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PICK'S FORMULA AND GENERALIZED EHRHART QUASI-POLYNOMIALS

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Abstract. By virtue of Pick's formula, the generalized Ehrhart quasi-polynomial of the triangulation $\mathcal{P}(n) \subset \mathbb{R}^2$ with the vertices $(0,0),(u(n),0),(0,v(n))$, where $u(x)$ and $v(x)$ belong to $\mathbb{Z}[x]$ and where $n = 1,2,\ldots$, will be computed.

Key words: Generalized Ehrhart quasi-polynomial, Pick's formula.

Abstrak. Dengan menggunakan formulasi Pick, disajikan perhitungan dari polinomial quasi Ehrhart yang diperumum dari triangulasi $\mathcal{P}(n) \subset \mathbb{R}^2$ dengan verteks $(0,0),(u(n),0),(0,v(n))$ dimana $u(x), v(x) \in \mathbb{Z}[x]$ dan $n = 1,2,\ldots$.

Kata kunci: Generalized Ehrhart quasi-polynomial, Pick's formula.

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1. Introduction

The enumeration of the integer points belonging to a rational convex polytope is one of the most traditional topics in combinatorics.

Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. Recall that a numerical function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a *quasi-polynomial* if there exist an integer $s \geq 1$, called a *period* of f, and polynomials $f_0(x), \ldots, f_{s-1}(x)$ belonging to $\mathbb{Q}[x]$ such that $f(n) = f_i(n)$ when $n \equiv i \pmod{s}$. Furthermore, the *quasi-period* of f is the smallest integer $r \geq 1$ such that there exist subsets $A_1, \ldots, A_r$ of $\mathbb{Z}_{\geq 0}$ with $\mathbb{Z}_{\geq 0} = A_1 \cup \cdots \cup A_r$ and polynomials $g_1(x), \ldots, g_r(x)$ belonging to $\mathbb{Q}[x]$ for which $f(n) = g_i(n)$ when $n \in A_i$.

A typical example of a quasi-polynomial is the function $\sharp(n\mathcal{P} \cap \mathbb{Z}^d)$, called the *Ehrhart quasi-polynomial* ([1], [3]), arising from a rational convex polytope $\mathcal{P} \subset \mathbb{R}^d$.

More generally, given polynomials $w_i^{(j)}(x) \in \mathbb{Z}[x]$, $1 \leq i \leq q$ and $1 \leq j \leq d$, we introduce $v_i(n) \in \mathbb{Z}^d$, $n = 1, 2, \ldots$, by setting $v_i(n) = (w_i^{(1)}(n), \ldots, w_i^{(d)}(n))$. Write $\mathcal{P}_{\{w_i^{(j)}\}}(n) \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\{v_1(n), \ldots, v_q(n)\}$. It follows from [2] that the numerical function $\sharp(\mathcal{P}_{\{w_i^{(j)}\}}(n) \cap \mathbb{Z}^d)$ is a quasi-polynomial, which is called the *generalized Ehrhart quasi-polynomial* of $\{\mathcal{P}_{\{w_i^{(j)}\}}(n)\}_{n=1,2,\ldots}$.

We now come to a basic problem which we are interested in. Let $\mathbf{0}$ be the origin of $\mathbb{R}^d$ and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the canonical unit coordinate vectors of $\mathbb{R}^d$.

Problem 1. Given arbitrary integers $r \geq 1, e \geq 1$ and $d \geq 2$, find polynomials $v_1(x), \ldots, v_d(x)$ belonging to $\mathbb{Z}[x]$ with each $\deg(v_i(x)) = e$ such that the quasi-period of the generalized Ehrhart quasi-polynomial $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^d)$ of $\{\mathcal{P}(n)\}_{n=1,2,\ldots}$ is r, where $\mathcal{P}(n) \subset \mathbb{R}^d$ is the simplex with the vertices $\mathbf{0}$, $v_1(n)\mathbf{e}_1, \ldots, v_d(n)\mathbf{e}_d$.

In the present paper, by virtue of Pick's formula, an answer to Problem 1 for $d = 2$ can be given.

2. main result

The following theorem is the main result in this paper.

Theorem 2. *Given arbitrary integers $r \geq 1, e \geq 1$ and $s \geq 1$, there exist polynomials $u(x)$ and $v(x)$ belonging to $\mathbb{Z}[x]$ with $\deg(u(x)) = \deg(v(x)) = e$ for which the quasi-period of the generalized Ehrhart quasi-polynomial $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is r, where $\mathcal{P}(n) \subset \mathbb{R}^2$ is a triangle with the vertices $(0,0), (u(n), 0), (0, v(n))$, and the smallest period of $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is bigger than s.*

Proof. Fix a prime number $p > 1$. Let $u(x) = x^e$ and $v(x) = x^e + p^{e(r-1)}$. Write $A(\mathcal{P}(n))$ for the area of $\mathcal{P}(n)$. Let $I(\mathcal{P}(n))$ and $B(\mathcal{P}(n))$ denote the number of integer points belonging to the interior of $\mathcal{P}(n)$ and the number of integer points belonging to the boundary of $\mathcal{P}(n)$, respectively. Pick's formula guarantees that

$$A(\mathcal{P}(n)) = I(\mathcal{P}(n)) + \frac{1}{2}B(\mathcal{P}(n)) - 1.$$

Moreover, one has $A(\mathcal{P}(n)) = u(n)v(n)/2$. Let $f(n) = \sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$. Since $f(n) = I(\mathcal{P}(n)) + B(\mathcal{P}(n))$, it follows that

$$f(n) = \frac{1}{2}u(n)v(n) + \frac{1}{2}B(\mathcal{P}(n)) + 1.$$

Let $\mathcal{H}(n) \subset \mathbb{R}^2$ denote the segment which is the convex hull of $\{(u(n), 0), (0, v(n))\}$ and $g(n) = \sharp(\mathcal{H}(n) \cap \mathbb{Z}^2)$. Since $B(\mathcal{P}(n)) = g(n) + u(n) + v(n) - 1$, it follows that

$$f(n) = \frac{1}{2}(u(n)v(n) + u(n) + v(n) + 1) + \frac{1}{2}g(n).$$

Now, what we must show is that the quasi-period of the quasi-polynomial $g(n)$ is equal to r. One has

$$g(n) = \sharp\left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \,:\, \frac{x}{u(n)} + \frac{y}{v(n)} = 1\right\}.$$

Let $h(n)$ denote the greatest common divisor of $u(n) = n^e$ and $v(n) = n^e + p^{e(r-1)}$. In other words, $h(n)$ is the greatest common divisor of n^e and $p^{e(r-1)}$. Writing $u(n) = h(n)u_0(n)$ and $v(n) = h(n)v_0(n)$, it follows that

$$g(n) = \sharp\left\{(x, y) \in \mathbb{Z}^2_{\geq 0} \,:\, \frac{x}{u_0(n)} + \frac{y}{v_0(n)} = h(n)\right\}.$$

Since $u_0(n)$ and $v_0(n)$ are relatively prime, one has $g(n) = h(n) + 1$. We claim that the quasi-period of the quasi-polynomial $h(n)$ is equal to r. Let k denote the biggest integer for which n is divided by p^k. Then

- if $k = 0$, then n^e and $p^{e(r-1)}$ are relatively prime and $h(n) = 1$;
- if $1 \leq k \leq r - 2$, then $h(n) = p^{ek}$;
- if $k \geq r - 1$, then $h(n) = p^{e(r-1)}$.

Thus the quasi-period of $h(n)$ is equal to r, as desired.

We claim that the smallest period of $h(n)$ is p^{r-1}. Let $n \equiv b \pmod{p^{r-1}}$, where $0 \leq b < p^{r-1}$. When $b = 0$, one has $h(n) = p^{e(r-1)}$. Let $1 \leq b < p^{r-1}$ and ℓ the biggest integer for which n is divided by p^ℓ, where $0 \leq \ell \leq r - 2$. When $\ell = 0$, one has $h(n) = 1$. When $1 \leq \ell \leq r - 2$, one has $h(n) = p^{ek}$. Hence the smallest period of $h(n)$ is p^{r-1}. Finally, if p is large enough, then the smallest period of $h(n)$ is bigger than s, as required. □

3. Examples

As the end of this paper, we give some examples.

In Theorem 2, when $s \geq r$, it would, of course, be of interest to find $u(x)$ and $v(x)$ belonging to $\mathbb{Z}[x]$ for which the quasi-period of the generalized Ehrhart quasi-polynomial $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is r and the smallest period of $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is s.

Example 3. Let $e = 3, r = 4$ and $p = 2$ in the proof of Theorem 2. Thus $u(x) = x^3$ and $v(x) = x^3 + 2^9$. Then

$$\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2) = \begin{cases} \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{3}{2}, & n : \text{odd}, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^2, & n = 2 \cdot a, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^5, & n = 2^2 \cdot a, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^8, & n = 2^k \cdot a,\ k \geq 3, \end{cases}$$

where $a \geq 1$ is an odd integer. Furthermore,

$$\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2) = \begin{cases} \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{3}{2}, & n \equiv 1,3,5,7 \pmod 8, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^2, & n \equiv 2,6 \pmod 8, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^5, & n \equiv 4 \pmod 8, \\ \frac{n^6}{2} + 2^8(n^3+1) + n^3 + \frac{1}{2} + 2^8, & n \equiv 0 \pmod 8). \end{cases}$$

Thus the quasi-period of the quasi-polynomial $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is equal to 4, while its smallest period is 8.

Example 4. Let $u(x) = x^2 + 3x + 2$ and $v(x) = x^2 + 4x + 1$. Write $h(n)$ for the greatest common divisor of $u(n)$ and $v(n)$. Let $u(n) = h(n)u_0(n)$ and $v(n) = h(n)v_0(n)$. Then $n = h(n)(v_0(n) - u_0(n)) + 1$. Thus

$$h(n)(u_0(n) - h(n)(v_0(n) - u_0(n))^2 - 5(v_0(n) - u_0(n))) = 6.$$

Hence $h(n) \in \{1, 2, 3, 6\}$. A routine computation shows that

$$h(n) = \begin{cases} 1, & n = 6k - 4 \text{ or } 6k, \\ 2, & n = 6k - 3 \text{ or } 6k - 1, \\ 3, & n = 6k - 2, \\ 6, & n = 6k - 5. \end{cases}$$

Following the proof of Theorem 2, one has

$$\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2) = \begin{cases} \frac{n^4}{2} + \frac{7n^3}{2} + \frac{17n^2}{2} + 9n + 4, & n = 6k - 4 \text{ or } 6k, \\ \frac{n^4}{2} + \frac{7n^3}{2} + \frac{17n^2}{2} + 9n + \frac{9}{2}, & n = 6k - 3 \text{ or } 6k - 1, \\ \frac{n^4}{2} + \frac{7n^3}{2} + \frac{17n^2}{2} + 9n + 5, & n = 6k - 2, \\ \frac{n^4}{2} + \frac{7n^3}{2} + \frac{17n^2}{2} + 9n + \frac{13}{2}, & n = 6k - 5. \end{cases}$$

Thus the quasi-period of the quasi-polynomial $\sharp(\mathcal{P}(n) \cap \mathbb{Z}^2)$ is equal to 4, while its smallest period is 6.

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