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BOUNDS ON ENERGY AND LAPLACIAN ENERGY OF GRAPHS

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Abstract. Let G be simple graph with n vertices and m edges. The energy $E(G)$ of G, denoted by $E(G)$, is defined to be the sum of the absolute values of the eigenvalues of G. In this paper, we present two new upper bounds for energy of a graph, one in terms of m,n and another in terms of largest absolute eigenvalue and the smallest absolute eigenvalue. The paper also contains upper bounds for Laplacian energy of graph.

Key words and Phrases: Adjacency matrix, Laplacian matrix, Energy of graph, Laplacian energy of graph.

Abstrak. Misalkan G adalah graf sederhana dengan n titik dan m sisi. Energi $E(G)$ dari G, dinotasikan dengan $E(G)$, didefinisikan sebagai jumlahan dari nilai mutlak dari nilai-nilai eigen G. Pada paper ini, kami menyatakan dua batas atas baru untuk energi dari graf, satu batas dalam suku m, n dan batas yang lain dalam suku nilai eigen mutlak terbesar dan terkecil. Paper ini juga memuat batas atas untuk energi Laplace dari graf.

Kata kunci: Matriks ketetanggaan, matriks Laplace, energi dari graf, energi Laplace dari graf.

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1. Introduction

The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let G be a graph with n vertices $\{v_1, v_2, ..., v_n\}$ and m edges and $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e.,$E(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on the mathematical aspects of the theory of graph energy see the papers [2, 3, 8] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [10] and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 9]. The bounds for eigenvalues of graph can be found in [1,13].

Definition 1.1. *Let G be a graph with n vertices and m edges. The* ***Laplacian matrix*** *of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n whose elements are defined as*

$$L_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i & \text{if } i = j \end{cases}$$

where d_i is the degree of the vertex v_i.
Eigenvalues of L is called eigenvalues of G.

Definition 1.2. *Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Laplacian eigenvalues of G.* ***Laplacian energy*** *LE(G) of G is defined as* $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

The matrix L is positive semi-definite and therefore its eigenvalues are non-negative. The least eigenvalue is always equal to zero. The second largest eigenvalue is called the algebraic connectivity of G. The basic properties including various upper and lower bounds for Laplacian energy have been established in [7, 11, 12, 13].

2. Main Results

2.1. **Energy of graph.** We denote the decreasing order of the the absolute value of eigenvalues of G by $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$. The following are the elementary results that follows from this notation.

(1) $\rho_i =| \lambda_k |$ for some k
(2) $\rho_i \geq \lambda_i$ for all i
(3) $E(G) = \sum_{i=1}^{n} \rho_i$

(4) $\rho_n \leq \sum_{i=1}^{n} \rho_i = E(G)$

(5) By Cauchy-Schwarz inequality

$$\Big(\sum_{i=1}^{n} \lambda_i \rho_i\Big)^2 \leq \Big(\sum_{i=1}^{n} \rho_i^2\Big)\Big(\sum_{i=1}^{n} \lambda_i^2\Big)$$

$$\sum_{i=1}^{n} \lambda_i \rho_i \leq \sqrt{(2m)(2m)}$$

Therefore $\sum_{i=1}^{n} \lambda_i \rho_i \leq 2m$, equality holds if $\rho_i = \lambda_i$.

(6) Let G and H be any two graphs with same n vertices each. Let their number of edges be respectively m_1 and m_2. If $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$ and $\rho_1^{'} \geq \rho_2^{'} \geq ... \geq \rho_n^{'}$ are their the absolute value of eigenvalues then

$$\sum_{i=1}^{n} \rho_i \rho_i^{'} \leq \sqrt{\Big(\sum_{i=1}^{n} \rho_i^2\Big)\Big(\sum_{i=1}^{n} \rho_i^2\Big)}$$

$$\leq \sqrt{(2m_1)(2m_2)}$$

$$\therefore \quad \sum_{i=1}^{n} \rho_i \rho_i^{'} \leq 2\sqrt{m_1 m_2}$$

(7) Since λ_1 is always positive, so $\rho_1 = \lambda_1 \geq \frac{2m}{n}$

(8) Since $n\rho_n^2 \leq \rho_1^2 + \rho_2^2 + ... + \rho_n^2 = 2m$ which implies $\rho_n \leq \sqrt{\frac{2m}{n}}$

Theorem 2.1. *Let G be a graph with n vertices and m edges. Let $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$ be the the absolute value of eigenvalues of G then $\rho_n \leq \sqrt{\dfrac{2m(n-1)}{n}}$.*

Proof. We know that $E(G) = \sum_{i=1}^{n} \rho_i$ and $\sum_{i=1}^{n} \rho_i^2 = 2m$

Since $\rho_n \leq \rho_i \ \forall i \ \therefore \ \rho_n \leq \sum_{i=1}^{n-1} \rho_i$

By Cauchy Schwarz inequality $\Big(\sum_{i=1}^{n-1} \rho_i\Big)^2 \leq \sum_{i=1}^{n-1} 1^2 \sum_{i=1}^{n-1} \rho_i^2$

$$= (n-1)\sum_{i=1}^{n-1} \rho_i^2$$

$$\Rightarrow \sum_{i=1}^{n-1} \rho_i^2 \geq \frac{1}{(n-1)}\Big(\sum_{i=1}^{n-1} \rho_i\Big)^2$$

$$2m - \rho_n^2 \geq \frac{1}{(n-1)}\Big(\sum_{i=1}^{n-1} \rho_i\Big)^2$$
$$\geq \frac{1}{(n-1)}\ \rho_n^2$$
$$\Rightarrow \rho_n \leq \sqrt{\frac{2m(n-1)}{n}}$$

which is an upper bound for the smallest absolute eigenvalue of the graph G □

Theorem 2.2. *Let G be a graph with n vertices and m edges. Let $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$ be the the absolute value of eigenvalues of G. If ρ_1 is repeated k times then*

$$\rho_1 \leq \frac{1}{k(p-1)}\ \Big(\sqrt{2mkp(p-1)} - \sum_{i=k+1}^{kp} \rho_i\Big) \text{ where } kp \leq n \text{ and } p \neq 1,\ k \neq 0.$$

Proof. Let $H = \Big(\bigcup_k K_p\Big) \cup \Big(K_{n-kp}\Big)^c$ where $kp \leq n$

That is H is the union of graphs K_p, repeated k times and a graph $(K_{n-kp})^c$.

The number of vertices of H is n and the number of edges is $\frac{kp(p-1)}{2}$.
Its the absolute value of eigenvalues spectrum is
$\begin{pmatrix} p-1 & 1 & 0 \\ k & k(p-1) & (n-kp) \end{pmatrix}$.

By Cauchy Schwarz inequality
$\rho_1(p-1)+...+\rho_k(p-1)+\rho_{k+1}(1)+...+\rho_{kp}(1)+\rho_{kp+1}(0)+...+\rho_n(0) \leq 2\sqrt{m\frac{kp(p-1)}{2}}$

But $\rho_1 = \rho_2 = ... = \rho_k$

$$\therefore (p-1)k\rho_1 + \sum_{i=k+1}^{kp} \rho_i \leq 2\sqrt{m\frac{kp(p-1)}{2}}$$
$$\rho_1 \leq \frac{1}{k(p-1)}\ \Big(\sqrt{2mkp(p-1)} - \sum_{i=k+1}^{kp} \rho_i\Big).\text{ Here } (p \neq 1, k \neq 0)$$

□

Corollary 2.3. *If $kp = n$, then by the above theorem*

$$(n-k)\rho_1 + \sum_{i=k+1}^{n} \rho_i \leq \sqrt{\frac{2mn(n-k)}{k}}$$
$$(n-k)\rho_1 + E(G) - k\rho_1 \leq \sqrt{\frac{2mn(n-k)}{k}}$$

$$(n-2k)\rho_1 + E(G) \leq \sqrt{\frac{2mn(n-k)}{k}}$$

$$E(G) \leq \sqrt{\frac{2mn(n-k)}{k}} - (n-2k)\rho_1$$

Also if $p = 2$ *and* $2k = n$ *then the upper bound for energy of graph is*

$$E(G) \leq \sqrt{\frac{2mn(2k-k)}{k}}$$

$$E(G) \leq \sqrt{2mn}.$$

Corollary 2.4. *If* $kp = n-1$*, then we get the following result.*

$$E(G) - \rho_n \leq \sqrt{\frac{2m(n-1)(n-1-k)}{k}} - (n-1-2k)\rho_1$$

$$E(G) \leq \sqrt{\frac{2m(n-1)(n-1-k)}{k}} - (n-1-2k)\rho_1 + \rho_n.$$

Also if $p = 2$ *and* $2k = n-1$ *then the upper bound for energy of graph is*

$$E(G) \leq \sqrt{2m(n-1)} + \rho_n.$$

Corollary 2.5. *If* $k = 1$*, then* $E(G) \leq \sqrt{2mn(n-1)} - (n-2)\rho_1$ *for* $p = n$*. and* $E(G) \leq \sqrt{2m(n-1)(n-2)} - (n-3)\rho_1 + \rho_n$ *for* $p = n-1$*.*

Corollary 2.6. *Since* $\rho_1 \geq \frac{2m}{n}$ *and* $\rho_n \leq \sqrt{\frac{2m}{n}}$ *we get new upper bound for energy of graph in term of* m *and* n

$$E(G) \leq \sqrt{\frac{2mn(n-k)}{k}} - (n-2k)\frac{2m}{n} \text{ for } pk = n.$$

$$E(G) \leq \sqrt{\frac{2m(n-1)(n-1-k)}{k}} - (n-1-2k)\frac{2m}{n} + \sqrt{\frac{2m}{n}} \text{ for } pk = n-1.$$

Corollary 2.7. *For a* r*-regular graph* $m = \frac{rn}{2}$ *and* $\rho_1 = r$ *we have the following upper bound*

$$E(G) \leq n\sqrt{\tfrac{r(n-k)}{k}} - (n-2k)r \text{ for } pk = n.$$

$$E(G) \leq \sqrt{\tfrac{rn(n-1)(n-1-k)}{k}} - (n-1-2k)r + \sqrt{r} \text{ for } pk = n-1.$$

Theorem 2.8. *Let G be a graph with n vertices and m edges. Let $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$ be the the absolute value of eigenvalues of G. If ρ_1 is repeated k times then* $\rho_1 \leq \frac{1}{k}\Big(2\sqrt{mk} - \sum_{i=k+1}^{2k} \rho_i\Big)$. $(k \neq 0)$

Proof. Here we compare the absolute value of eigenvalues of G with absolute eigenvalue of the graph $H = \Big(\bigcup_k K_{p,q}\Big)$.

Select p and q such that $n = k(p+q)$. The number of vertices of H is n and the number of edges is kpq. Its the absolute value of eigenvalues spectrum are

$$\begin{pmatrix} \sqrt{pq} & 0 \\ 2k & (n-2k) \end{pmatrix}.$$

By Cauchy Schwarz inequality

$$\rho_1\sqrt{pq} + ... + \rho_k\sqrt{pq} + \rho_{k+1}\sqrt{pq} + ... + \rho_{2k}\sqrt{pq} + \rho_{2k+1}(0) + ... + \rho_n(0) \leq 2\sqrt{mkpq}$$

But $\rho_1 = \rho_2 = ... = \rho_k$

$$\therefore \quad \rho_1 k\sqrt{pq} + \sqrt{pq}\sum_{i=k+1}^{2k} \rho_i \leq 2\sqrt{mkpq}$$

$$\rho_1 k + \sum_{i=k+1}^{2k} \rho_i \leq 2\sqrt{mk}$$

$$\rho_1 \leq \frac{1}{k}\Big(2\sqrt{mk} - \sum_{i=k+1}^{2k} \rho_i\Big).$$

□

Corollary 2.9. *If $p = q = 1$ and $2k = n$ then*

$$\rho_1 k + \sum_{i=k+1}^{n} \rho_i \leq 2\sqrt{m\frac{n}{2}}$$

i.e., $E(G) \leq \sqrt{2mn}$.

Corollary 2.10. *If $p = q = 1$ and $2k = n-1$ then*

$$\rho_1 k + \sum_{i=k+1}^{n-1} \rho_i \leq 2\sqrt{m\frac{(n-1)}{2}}$$

$\Rightarrow E(G)$ - $\rho_n \leq \sqrt{2m(n-1)}$

i.e., $E(G) \leq \sqrt{2m(n-1)} + \rho_n$

i.e., $E(G) \leq \sqrt{2m(n-1)} + \sqrt{\frac{2m}{n}}$.

Corollary 2.11. *For* $k = 1$, $\rho_1 + \rho_2 \leq 2\sqrt{m}$.

Using the above corollary we obtain another bound for energy of graphs.

Theorem 2.12. *Let* G *be a graph with* n *vertices and* m *edges and* $2m \geq n$. *If the first absolute eigenvalue,* ρ_1 *not repeated then* $E(G) \leq \sqrt{m}(2 + \sqrt{2n-4})$

Proof. Cauchy Schwarz inequality for $(n-2)$ terms is

$$\Big(\sum_{i=3}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=3}^{n} a_i^2\Big)\Big(\sum_{i=3}^{n} b_i^2\Big)$$

Put $a_i = \rho_i$ and $b_i = 1$

$$\sum_{i=3}^{n} \rho_i \leq \sqrt{\Big(\sum_{i=3}^{n} \rho_i^2\Big)\Big(\sum_{i=3}^{n} 1\Big)}$$

$$E(G) - (\rho_1 + \rho_2) \leq \sqrt{(2m - (\rho_1^2 + \rho_2^2))(n-2)}$$

$$E(G) \leq (\rho_1 + \rho_2) + \sqrt{n-2}\sqrt{(2m - (\rho_1^2 + \rho_2^2))}$$

But $\rho_1 + \rho_2 \leq 2\sqrt{m}$ $\therefore$ $E(G) \leq 2\sqrt{m} + \sqrt{n-2}\sqrt{(2m - (\rho_1^2 + \rho_2^2))}$

We maximize the function $f(x,y) = 2\sqrt{m} + \sqrt{n-2}\sqrt{(2m - (x^2 + y^2))}$

Then $f_x = \dfrac{-\sqrt{n-2}\,x}{\sqrt{(2m - (x^2+y^2))}}$ and $f_y = \dfrac{-\sqrt{n-2}\,y}{\sqrt{(2m - (x^2+y^2))}}$

For maxima value $f_x = 0$ and $f_y = 0$ which implies $(x,y) \equiv (0,0)$

$f_{xx} = \dfrac{-\sqrt{n-2}(2m - y^2)}{(2m - (x^2+y^2))^{\frac{3}{2}}}$, $f_{yy} = \dfrac{-\sqrt{n-2}(2m - x^2)}{(2m - (x^2+y^2))^{\frac{3}{2}}}$, $f_{xy} = \frac{\sqrt{n-2}xy}{(2m-(x^2+y^2))^{\frac{3}{2}}}$

At $(x,y) \equiv (0,0)$, $f_{xx} = -\sqrt{\frac{n-2}{2m}}$, $f_{yy} = -\sqrt{\frac{n-2}{2m}}$, $f_{xy} = 0$ and $\Delta = f_{xx}f_{yy} - (f_{xy})^2 = \frac{n-2}{2m}$

Thus $f(x,y)$ attains maximum value at $(0,0)$ $\therefore$ $f(0,0) = \sqrt{m}(2 + \sqrt{2n-4})$

$$E(G) \leq \sqrt{m}(2 + \sqrt{2n-4}).$$ □

2.2. **Laplacian energy of graph.** Analogous to the bounds for energy of graphs, now we obtain bounds for Laplacian energy of graphs.

Theorem 2.13. *Let* G *and* H *are two graphs with* n *vertices each. Let their number of edges be respectively be* m_1 *and* m_2. *If* $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ *represent*

absolute Laplacian eigenvalues of G and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *eigenvalues of H then*
$\sum_{i=1}^{n} \sigma_i\lambda_i \leq \sqrt{(2m_2)\Big(2m_1 + \sum_{i=1}^{n}\Big(d_i(G)\Big)^2\Big)}$
where $d_i(G)$ *is the degree of the vertex* v_i.

Proof. By Cauchy Schwarz inequality

$$\sum_{i=1}^{n} \sigma_i\lambda_i \leq \sqrt{\Big(\sum_{i=1}^{n}\sigma_i^2\Big)\Big(\sum_{i=1}^{n}\lambda_i^2\Big)}$$

But $$\sum_{i=1}^{n}\sigma_i^2 = \Big(2m_1 + \sum_{i=1}^{n}\Big(d_i(G)\Big)^2\Big)$$

$$\therefore \sum_{i=1}^{n} \sigma_i\lambda_i \leq \sqrt{(2m_2)\Big(2m_1 + \sum_{i=1}^{n}\Big(d_i(G)\Big)^2\Big)}.$$ □

Theorem 2.14. *Let G be a graph with n vertices and m edges. Let* $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ *be the absolute Laplacian eigenvalues of G. If* σ_1 *is repeated k times then*

$$\sigma_1 \leq \frac{1}{k(p-1)}\Big(\sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)\frac{kp(p-1)}{2}} - \sum_{i=k+1}^{kp}\sigma_i\Big)$$

where $kp \leq n, \quad k \neq 0, p \neq 1$

Proof. Let $H = \Big(\bigcup_k K_p\Big) \cup \Big(K_{n-kp}\Big)^c$ where $kp \leq n$

That is H is union of graphs K_p, repeated k times and a graph $(K_{n-kp})^c$.

The number of vertices of H is n and the number of edges is $\frac{kp(p-1)}{2}$.
Its the absolute value of eigenvalues spectrum is

$$\begin{pmatrix} p-1 & 1 & 0 \\ k & k(p-1) & (n-kp) \end{pmatrix}.$$

By Cauchy Schwarz inequality
$\sigma_1(p-1) + \sigma_2(p-1) + ... + \sigma_k(p-1) + \sigma_{k+1}(1) + \sigma_{k+2}(1) + ... + \sigma_{kp}(1) + \sigma_{kp+1}(0) + ... + \sigma_n(0) \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)\frac{kp(p-1)}{2}}$

But $\sigma_1 = \sigma_2 = ... = \sigma_k$

$$(p-1)k\sigma_1 + \sum_{i=k+1}^{kp}\sigma_i \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)\frac{kp(p-1)}{2}}$$

$$k\sigma_1 \leq \frac{1}{p-1}\ \Big(\sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{kp(p-1)}{2}} - \sum_{i=k+1}^{kp}\sigma_i\Big)$$

$$\sigma_1 \leq \frac{1}{k(p-1)}\ \Big(\sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)\frac{kp(p-1)}{2}} - \sum_{i=k+1}^{kp}\sigma_i\Big). \qquad \square$$

Corollary 2.15. *If $kp = n$ then by the above theorem*

$$(n-k)\sigma_1 + \sum_{i=k+1}^{kp}\sigma_i \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)\frac{n(n-k)}{k}}$$

$$(n-k)\sigma_1 + LE(G) - k\sigma_1 \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{n(n-k)}{k}}$$

$$(n-2k)\sigma_1 + LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{n(n-k)}{k}}$$

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{n(n-k)}{k}} - (n-2k)\sigma_1.$$

Also if $p = 2$ and $2k = n$ then the upper bound for Laplacian energy of graph is

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{n(2k-k)}{k}}$$

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)n}.$$

Corollary 2.16. *If $kp = n-1$ we get the following result.*

$$LE(G) - \sigma_n \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{(n-1)(n-1-k)}{k}} - (n-1-2k)\sigma_1$$

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{(n-1)(n-1-k)}{k}} - (n-1-2k)\sigma_1 + \sigma_n$$

Also if $p = 2$ and $2k = n-1$ then we get the following upper bound for Laplacian energy of graph

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)\frac{(n-1)(2k-k)}{k}} + \sigma_n$$

$$LE(G) \leq \sqrt{\Big(2m + \textstyle\sum_{i=1}^{n}(d_i(G))^2\Big)(n-1)} + \sigma_n.$$

Corollary 2.17. *If $k = 1$ then the upper bounds changes to*
$LE(G) \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)n(n-1)} - (n-2)\sigma_1$ *for* $p = n$

$LE(G) \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)(n-1)(n-2)} - (n-3)\sigma_1 + \sigma_n$ *for* $p = n-1$.

Theorem 2.18. *Let G be a graph with n vertices and m edges. Let $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ be the absolute Laplacian eigenvalues of G. If σ_1 is repeated k times then*

$$\sigma_1 \leq \frac{1}{k}\Big(\sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)2k} - \sum_{i=k=1}^{2k}\sigma_i\Big) \qquad (k \neq 0).$$

Proof. Here we compare absolute Laplacian eigenvalues of G with absolute eigenvalue of graph $H = \Big(\bigcup_k K_{p,q}\Big)$.

Select p and q such that $n = k(p+q)$. The number of vertices of H is n and the number of edges is kpq. Its the absolute value of eigenvalues spectrum is $\begin{pmatrix} \sqrt{pq} & 0 \\ 2k & (n-2k) \end{pmatrix}$.

By Cauchy Schwarz inequality
$\sigma_1\sqrt{pq} + ... + \sigma_k\sqrt{pq} + \sigma_{k+1}\sqrt{pq} + ... + \sigma_{2k}\sqrt{pq} + \sigma_{2k+1}(0) + ... + \sigma_n(0) \leq \sqrt{(2m + \sum_{i=1}^{n}(d_i(G))^2)2kpq}$

But $\sigma_1 = \sigma_2 = ... = \sigma_k$

$$\therefore \sigma_1 k\sqrt{pq} + \sqrt{pq}\sum_{i=k+1}^{2k}\sigma_i \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)2kpq}$$

$$\sigma_1 k + \sum_{i=k+1}^{2k}\sigma_i \leq \sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)2k}$$

$$\sigma_1 \leq \frac{1}{k}\Big(\sqrt{\Big(2m + \sum_{i=1}^{n}(d_i(G))^2\Big)2k} - \sum_{i=k=1}^{2k}\sigma_i\Big).$$

□

Corollary 2.19. *If $2k = n$ then by above theorem*

$$LE(G) \leq \sqrt{(2m + \sum_{i=1}^{n}(d_i(G))^2)n}.$$

Corollary 2.20. *If $2k = (n-1)$ then by above theorem*

$$LE(G) \leq \sqrt{(2m + \sum_{i=1}^{n}(d_i(G))^2)(n-1)} + \sigma_n.$$

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