$\{y, z\}$	$\{y, z\}$

Then the hyperoperation "$\cdot$"is not a left strongly associative nor a right strongly associative and so H is not a strongly associative hypergroup.

Example 2.5. *Consider $H = \{a, b, c\}$ and define $\cdot$ on H with the help of the following table:*

$\cdot$	a	b	c
a	a	$\{a, b\}$	H
b	a	$\{a, b\}$	H
c	H	H	H

H is a right strongly associative hypergroup and it is not a left strongly associative hypergroup.

Proposition 2.6. *If the hyperoperation $\circ$ is strongly associative on H, then $(H, \circ)$ is a semihypergroup.*

Proof. Let $\circ$ be a SASS hyperoperation on H and $(x, y, z) \in H^3$. Then we have $\{x \circ t | t \in y \circ z\} = \{s \circ z | s \in x \circ y\}$. Hence $(x \circ y) \circ z = x \circ (y \circ z)$. □

Proposition 2.7. *If H is a strongly associative hypergroup and there exists $x \in H$ such that for all $y \in H$, $y \circ x = H$, then H is a total hypergroup.*

Proof. Suppose that $\alpha, y \in H$. According to the strongly associativity of H, for all $t \in y \circ x$, $\alpha \circ t = H$, since for all $s \in \alpha \circ y$, we have $s \circ x = H$. Therefore H is a total hypergroup. □

Similarly, if H is a strongly associative hypergroup and there exists $x \in H$ such that for all $y \in H$, $x \circ y = H$, then H is a total hypergroup.

Example 2.8. *Let $H = \{e, a, b\}$. We define the following hypergroup structure on H:*

$\circ$	e	a	b
e	$\{e, a\}$	H	H
a	H	$\{e, a\}$	$\{e, a\}$
b	H	$\{e, a\}$	$\{e, a\}$

Then H is a strongly associative hypergroup which is not a total hypergroup.

Proposition 2.9. *Let $(H, \circ)$ be a strongly associative hypergroup and $x, y, z \in H$. If $y \circ z = \{t\}$ and $\{u, v\} \subseteq x \circ y$, then $u \circ z = v \circ z$.*

Proof. Since H is e strongly associative, $x \circ t = s \circ z$ for all $s \in x \circ y$. Therefore $u \circ z = v \circ z$. □

Corollary 2.10. *If $(H, \circ)$ is a hypergroup and there exist $x, y, z \in H$ such that $y \circ z = \{t\}$ and $x \circ y = H$, then H is not a strongly associative hypergroup.*

Proof. Let $(H, \circ)$be a hypergroup. By the previous proposition, for all $\alpha \in H$, $\alpha \circ z = H$ and by proposition 2.7. H is a total hypergroup, a contradiction. □

According to [4], [3], a K_H-semihypergroup is a semihypergroup constructed from a semihypergroup $(H, \circ)$ and a family $\{A(x)\}_{x \in H}$ of non-empty and mutually disjoint subsets of H. Set $K_H = \cup_{x\in H} A(x)$ and define the hyperoperation $*$ on K_H as follows:

$$\forall (a,b) \in K_H^2; a \in A(x), b \in A(y),\ a * b = \cup_{z\in x\circ y} A(z).$$

Theorem 2.11. [3] $(H,\circ)$ *is a hypergroup if and only if* $(K_H,*)$ *is a hypergroup.*

Theorem 2.12. *If* H*is a semihypergroup, then* $K = K_H$ *is a strongly associative semihypergroup if and only if* H *is a strongly associative semihypergroup.*

Proof. Let H be a strongly associative semihypergroup and $\{a,b,c\} \subseteq K$. Then there exist $\{x,y,z\} \subseteq H$ such that $a \in A(x), b \in A(y), c \in A(z)$. Let $t \in b{*}c$. There exists $\alpha \in y{\circ}z$ such that $t \in A_\alpha$. Since H is strongly associative, there exists $r \in x{\circ}y$ such that $x \circ \alpha = r \circ z$. Let $s \in A_r$. We have $a * t = \cup_{l\in x\circ\alpha} A_l = \cup_{l\in r\circ\ z} A_l = s * c$ and so $K = K_H$ is a strongly associative semihypergroup.

For the converse, let $K = K_H$ be a strongly associative semihypergroup and $\{x,y,z\} \subseteq H$. Let $\{a,b,c\} \subseteq K$ be such that $a \in A(x), b \in A(y), c \in A(z)$ and$t \in y \circ z$. Thus $A_t \subseteq \cup_{l\in y\circ z} A_l = b{*}c$. Let $t' \in A_t \subseteq b * c$. There exists $s' \in a * b$ such that $a * t' = s' * c$ and thus $\cup_{\beta\in x\circ t} A_\beta = \cup_{\tau\in s\circ z} A_\tau$, where $s \in H$ and $s' \in A_s$. So $x \circ t = s \circ z$, since the family $\{A(x)\}_{x\in H}$ contains mutually disjoint subsets of H. □

Corollary 2.13. *If* $(H,\circ)$ *is a complete semihypergroup, then it is strongly associative.*

Notice that there are semihypergroups that are not complete, but they are strongly associative. Example 2.3. presents such a semihypergroup.

Let $(H_1,\cdot)$ and $(H_2,\circ)$ be two hypergroups. On $H_1 \times H_2$ we can define a hyperproduct as follows: $(x_1,y_1) * (x_2,y_2) = \{(x,y) | x \in x_1 \cdot x_2, y \in y_1 \circ y_2\}$. This is the direct product of H_1 and H_2 and it is clearly a hypergroup.

Theorem 2.14. *If* H_1 *and* H_2 *are two strongly associative hypergroups, then* $H_1 \times H_2$ *is a strongly associative hypergroup.*

Proof. Let $\{(x_1,x_2),(y_1,y_2),(z_1,z_2)\} \subseteq H_1 \times H_2$ and $(t_1,t_2) \in (y_1,y_2) * (z_1,z_2)$. Therefore $t_1 \in y_1 \cdot z_1$ and $t_2 \in y_2 \circ z_2$. Since H_1 and H_2 are strongly associative, there exist $s_1 \in x_1 \cdot y_1$ and $s_2 \in x_2 \circ y_2$ such that $x_1 \cdot t_1 = s_1 \cdot z_1$ and $x_2 \circ t_2 = s_2 \circ z_2$. Thus $(t_1,t_2) \in (y_1,y_2) * (z_1,z_2)$ and $(x_1,x_2) * (t_1,t_2) = (s_1,s_2) * (z_1,z_2)$. Therefore $H_1 \times H_2$ is a left strongly associative hypergroup.

Similarly, it can be checked that $H_1 \times H_2$ is a right strongly associative hypergroup and this completes the proof. □

3. On Strongly Associative Complementable (Semi)Hypergroups

In this section, we show that the complement hypergroup of a complete hypergroup is strongly associative and a polygroup is strongly associative if and only if it is group.

Let $(H, \circ)$ be a semihypergroup such that $x \circ y \neq H$, for all $(x, y) \in H^2$.

According to [1], the complement of $(H, \circ)$ is the hypergroupoid $(H, \circ^c)$ endowed with the complement hyperoperation: $x \circ^c y = H - x \circ y$.

The semihypergroup $(H, \circ)$ is complementable if its complement $(H, \circ^c)$ is a semihypergroup too.

$(H, \circ^c)$ is called the complement semihypergroup of $(H, \circ)$.

Theorem 3.1. *([1], 5.2.) The hypergroup K_H is complementable if and only if H is complementable.*

Proposition 3.2. *All groups of order at least 2 are complementable.*

Proof. Let $(G, \cdot)$ be a group. If $|G| = 2$ then $(G, \cdot^c)$ is a group, too. Now suppose that $|G| \geq 3$ and $x, y, z \in G$. We have

$$(x \cdot^c y) \cdot^c z = \bigcup_{u \in x \cdot^c y} u \cdot^c z \supseteq u_1 \cdot^c z \cup u_2 \cdot^c z$$

where $u_1 \neq u_2$ and $u_1, u_2 \in x \cdot^c y$. Since $(G, \cdot)$ is group, $u_1 \cdot^c z \cup u_2 \cdot^c z = G$. Hence $(x \cdot^c y) \cdot^c z = G$ and similarly $x \cdot^c (y \cdot^c z) = G$, whence $(x \cdot^c y) \cdot^c z = x \cdot^c (y \cdot^c z)$. □

Corollary 3.3. *All non-total complete hypergroups of order at least 2 are complementable.*

Proof. It follows by the previous proposition and Theorems 1.1 and 3.1. □

Theorem 3.4. *The complement of a complete hypergroup is a strongly associative hypergroup.*

Proof. Let $(H, \circ)$ be a non-total complete hypergroup. If $|H| = 2$ then $(H, \circ^c)$ is a group and so, it is strongly associative.

Now suppose that $\{x, y, z\} \subseteq H$ and $t \in y \circ^c z$, so $t \notin y \circ z$. If $s \in x \circ t \circ z'$, where $z' \in i(z)$ (the set of inverse elements of z in $(H, \circ)$) it follows that $s \in x \circ^c y$. If $s \notin x \circ^c y$, then $s \in x \circ y \cap x \circ t \circ z'$. Since H is complete, $x \circ y = x \circ t \circ z'$ and so

$$\begin{aligned} x' \circ x \circ y =& x' \circ x \circ t \circ z' \\ =& e \circ t \circ z' \\ =& t \circ z' \end{aligned}$$

where $x' \in i(x)$, therefore $t \in y \circ z$ which is a contradiction.

Therefore, $x \circ^c t = s \circ^c z$, therefore $(H, \circ^c)$ is strongly associative on the right. Similarly it is strongly associative on the left and so it is a strongly associative hypergroup. □

There exist strongly associative semihypergroups of which complement semihypergroups are not strongly associative. The next example is such a semihypergroup.

Example 3.5. [1] *Set* $H = \{e, a, b\}$*. Consider the semihypergroup* $(H, \circ)$ *endowed with the hyperoperation* $\circ$ *defined as follows:*

$\circ$	e	a	b
e	$\{a, b\}$	b	b
a	b	b	b
b	b	b	b

Notice that H *is a strongly associative complementable semihypergroup, of which complement, defined as follows*

$\circ^c$	e	a	b
e	e	$\{e, a\}$	$\{e, a\}$
a	$\{e, a\}$	$\{e, a\}$	$\{e, a\}$
b	$\{e, a\}$	$\{e, a\}$	$\{e, a\}$

is a semihypergroup, which it is not strongly associative.

Corollary 3.6. *If* $(G, \cdot)$ *is a group, then* $(G, \cdot^c)$ *is a strongly associative hypergroup.*

A polygroup is a system $\zeta =< P, \cdot, ^{-1} >$, where $e \in P$ and $^{-1}$ is a unitary operation on P, while $\cdot$ maps $P \times P$ into the set of all non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(P1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(P2) $e \cdot x = x \cdot e = x$;
(P3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$ (see [7]).

[7] contains some definitions and results about polygroups.

Lemma 3.7. *A polygroup* $\langle P, \cdot, e, ^{-1} \rangle$ *is a group if and only if for all element* a *of* P, $|a \cdot a^{-1}| = 1$.

Proof. Set $\{c, b, \alpha, \beta\} \subseteq P$ and $\{\alpha, \beta\} \subseteq c.b$. We have

$$\begin{aligned} &\alpha \in c.b \\ \Rightarrow &b \in c^{-1}\alpha \\ \Rightarrow &\beta \in c.b \subseteq cc^{-1}\alpha = \alpha \\ \Rightarrow &\alpha = \beta. \end{aligned}$$

Thus, for all $(c, b) \in P^2$, $|c \cdot b| = 1$ and hence P is a group. □

Proposition 3.8. *A polygroup* P *is strongly associative if and only if it is a group.*

Proof. Suppose that $\langle P, \cdot, e, ^{-1} \rangle$ is a strongly associative polygroup, which it is not a group. By the above lemma, there exist $f, a \in P$ such that $f \cdot f^{-1} = \{e, a\}$. We have $a \in f \cdot f^{-1}$, $e \cdot f = f$ and $e \cdot a \neq f \cdot f^{-1}$. In other words, P is not a strongly associative polygroup, which is false. The converse is obvious. □

4. On Strongly Rosenberg Hypergroups

In this section, we prove that a Rosenberg hypergroup is strongly associative if and only if it is a total hypergroup. In [12], Rosenberg associated a partial hypergroupoid $H_\rho = (H, \circ_\rho)$ with a binary relation ρ defined on a set H, in the following way:

$$x \circ_\rho x = \{z \in H | (x, z) \in \rho\}, \; x \circ_\rho y = x \circ_\rho x \cup y \circ_\rho y,$$

for all $(x, y) \in H^2$.

Theorem 4.1. [12] *H_ρ is a hypergroup if and only if*

(1) *ρ has full domain;*
(2) *ρ has full range;*
(3) *$\rho \subset \rho^2$;*
(4) *If $(a, x) \in \rho^2$, then $(a, x) \in \rho$, whenever x is an outer element of ρ.*

An element $x \in H$ is called outer element of ρ if there exists $h \in H$ such that $(h, x) \notin \rho^2$.

Lemma 4.2. *Let H be a non-empty set such that $|H| \geq 2$ and*

$$\rho = \{(x, x) | x \in H\}.$$

Then the associated Rosenberg hypergroup $(H, \circ_\rho)$ of ρ is not a strongly associative hypergroup.

Proof. Let $x, y, z \in H$. We have $x \circ y \neq x \circ z$ and $x \circ y \neq y \circ z$, whence it follows the conlusion. □

Theorem 4.3. *A Rosenberg hypergroup $(H, \circ_\rho)$ is strongly associative if and only if it is a total hypergroup.*

Proof. Let $(H, \circ_\rho)$ be a strongly associative Rosenberg hypergroup. H has full range, so for all element $y \in H$ there exists $x \in H$ such that $y \in x \circ x$. Therefore $y \in x \circ y$. Since H is strongly associative, there exists $\beta \in x \circ y$ such that $y \circ y = x \circ \beta = x \circ x \cup \beta \circ \beta$. Therefore $x \circ x \subseteq y \circ y$ and so $y \in y \circ y$. Hence we obtain that ρ is reflexive. By Lemma 4.2. there are $x, y \in H$ $(x \neq y)$ such that $y \in x \circ x$. Hence $x \circ x \subseteq y \circ y$ and $\{x, y\} \subseteq x \circ x \cap y \circ y$.

Now set $z \in H$. We have $x \in y \circ z$. Since H is strongly associative, there exists $\alpha \in x \circ y$ such that $x \circ x = \alpha \circ z$. Therefore $z \in x \circ x$ and so $H = x \circ x$ and $H = y \circ y$. Set $\alpha \in H - \{x, y\}$. Then $\alpha \in x \circ x$. There exists $a \in x \circ \alpha$ such that $\alpha \circ \alpha = x \circ a$, consequently $x \circ x \subseteq \alpha \circ \alpha$ and so $\alpha \circ \alpha = H$. Thus we conclude that H is a total hypergroup. The converse is obvious. □

5. Derived Non-Strongly Associative Hypergroups from Groups

Definition 5.1. [5] *We say that two partial hypergroupoids $(H, \circ_1)$ and $(H, \circ_2)$ are weak mutually associative or w.m.a., if for all $(x, y, z) \in H^3$, we have:*

$$(x \circ_1 y) \circ_2 z \cup (x \circ_2 y) \circ_1 z = x \circ_1 (y \circ_2 z) \cup x \circ_2 (y \circ_1 z).$$

Definition 5.2. *We say that two partial hypergroupoids* $(H, \circ_1)$ *and* $(H, \circ_2)$ *are mutually associative or m.a., if for all* $(x, y, z) \in H^3$*, we have* $(x \circ_1 y) \circ_2 z = x \circ_2 (y \circ_1 z)$ *and* $(x \circ_2 y) \circ_1 z = x \circ_1 (y \circ_2 z)$.

If $(G, \cdot)$ and $(G, \circ)$ are groupoids, we can define a hyperoperation $\star$ on G as follows:

$$x \star y = \{x \cdot y, x \circ y\},$$

for all $(x, y) \in G^2$.

From now on, we call $\star$ the derived hyperoperation from $(G, \cdot)$ and $(G, \circ)$.

Proposition 5.3. *If* $(G, \cdot)$ *and* $(G, \circ)$ *are groups, such that their operations are mutually associative, then the derived hyperoperation from* $(G, \cdot)$ *and* $(G, \circ)$ *is strongly associative.*

Proof. Set $(x, y, z) \in G^3$. Then $(x \star y) \star z = \cup\{(x \cdot y) \star z, (x \circ y) \star z\}$ and $x \star (y \star z) = \cup\{x \star (y \cdot z), x \star (y \circ z)\}$. Since the groups $(G, \cdot)$ and $(G, \circ)$ are m.a we conclude that $(x \cdot y) \star z = x \star (y \cdot z)$ and $(x \circ y) \star z = x \star (y \circ z)$, hence $\star$ is a strongly associative hyperoperation. □

Example 5.4. *Set* $G = \{e, a, b, c\}$ *endowed with the following operations:*

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$\circ$	e	a	b	c
e	a	b	c	e
a	b	c	e	a
b	c	e	a	b
c	e	a	b	c

$(G, \cdot)$ *and* $(G, \circ)$ *are groups and their operations are mutually associative. Moreover, the derived hyperoperation from* $(G, \cdot)$ *and* $(G, \circ)$*, described bellow, is strongly associative.*

$\star$	e	a	b	c
e	e, a	a, b	b, c	e, c
a	a, b	b, c	e, c	e, a
b	b, c	e, c	e, a	a, b
c	e, c	e, a	a, b	b, c

Let m be a natural number and $(Z_m, +)$ be the cyclic group of order m. Let x be a real number and $[x]$ be the m-class of x. Then we obtain the following

Theorem 5.5. [10] *Let* n *be a natural number and set* $H_n = \{0, 1, \cdots, 2n-1\}$. *Then the structure* $(H_n, \oplus)$ *is a group isomorphic to* $(Z_{2n}, +)$*, where* $\oplus$ *is defined by:*

$$x \oplus y = x + y - 2n[(x + y)/2n], \quad \forall (x, y) \in H_n^2.$$

Theorem 5.6. [10] *Let n be a natural number and $H_n = \{0, 1, \cdots, 2n-1\}$. We define an operation $\otimes$ on H_n by:*

$$x \otimes y = \begin{cases} x + y - n[(x+y)/n], & \text{if } 0 \leq x < n, \quad 0 \leq y < n \\ x + y - n[(x+y)/n] + n, & \text{if } 0 \leq x < n, \quad n \leq y < 2n \\ x + y - n[(x+y)/n], & \text{if } n \leq x < 2n, \quad n \leq y < 2n \\ x + y - n[(x+y)/n] + n, & \text{if } n \leq x < 2n, \quad 0 \leq y < n \end{cases}$$

Then $(H_n, \otimes)$ is a group and $(H_n, \otimes) \cong (Z_2 \times Z_n, +)$.

Theorem 5.7. *Let n be a natural number and $H_n = \{0, 1, \cdots, 2n-1\}$. Then the groups $(H_n, \oplus)$ and $(H_n, \otimes)$ are not mutually associative.*

Proof. We prove that there exists $(x, y, z) \in H_n^3$ such that $(x \oplus y) \otimes z \neq x \otimes (y \oplus z)$. To this end, set $0 \leq x < n$, $0 \leq y < n$, $0 \leq z < n$, so $0 \leq x + y < 2n$ and $0 \leq y + z < 2n$. If $n \leq x + y < 2n$ and $0 \leq y + z < n$, then $(x \oplus y) \otimes z = (x+y) \otimes z = (x+y) + z - n[(x+y) + z/n] + n$, but $x \otimes (y \oplus z) = x + (y+z) - n[x + (y+z)/n]$. Hence $(x \oplus y) \otimes z \neq x \otimes (y \oplus z)$. $\square$

Proposition 5.8. *Let n be a natural number and $H_n = \{0, 1, \cdots, 2n-1\}$. Then $(H_n, \star)$ is a non-strongly associative hypergroup, where $\star$ is the derived hyperoperation from $(H_n, \oplus)$ and $(H_n, \otimes)$.*

Proof. We prove that there exists $(x, y, z) \in H_n^3$ and $t \in x \star y$ such that for all $s \in y \star z$ we have $t \star z \neq x \star s$. To this end, set $0 \leq x < n$, $0 \leq y < n$, $0 \leq z < n$, so $0 \leq x + y < 2n$ and $0 \leq y + z < 2n$. If $n \leq x + y < 2n$ and $0 \leq y + z < n$, then $x \star y = \{x + y, x + y - n\}$ and $y \star z = \{y + z\}$. Let $t = x + y - n$. We have $t \star z = \{x + y - n + z\}$, but $x \star (y + z) = \{x + y + z, x + y + z - n\}$. Hence for all $s \in y \star z$, $t \star z \neq x \star s$. Therefore $(H_n, \star)$ is a non-strongly associative hypergroup. $\square$

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