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TWO NICE DETERMINANTAL EXPRESSIONS AND A RECURRENCE RELATION FOR THE APOSTOL–BERNOULLI POLYNOMIALS

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Abstract. In the paper, the authors establish two nice determinantal expressions and a recurrence relation for the Apostol–Bernoulli polynomials.

Key words and Phrases: Apostol–Bernoulli polynomial; determinantal expression; recurrence relation; determinant; derivative of a ratio between two functions.

Abstrak. Pada makalah ini, para penulis menyajikan dua pernyataan berbentuk determinan dan sebuah relasi rekurensi untuk suku banyak Apostol-Bernoulli.

Kata kunci: Suku banyak Apostol–Bernoulli, ekspresi berbentuk determinan, relasi rekurensi, determinan, turunan dari rasio dua fungsi.

1. Introduction

It is well-known that the Bernoulli numbers B_k, the Bernoulli polynomials $B_k(u)$, and the Apostol–Bernoulli polynomials $B_k(u,z)$ for $k \ge 0$ can be generated respectively by

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}=1-\frac{x}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{x^{2k}}{(2k)!},\quad |x|<2\pi,$$

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$$\frac{xe^{ux}}{e^x-1}=\sum_{k=0}^{\infty}B_k(u)\frac{x^k}{k!},\quad |x|<2\pi,$$

and

$$\frac{xe^{ux}}{ze^x-1}=\sum_{k=0}^{\infty}B_k(u,z)\frac{x^k}{k!},\quad |x|<\begin{cases}2\pi, & z=1;\\ |\ln z|, & z\ne 1.\end{cases} \tag{1}$$

It is clear that these notions have the relations

$$B_k=B_k(0)\quad\text{and}\quad B_k(u)=B_k(u,1).$$

In [1, 2], Apostol connected special values of the Lerch zeta functions with the Apostol–Bernoulli polynomials $B_k(u,z)$. In [8], Luo gave a relation between the λ-power sums and the Apostol–Bernoulli polynomials $B_k(u,z)$, which generalize J. Bernoulli's formula on the representation of power sums in terms of the Bernoulli polynomials $B_k(u)$. In [7], Kim and Hu obtained the sums of products identity for the Apostol–Bernoulli numbers $B_k(u,z)$, which is an analogue of the classical sums of products identity for the Bernoulli numbers B_k dating back to Euler.

Let $p=p(x)$ and $q=q(x)\ne0$ be two differentiable functions. Then

$$\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}}\begin{vmatrix} p & q & 0 & \cdots & 0\\ p' & q' & q & \cdots & 0\\ p'' & q'' & 2q' & \cdots & 0\\ \hdotsfor{5}\\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \cdots & q\\ p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \cdots & \binom{k}{k-1}q' \end{vmatrix},\quad k\ge0. \tag{2}$$

See [3, p. 40]. We can rewrite the formula (2) as

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}(x)}\big|W_{(k+1)\times(k+1)}(x)\big|, \tag{3}$$

where $|W_{(k+1)\times(k+1)}(x)|$ denotes the determinant of the $(k+1)\times(k+1)$ matrix

$$W_{(k+1)\times(k+1)}(x)=\begin{pmatrix}U_{(k+1)\times1}(x) & V_{(k+1)\times k}(x)\end{pmatrix},$$

the quantity $U_{(k+1)\times1}(x)$ is a $(k+1)\times1$ matrix whose elements $u_{\ell,1}(x)=p^{(\ell-1)}(x)$ for $1\le\ell\le k+1$, and $V_{(k+1)\times k}(x)$ is a $(k+1)\times k$ matrix whose elements

$$v_{i,j}(x)=\begin{cases}\dbinom{i-1}{j-1}q^{(i-j)}(x), & i-j\ge0\\ 0, & i-j<0\end{cases}$$

for $1\le i\le k+1$ and $1\le j\le k$. For more information, please refer to related texts in the recently published papers [6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 25, 26] and the closely related references therein.

The determinant expressions for the classical Bernoulli polynomials $B_k(u)$ have a long history, see [5, p. 53].

Applying the formula (3) to $p(x) = 1$ and $q(x) = \int_0^1 e^{x(s-u)}\,\mathrm{d}\,s$, Qi and Chapman [13, Theorem 1.2] obtained the determinantal expressions

$$B_k(u) = (-1)^k \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \left[(1-u)^{\ell-m+1} - (-u)^{\ell-m+1}\right] \right|_{1\le \ell\le k, 0\le m\le k-1}$$

and

$$B_k = (-1)^k \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \right|_{1\le \ell\le k, 0\le m\le k-1},$$

where $|\cdot|_{1\le \ell\le k, 0\le m\le k-1}$ denotes a $k\times k$ determinant.

Influenced by the paper [13], Hu and Kim [6, Theorem 1.10] established

$$B_{k+1}(u,z) = \frac{(-1)^k(k+1)}{(z-1)^{k+1}} \left| \binom{\ell}{m} \left[z(z-u)^{\ell-m} - (-u)^{\ell-m}\right] \right|_{1\le \ell\le k, 0\le m\le k-1} \tag{4}$$

and

$$B_{k+1}(z) = \frac{(-1)^k(k+1)}{(z-1)^{k+1}} \left| \binom{\ell}{m} (z-\delta_{\ell,m}) \right|_{1\le \ell\le k, 0\le m\le k-1} \tag{5}$$

for $k \in \mathbb{N}$ and $z \ne 1$, where

$$\delta_{\ell,m} = \begin{cases} 1, & \ell = m \\ 0, & \ell \ne m \end{cases}$$

denotes the Kronecker delta. When deriving (4) and (5), Hu and Kim used the Leibnitz theorem for differentiation of a product and applied the formula (3) to $p(x) = 1$ and $q(x) = ze^{(1-u)x} - e^{-ux}$.

In this paper, we will apply the formula (3) again to establish two nice determinantal expressions and consequently derive a recurrence relation of the Apostol–Bernoulli polynomials $B_k(u,z)$ for $k \in \mathbb{N}$ and $z \ne 1$.

2. Main results and their proofs

Our main results, two nice determinantal expressions and a recurrence relation of the Apostol–Bernoulli polynomials $B_k(u,z)$ for $k \in \mathbb{N}$ and $z \ne 1$, can be stated as the following theorem.

Theorem 2.1. *The Apostol–Bernoulli polynomials $B_k(u,z)$ for $k \in \mathbb{N}$ and $z \ne 1$ can be determinantally expressed by*

$$B_k(u,z) = \frac{(-1)^{k-1}k}{(z-1)^k} \begin{vmatrix} 1 & z-1 & 0 & \cdots & 0 & 0 & 0 \\ u & z & z-1 & \cdots & 0 & 0 & 0 \\ u^2 & z & \binom{2}{1}z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{k-3} & z & \binom{k-3}{1}z & \cdots & \binom{k-3}{k-4}z & z-1 & 0 \\ u^{k-2} & z & \binom{k-2}{1}z & \cdots & \binom{k-2}{k-4}z & \binom{k-2}{k-3}z & z-1 \\ u^{k-1} & z & \binom{k-1}{1}z & \cdots & \binom{k-1}{k-4}z & \binom{k-1}{k-3}z & \binom{k-1}{k-2}z \end{vmatrix} \tag{6}$$

and

$$B_k(u,z)=\frac{(-1)^{k+1}}{(z-1)^k}\begin{vmatrix}1 & z-1 & 0 & \cdots & 0 & 0\\ 2u & \binom{2}{1}z & z-1 & \cdots & 0 & 0\\ 3u^2 & \binom{3}{1}z & \binom{3}{2}z & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (k-2)u^{k-3} & \binom{k-2}{1}z & \binom{k-2}{2}z & \cdots & z-1 & 0\\ (k-1)u^{k-2} & \binom{k-1}{1}z & \binom{k-1}{2}z & \cdots & \binom{k-1}{k-2}z & z-1\\ ku^{k-1} & \binom{k}{1}z & \binom{k}{2}z & \cdots & \binom{k}{k-2}z & \binom{k}{k-1}z\end{vmatrix}. \quad (7)$$

Consequently, the Apostol–Bernoulli polynomials $B_k(u,z)$ for $k\in\mathbb{N}$ and $z\ne1$ satisfy the recurrence relation

$$B_k(u,z)=\frac{z}{1-z}\left[kB_{k-1}(u,z)+\sum_{r=1}^{k-2}\binom{k}{r}B_r(u,z)-\frac{ku^{k-1}}{z}\right],\quad k\ge2. \quad (8)$$

Proof of Theorem 2.1. Applying the formula (3) to $p(x)=xe^{ux}$ and $q(x)=ze^x-1$ for $z\ne1$ gives

$$u_{\ell,1}(x)=(xe^{ux})^{(\ell-1)}\to(\ell-1)u^{\ell-2},\quad x\to0$$

for $\ell\ge1$ and

$$v_{i,j}(x)=\begin{cases}\dbinom{i-1}{j-1}(ze^x-1)^{(i-j)}, & i-j\ge0\\ 0, & i-j<0\end{cases}\to\begin{cases}z-1, & i-j=0\\ \dbinom{i-1}{j-1}z, & i-j>0\\ 0, & i-j<0\end{cases} \quad (9)$$

as $x\to0$ for $i,j\ge1$. As a result, the Apostol–Bernoulli polynomials $B_k(u,z)$ for $k\ge1$ can be expressed as

$$B_k(u,z)=\lim_{x\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{xe^{ux}}{ze^x-1}\right)$$
$$=\frac{(-1)^k}{(z-1)^{k+1}}\begin{vmatrix}0 & z-1 & 0 & \cdots & 0 & 0 & 0\\ 1 & z & z-1 & \cdots & 0 & 0 & 0\\ 2u & z & \binom{2}{1}z & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (k-2)u^{k-3} & z & \binom{k-2}{1}z & \cdots & \binom{k-2}{k-3}z & z-1 & 0\\ (k-1)u^{k-2} & z & \binom{k-1}{1}z & \cdots & \binom{k-1}{k-3}z & \binom{k-1}{k-2}z & z-1\\ ku^{k-1} & z & \binom{k}{1}z & \cdots & \binom{k}{k-3}z & \binom{k}{k-2}z & \binom{k}{k-1}z\end{vmatrix}$$

$$= \frac{(-1)^{k+1}}{(z-1)^k} \begin{vmatrix} 1 & z-1 & 0 & \cdots & 0 & 0 & 0 \\ 2u & \binom{2}{1}z & z-1 & \cdots & 0 & 0 & 0 \\ 3u^2 & \binom{3}{1}z & \binom{3}{2}z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (k-2)u^{k-3} & \binom{k-2}{1}z & \binom{k-2}{2}z & \cdots & \binom{k-2}{k-3}z & z-1 & 0 \\ (k-1)u^{k-2} & \binom{k-1}{1}z & \binom{k-1}{2}z & \cdots & \binom{k-1}{k-3}z & \binom{k-1}{k-2}z & z-1 \\ ku^{k-1} & \binom{k}{1}z & \binom{k}{2}z & \cdots & \binom{k}{k-3}z & \binom{k}{k-2}z & \binom{k}{k-1}z \end{vmatrix}.$$

The determinantal expression (7) is thus proved.

Since $B_0(u,z)=0$ for $z\ne 1$, the equation (1) can be rewritten as

$$\frac{e^{ux}}{ze^x-1} = \sum_{k=0}^{\infty} B_{k+1}(u,z)\frac{x^k}{(k+1)!}, \quad |x| < \begin{cases} 2\pi, & z=1; \\ |\ln z|, & z\ne 1. \end{cases}$$

This implies that

$$\frac{B_{k+1}(u,z)}{k+1} = \lim_{x\to 0} \frac{\mathrm{d}^k}{\mathrm{d}\, x^k}\left(\frac{e^{ux}}{ze^x-1}\right), \quad k\ge 0.$$

Further applying the formula (3) to $p(x)=e^{ux}$ and $q(x)=ze^x-1$ for $z\ne 1$ gives

$$u_{\ell,1}(x) = (e^{ux})^{(\ell-1)} \to u^{\ell-1}, \quad x\to 0$$

for $\ell\ge 1$ and (9). Therefore, we have

$$\frac{B_{k+1}(u,z)}{k+1} = \frac{(-1)^k}{(z-1)^{k+1}} \begin{vmatrix} 1 & z-1 & 0 & \cdots & 0 & 0 & 0 \\ u & z & z-1 & \cdots & 0 & 0 & 0 \\ u^2 & z & \binom{2}{1}z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{k-2} & z & \binom{k-2}{1}z & \cdots & \binom{k-2}{k-3}z & z-1 & 0 \\ u^{k-1} & z & \binom{k-1}{1}z & \cdots & \binom{k-1}{k-3}z & \binom{k-1}{k-2}z & z-1 \\ u^k & z & \binom{k}{1}z & \cdots & \binom{k}{k-3}z & \binom{k}{k-2}z & \binom{k}{k-1}z \end{vmatrix}$$

for $k\ge 0$. The determinantal expression (6) is thus proved.

Let $M_0=1$ and

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \cdots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. In [4, p. 222, Theorem], it was proved that the sequence M_n for $n \ge 0$ satisfies $M_1 = m_{1,1}$ and

$$M_n = m_{n,n} M_{n-1} + \sum_{r=1}^{n-1} (-1)^{n-r} m_{n,r} \left(\prod_{j=r}^{n-1} m_{j,j+1} \right) M_{r-1}, \quad n \ge 2.$$

See also [17, Lemma 2], [18, Lemma 5], [24, Lemma 2], and [25, Remark 3]. Applying this conclusion to determinants in (6) and (7) readily produces the same recurrence relation (8) respectively. The proof of Theorem 2.1 is complete. □

Remark 2.1. *This paper is a slightly modified version of the preprint* [11].

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