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TWO ASPECTS OF A GENERALIZED FIBONACCI SEQUENCE

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Abstract. In this paper we study the so-called generalized Fibonacci sequence: $x_{n+2} = \alpha x_{n+1} + \beta x_n, n \in \mathbb{N}$. We derive an open domain around the origin of the parameter space where the sequence converges to 0. The limiting behavior on the boundary of this domain are: convergence to a nontrivial limit, k-periodic ($k \in \mathbb{N}$), or quasi-periodic. We use the ratio of two consecutive terms of the sequence to construct a rational approximation for algebraic numbers of the form: $\sqrt{r}, r \in \mathbb{Q}$. Using a similar idea, we extend this to higher dimension to construct a rational approximation for $\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}} + d$.

Key words and Phrases: Generalized Fibonacci sequence, convergence, discrete dynamical system, rational approximation.

Abstrak. Dalam makalah ini barisan Fibonacci yang diperumum: $x_{n+2} = \alpha x_{n+1} + \beta x_n, n \in \mathbb{N}$, dipelajari. Di sekitar titik asal dari ruang parameter, sebuah daerah (himpunan terhubung sederhana) yang buka diturunkan. Pada daerah ini barman tersebut konvergen ke 0. Perilaku barisan untuk n yang besar, pada batas daerah juga diturunkan yaitu: konvergen ke titik limit tak trivial, periodik-k ($k \in \mathbb{N}$), atau *quasi-periodic*. Dengan menghitung perbandingan dari dua suku berturutan dari barisan, sebuah hampiran dengan menggunakan bilangan rasional untuk: $\sqrt{r}, r \in \mathbb{Q}$ dikonstruksi. Ide yang serupa digunakan untuk mengkonstruksi hampiran rasional untuk $\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}} + d$.

Kata kunci: Barisan Fibonacci diperumum, kekonvergenan, sistem dinamik diskrit, hampiran rasional.

1. Introduction

Arguably, one of the most studied sequence in the history of mathematics is the Fibonacci sequence. The sequence, which is generated using a recursive

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formula:

$$x_{n+2} = x_{n+1} + x_n, n \in \mathbb{N},$$

with: $x_1 = 0$ and $x_2 = 1$, can be found in the Book of Calculation (*Liber Abbaci*) by Leonardo of Pisa (see [5], pp. 307-309) as a solution to the problem: "How many pairs of rabbits can be bred in one year from one pair?". This sequence has beautiful properties, among others: its ratio:

$$\frac{x_{n+1}}{x_n}, \quad n > 1,$$

converges to the golden ratio: $\frac{1}{2}\left(1+\sqrt{5}\right)$.

Generalization of the Fibonacci sequence has been done using various approaches. One usually found in the literature that the generalization is done by varying the initial conditions: x_1 and x_2 (see for example: [11]). Another type of generalization is the one that can be found already in 1878 [8] (for more recent literature see [4]), by considering the so called: *a two terms recurrence*, i.e.

$$x_{n+2} = \alpha x_{n+1} + \beta x_n, n \in \mathbb{N}.$$

A wonderful exposition about properties of the generalized Fibonacci sequence can be found in [4, 6]. The application of this type of generalized Fibonacci sequence can be found in various fields of science, among others: in computer algorithm [1] and in probability theory [2].

A further generalization of the *two terms recurrence* can be found in [3], where the authors there use different formula for the even numbered and the odd numbered term of the sequence. Another interesting variation can be found in [7, 9, 12] where the authors there consider the two terms recurrence modulo an integer. One of the most striking results is that a connection with Fermat's Last Theorem has been discovered.

Summary of the results. In this paper, we follow the generalization in [8, 4], i.e. the two terms recurrence, and the sequence shall be called the generalized Fibonacci sequence. Following [10], we concentrate on the issue of convergence of the generalized Fibonacci sequence for various value of α and β.

We rewrite the generalized Fibonacci sequence as a two dimensional discrete linear dynamical system:

$$\boldsymbol{v}_{n+1} = A(\alpha, \beta)\boldsymbol{v}_n, n \in \mathbb{N}.$$

There are three cases: the case where $A(\alpha, \beta)$ has two real eigenvalues, has one real eigenvalue and a pair of complex eigenvalues. For the first case, the iteration can be represented as linear combination of the eigenvectors of $A(\alpha, \beta)$. For the second case, we rewrite $A(\alpha, \beta)$ as a sum to two commuting matrices: a semi-simple matrix and a nilpotent matrix. For the third case, $A(\alpha, \beta)$ is represented as VRV^{-1} where R is a rotation matrix. These are all standard techniques in linear algebra.

Using this, we derive an open domain in (α, β)-plane where the generalized Fibonacci sequence converges to 0. It turns out that the boundary of this open domain is also interesting to analyze. On this boundary, the limiting behavior of the Fibonacci sequence could be quite different. The boundary consists of three lines. On one of those lines the generalized Fibonacci sequence converges to a non-trivial limit. On the other two lines the behavior is either 2-periodic, or k-periodic or quasi-periodic.

Another aspect that we discuss in this paper is the ratio of generalized Fibonacci sequence. As is mentioned, in the classical Fibonacci sequence, the ratio converges to the golden ratio. For the generalized Fibonacci sequence, the limit of the ratio is an algebraic number of the form:

$$\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 + 4\beta}}{2},$$

for some α and β. For rational α and β, the ratio of the generalized Fibonacci sequence is rational. Thus, we can use this iteration to construct a rational approximation of radicals. The rate of convergence of this iteration is also discussed in this paper.

We present also a generalization to higher dimension, i.e. by looking at *three terms recurrent*. The ratio of this generalization can be used to approximate a more sophisticated algebraic number.

Apart from a detail analysis and explicit computation in applying the knowledge in linear algebra, in this paper we also describe a few examples which are in confirmation with the analytical results.

2. A Generalized Fibonaci Sequence

Consider a sequence of real numbers which is defined by the following recursive formula

$$x_{n+2} = \alpha x_{n+1} + \beta x_n, \quad n \in \mathbb{N} \tag{1}$$

where α and β are real constants, and x_1 and x_2 are two real numbers called the initial conditions. This sequence is called: *generalized Fibonaci* sequence. By setting $y_n = x_{n+1}$, we can write (1) as a two dimensional discrete dynamical system

$$\boldsymbol{v}_{n+1} = A(\alpha, \beta)\boldsymbol{v}_n. \tag{2}$$

where

$$\boldsymbol{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \text{ and } A(\alpha, \beta) = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}.$$

Given, the initial condition: $\boldsymbol{v}_1 = (x_1, x_2)^T$, then the iteration (2) can be written as:

$$\boldsymbol{v}_n = A(\alpha, \beta)^{n-1}\boldsymbol{v}_1, n \in \mathbb{N}.$$

We are interested in the limiting behavior of (2) (and consequently (1)), as $n \to \infty$. It is clear that an asymptotically stable fixed point of the discrete system (2) is a limiting behavior of (1) as n goes to infinity. However, an asymptotically stable fixed point of (2) is an isolated point in $\mathbb{R}^2$, which is the origin: $(0, 0)^T \in \mathbb{R}^2$. In the next section we will use this technique to derive the domain Ω in (α, β)-plane where (1) converges to 0. Another interesting limiting behavior of (1) will be found on the boundaries of Ω.

Definition 2.1. *A sequence $\{x_n\} \subset \mathbb{R}$ is called p-periodic (or, periodic with period p), if there exists $p \in \mathbb{N}$ such that $x_n = x_{n+p}$, $\forall n \in \mathbb{N}$. Furthermore, if $m \in \mathbb{N}$ satisfies: $x_n = x_{n+p}$, $\forall n \in \mathbb{N}$, then m is divisible by p.*

A p-periodic solution will be denoted by:

$$\{\overline{x_1, x_2, \ldots, x_p}\}.$$

Clearly, a p-periodic sequence (with $p > 1$) does not satisfy the Cauchy criteria for convergence sequence, thus the sequence diverges. In the next definition we will describe what we mean by *converging to a periodic sequence.*

Definition 2.2. *A sequence $\{y_n\} \subset \mathbb{R}$ converges to $\{\overline{x_1, x_2, \ldots, x_p}\}$ if, for every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that, for every $k > N$*

$$\sqrt{(y_{M+k} - x_1)^2 + (y_{M+1+k} - x_2)^2 + \ldots + (y_{M+p+k} - x_p)^2} < \varepsilon.$$

In the next section, we will look at the eigenvectors of the matrix $A(\alpha, \beta)$ to derive the limiting behavior for (2). Furthermore, a periodic behavior will also be discussed.

3. Limiting behavior of the generalized Fibonaci sequence

Consider the equation for the eigenvalues of $A(\alpha, \beta)$, i.e.

$$\lambda^2 - \alpha\lambda - \beta = 0.$$

There are three cases to be considered, i.e. the case where: $\alpha^2 + 4\beta > 0$, the case where: $\alpha^2 + 4\beta = 0$, and the case where: $\alpha^2 + 4\beta < 0$.

3.1. The case where $\alpha^2 + 4\beta > 0$.

Let us consider the situation for $\alpha^2 + 4\beta > 0$, where the matrix $A(\alpha, \beta)$ has two different real eigenvalues. Those of eigenvalues are: $\lambda_1 = \frac{1}{2}\alpha + \frac{1}{2}\sqrt{\alpha^2 + 4\beta}$ and $\lambda_2 = \frac{1}{2}\alpha - \frac{1}{2}\sqrt{\alpha^2 + 4\beta}$, where their corresponding eigenvectors are:

$$\begin{pmatrix} 1 \\ \lambda_k \end{pmatrix}, k = 1, 2,$$

respectively. These eigenvectors are linearly independent, so we can write:

$$\boldsymbol{v}_1 = \theta_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \theta_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix},$$

where:

$$\theta_1 = \frac{\lambda_2 x_1 - y_1}{\sqrt{\alpha^2 + 4\beta}} \text{ and } \theta_2 = -\frac{\lambda_1 x_1 - y_1}{\sqrt{\alpha^2 + 4\beta}}.$$

Then,

$$\boldsymbol{v}_n = \theta_1 {\lambda_1}^{n-1} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \theta_2 {\lambda_2}^{n-1} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}. \tag{3}$$

The iteration (3) converges to $(0, 0)^T$ if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

Consider: $|\lambda_1| < 1$, or equivalently:

$$-2 < \alpha + \sqrt{\alpha^2 + 4\beta} \text{ and } \alpha + \sqrt{\alpha^2 + 4\beta} < 2.$$

From $-2 < \alpha + \sqrt{\alpha^2 + 4\beta}$ we derive:

$$-(2 + \alpha) < \sqrt{\alpha^2 + 4\beta}.$$

If $\alpha > -2$, then the inequality is satisfied for every β (whenever $\alpha^2 + 4\beta > 0$). If $\alpha < -2$ then

$$(2 + \alpha)^2 < \alpha^2 + 4\beta \text{ which implies } \beta - \alpha > 1.$$

From $\alpha + \sqrt{\alpha^2 + 4\beta} < 2$ we derive:

$$\sqrt{\alpha^2 + 4\beta} < (2 - \alpha).$$

Then, there are no possible solutions for the inequality, if $\alpha > 2$. If $\alpha < 2$, then

$$\alpha^2 + 4\beta < (2 - \alpha)^2 \text{ which implies } \alpha + \beta < 1.$$

These results are presented geometrically in Figure 1.

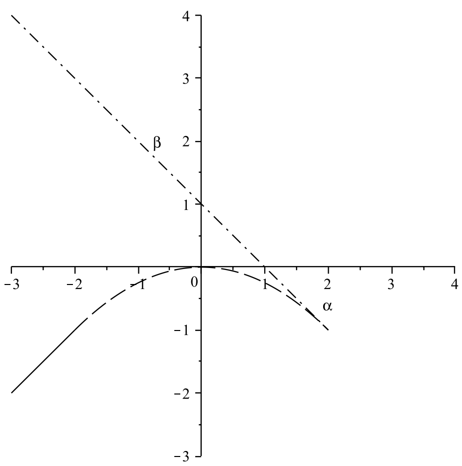

FIGURE 1. In this Figure, we have plotted the area in the $(\alpha, \beta)-$plane where $|\lambda_1| < 1$. The domain is bounded by the line $\alpha + \beta = 1$ which is plotted using a dashed and dotted line, the parabola: $\alpha^2 + 4\beta = 0$ which is plotted using a dashed line, and $\beta - \alpha = 1$ which is plotted using a solid line.

The previously described analysis can be repeated to derive the domain where $|\lambda_2| < 1$. However, we will use a symmetry argument to derive that domain. Note that, by writing $\alpha = -\gamma$ then

$$\lambda_2 = \frac{1}{2}\alpha - \frac{1}{2}\sqrt{\alpha^2 + 4\beta} = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\beta}.$$

Thus, the domain where $|\lambda_2| < 1$ can be achieved by taking the mirror image of the domain in Figure 1 with respect to the β-axis. Taking the intersection between the two domains, we derive the domain in (α, β)-plane where (2) converges to $(0, 0)$ for $\alpha^2 + 4\beta > 0$. The domain is plotted in Figure 2

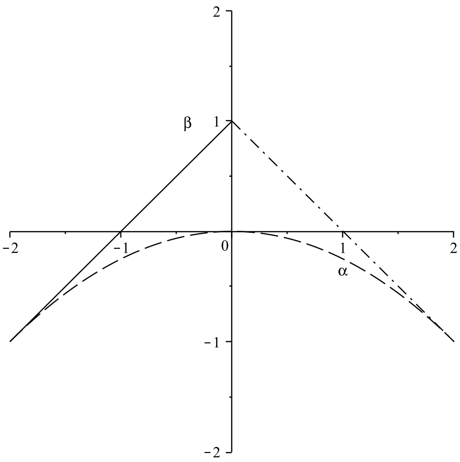

FIGURE 2. In this Figure, we have plotted the area in the $(\alpha, \beta)-$plane where $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The domain is bounded by the line $\alpha + \beta = 1$ which is plotted using a dashed and dotted line, the parabola: $\alpha^2 + 4\beta = 0$ which is plotted using a dashed line, and $\beta - \alpha = 1$ which is plotted using a solid line.

Let us now explore the boundaries of this domain. The boundaries are:

(1) $\alpha + \beta = 1$, for $0 \leq \alpha \leq 2$,
(2) $\beta - \alpha = 1$, for $-2 \leq \alpha < 0$, and
(3) $\alpha^2 + 4\beta = 0$ for $-2 < \alpha < 2$.

In 3.1.1 and 3.1.2 we will analyze the first two boundaries while the third will be treated in Subsection 3.2.

3.1.1. *The case where* $\alpha + \beta = 1$. Let $\beta = 1 - \alpha$. Then one of the eigenvalues of $A(\alpha, 1 - \alpha)$ is 1. The other eigenvalue is $\alpha - 1$. The eigenvectors corresponding to this eigenvalue are:

$$\kappa \begin{pmatrix} 1 \\ \alpha - 1 \end{pmatrix}, \quad 0 \neq \kappa \in \mathbb{R}.$$

Clearly, for $\alpha \neq 2$, the set of vectors:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \alpha - 1 \end{pmatrix} \right\}$$

is linearly independent. Then for any $x_1 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$, we can write:

$$\boldsymbol{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \theta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \theta_2 \begin{pmatrix} 1 \\ \alpha - 1 \end{pmatrix},$$

where

$$\theta_1 = x_1 + \frac{y_1 - x_1}{2 - \alpha} \text{ and } \theta_2 = -\frac{y_1 - x_1}{2 - \alpha}.$$

Since $\boldsymbol{v}_n = A^{n-1}\boldsymbol{v}_1$, we have:

$$\boldsymbol{v}_n = \theta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \theta_2(\alpha - 1)^{n-1} \begin{pmatrix} 1 \\ 1 - \alpha \end{pmatrix}. \tag{4}$$

Thus, the iteration (1) for $0 < \alpha < 2$ and $\beta = 1 - \alpha$ converges to:

$$x_1 + \frac{x_2 - x_1}{2 - \alpha},$$

(since $y_1 = x_2$).

When $\alpha = 0$, then (1) becomes:

$$x_{n+2} = x_n.$$

Thus, $\{x_n\}$ is either constant, that is when $x_1 = x_2$, or 2-periodic when $x_1 \neq x_2$. When $\alpha = 2$ then (1) becomes:

$$x_{n+2} = 2x_{n+1} - x_n.$$

If $x_1 = x_2 = \gamma \in \mathbb{R}$, then $x_3 = (2-1)\gamma = \gamma$. Then $x_n = \gamma$, for all $n \in \mathbb{N}$. Consider:

$$|x_{n+2} - x_{n+1}| = |2x_{n+1} - x_n - x_{n+1}| = |x_{n+1} - x_n|, \forall n \in \mathbb{N}.$$

Then the sequence (1) does not satisfy the Cauchy criteria for convergent sequence. Thus, the sequence (1) converge for $\alpha = 2$ and $\beta = -1$ if and only if $x_1 = x_2$.

Remark 3.1. *Let us now consider the cases where $\alpha < 0$ or $\alpha > 2$. In these cases, $|\alpha - 1| > 1$. Then clearly $\boldsymbol{v}_n$ in (4) diverges as n goes to infinity, except for $\theta_2 = 0$. Then, in these cases, the sequence (1) diverges except if $x_1 = x_2$.*

3.1.2. *The case where $\beta - \alpha = 1$.* Let $\beta = 1 + \alpha$. The eigenvalues of $A(\alpha, 1 + \alpha)$ are

$$1 + \alpha, \text{ and } -1,$$

with their corresponding eigenvectors:

$$\begin{pmatrix} 1 \\ 1 + \alpha \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

respectively. Writing:

$$\boldsymbol{v}_1 = \theta_1 \begin{pmatrix} 1 \\ 1 + \alpha \end{pmatrix} + \theta_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

where

$$\theta_1 = \frac{x_1 + y_1}{2 + \alpha} \text{ and } \theta_2 = \frac{y_1 - (1 + \alpha)x_1}{2 + \alpha}.$$

Then

$$\boldsymbol{v}_n = \theta_1(1+\alpha)^{n-1}\begin{pmatrix} 1 \\ 1+\alpha \end{pmatrix} + \theta_2(-1)^{n-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

For $-2 < \alpha < 0$ the iteration (2) converges to a periodic sequence:

$$\left\{\theta_2(-1)^{n-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}_{n=1}^{\infty}.$$

For $\alpha = 0$, the situation is similar with the case: $\alpha + \beta = 1$ for $\alpha = 0$. For $\alpha = -2$, then the eigenvalue of $A(-2,-1)$ is -1 with algebraic multiplicity 2 but geometric multiplicity 1. We will address this situation in the next subsection.

Remark 3.2. *For $\alpha < -2$ or $\alpha > 0$, the term: $(1+\alpha)^{n-1}$ grows without bound which implies that the iteration in (2) (and hence, (1)) diverges.*

3.2. **The case where $\alpha^2 + 4\beta = 0$.** For $\alpha^2 + 4\beta = 0$, the eigenvalue of $A(\alpha,\beta)$ is $\frac{\alpha}{2}$ with algebraic multiplicity two and geometric multiplicity one. Let

$$S = \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\alpha}{2}I$$

and $N = A(\alpha,\beta) - S$. Clearly:

$$S\,N = \frac{\alpha}{2}I\,\left(A(\alpha,\beta) - \frac{\alpha}{2}I\right) = A(\alpha,\beta)\,\frac{\alpha}{2}I - \left(\frac{\alpha}{2}I\right)^2 = \left(A(\alpha,\beta) - \frac{\alpha}{2}I\right)\,\frac{\alpha}{2}I = N\,S.$$

Furthermore:

$$N^2 = \begin{pmatrix} -\frac{\alpha}{2} & 1 \\ \beta & \frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} -\frac{\alpha}{2} & 1 \\ \beta & \frac{\alpha}{2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \alpha^2 + 4\beta & 0 \\ 0 & \alpha^2 + 4\beta \end{pmatrix} = 0.$$

Since $A(\alpha,\beta) = S + N$ then

$$\boldsymbol{v}_n = (S+N)^{n-1}\,\boldsymbol{v}_1.$$

Lemma 3.3. *For arbitrary $n \in \mathbb{N}$*

$$(S+N)^n = S^n + nS^{n-1}N.$$

Proof. The proof is done by induction on n. For $n = 2$,

$$(S+N)^2 = S^2 + SN + NS + N^2 = S^2 + 2SN.$$

If $(S+N)^{n-1} = S^{n-1} + (n-1)S^{n-2}N$, then

$$\begin{aligned} (S+N)^n &= (S+N)\,(S+N)^{n-1} \\ &= (S+N)\left(S^{n-1} + (n-1)S^{n-2}N\right) \\ &= S^n + (n-1)S^{n-1}N + NS^{n-1} + (n-1)NS^{n-2}N \\ &= S^n + nS^{n-1}N + (n-1)S^{n-2}N^2 \\ &= S^n + nS^{n-1}N. \end{aligned}$$

This ends the proof. □

Using Lemma 3.3 we conclude that:

$$\boldsymbol{v}_n = \left(\frac{\alpha}{2}\right)^{n-2} \left(\frac{\alpha}{2} I + (n-1)N\right) \boldsymbol{v}_1.$$

Convergence of this iteration to $(x, y) = (0, 0)$ is achieved when $-2 < \alpha < 2$. The situation for $\alpha = 2$ has been treated in Subsection 3.1.1. For $\alpha = -2$, this iteration diverges, as $(n-1)N$ grows without bound.

3.3. **The case where $\alpha + \beta \neq 1$ and $\alpha^2 + 4\beta < 0$.** For $\alpha^2 + 4\beta < 0$, then we have a pair of complex eigenvalues λ and $\overline{\lambda}$, which can be represented as:

$$\lambda = \rho \left(\cos\theta + \mathrm{i}\sin\theta\right),$$

where:

$$\rho = \left(\frac{1}{2}\alpha\right)^2 + \left(\frac{1}{2}\sqrt{-\alpha^2 - 4\beta}\right)^2 = -\beta.$$

From linear algebra, we know that: if $\mathbf{0} \neq \boldsymbol{v} \in \mathbb{C}^2$, satisfies:

$$A(\alpha, \beta) = \lambda \boldsymbol{v},$$

then: $V = (\mathrm{Re}(\boldsymbol{v})\ ,\ \mathrm{Im}(\boldsymbol{v})) \in \mathbb{R}^{2\times 2}$, satisfies:

$$A(\alpha, \beta) = \rho V R(\theta) V^{-1},$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then we have:

$$\boldsymbol{v}_n = A(\alpha, \beta)^{n-1} \boldsymbol{v}_1 = \rho^{n-1} V R\left((n-1)\theta\right) V^{-1} \boldsymbol{v}_1. \tag{5}$$

This iteration converges to $(0, 0)$ as n goes to infinity, if $0 < \rho < 1$. For $\rho > 1$, the iteration diverges as n goes to infinity.

Periodic and quasi-periodic behavior. Let us look at the case where $\rho = 1$. Since $\rho = -\beta$ then $\beta = -1$. Moreover:

$$\lambda = \frac{\alpha}{2} + \mathrm{i}\frac{1}{2}\sqrt{-\alpha^2 - 4\beta} = \cos\theta + \mathrm{i}\sin\theta,$$

then $\alpha = 2\cos\theta$.

If

$$\frac{\theta}{2\pi} = \frac{p}{q}, \quad p \in \mathbb{Z}, q \in \mathbb{N} \text{ with } \gcd(|p|, q) = 1,$$

then (5), and hence $\{x_n\}$, is q-periodic. Thus, for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $\gcd(|p|, q) = 1$,

$$x_{n+2} = 2\cos\left(\frac{2p}{q}\pi\right) x_{n+1} - x_n, \quad n \in \mathbb{N}$$

is q-periodic.

If

$$\frac{\theta}{2\pi} \notin \mathbb{Q},$$

then $\{x_n\}$ remains bounded for all n but the sequence diverges. The sequence $\{\boldsymbol{v}_n\}$ fills up densely a close curve in $\mathbb{R}^2$, see Figure 4 for an example. This behavior is known as quasi-periodic.

3.4. **Conclusion.** We conclude our discussion on the convergence of (1) with the following proposition. In Figure 3 we have presented the Proposition 3.4 geometrically.

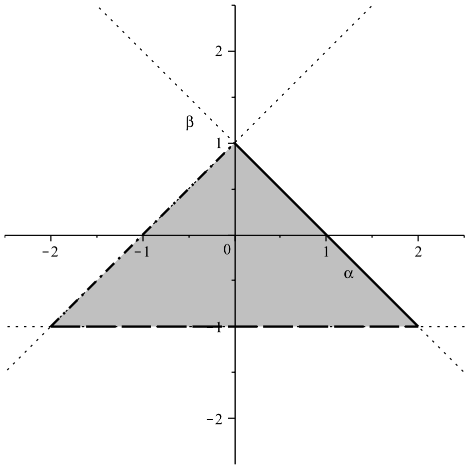

FIGURE 3. In this Figure, we have plotted the area in the $(\alpha, \beta)-$plane where the sequence (1) converges to zero. This is done by analyzing the magnitude of λ_1 and λ_2.

Proposition 3.4. *Given x_1, x_2, α and β real numbers. Consider the sequence*

$$x_{n+2} = \alpha x_{n+1} + \beta x_n, \quad n \in \mathbb{N}.$$

Then, if α and β is an element of an open domain Ω defined by:

$$\begin{cases} \alpha + \beta < 1 \\ \beta - \alpha < 1 \\ \beta > -1 \end{cases}$$

At the boundary: $\alpha + \beta = 1$, for $0 < \alpha < 2$, the sequence converges to:

$$x_1 + \frac{x_2 - x_1}{2 - \alpha}.$$

At the boundary: $\beta - \alpha = 1$, for $-2 < \alpha \leq 0$, the sequence goes to a periodic sequence:

$$\left\{ \overline{\frac{y_1 - (1+\alpha)x_1}{2+\alpha}, \frac{(1+\alpha)x_1 - y_1}{2+\alpha}} \right\}$$

At the boundary: $\beta = -1$, for $-2 < \alpha < 2$, the sequence (1) *takes the form:*

$$x_{n+2} = 2\cos(2\omega\pi)\, x_{n+1} - x_n, \quad n \in \mathbb{N};$$

the sequence is periodic whenever $\omega \in \mathbb{Q}$ and quasi-periodic *whenever $\omega \notin \mathbb{Q}$.*

4. Examples

Let us look at a view examples of iteration (1) for various combination of α and β. These examples are presented in Table 1. In the first column indicates the iteration number n. At the rows where $n = 1$ and $n = 2$, we put the initial conditions, namely $x_1 = 4$ and $x_2 = 3$. For $n \geq 3$, x_n is computed using the formula (1). To give an idea on the limiting behavior of the iteration, we listed the first 13th iterations, and a few later iterations (from $n = 64$ to 71). Indeed, for the purpose of illustration we have used only four decimals to represent real numbers.

Convergence to 0. The first example is for $\alpha = 0.2500$ and $\beta = 0.5000$. This is an example for the situation where $(\alpha, \beta) \in \Omega$, where (1) converges to 0. In this example: $\lambda_1 = 0.8431$ and $\lambda_2 = -0.5931$. This explains the slow convergence of the iteration. See 1 column two. The second example is for $\alpha = 1.6500$ and $\beta = -0.6806$ (see Table 1 column three) while the third example is for $\alpha = -1.500$ and $\beta = -0.5625$ (see Table 1 column four). These examples satisfy $\alpha^2 + 4\beta = 0$. In the second example: $\lambda_1 = \lambda_2 = 0.8200$ while in the third: $\lambda_1 = \lambda_2 = -0.7500$. In both of these examples, (1) converges to 0, just as the first example. The rate of the convergence in the third example is slightly better than in the first two, since the absolute value of the eigenvalue is smaller than in the first two examples.

Quasi-periodic behavior, periodic behavior, and eventually periodic behavior. Let us consider the fifth column and the sixth column of Table 1. In these examples, α and β satisfy: $\alpha^2 + 4\beta < 0$; thus the eigenvalues of $A(\alpha, \beta)$ are complex valued, i.e.:$\rho\,(\cos\theta + \mathrm{i}\sin\theta)$. In both examples, $\rho = 1$. The argument θ is approximately 0.8956 for the example in the fifth column while, for the example in the sixth column:

$$\theta = \frac{2\pi}{7} \approx 0.8975979011.$$

Thus, in the fifth column a quasi-periodic behavior is displayed, while in sixth: periodic behavior. If the periodic behavior exactly repeating it self after seven iterations, the quasi-periodic behavior shows that after seven iterations, it becomes closed to the starting point, but not exactly the same.

Another periodic behavior is displayed in column eight. In contrast with the periodic behavior in column sixth. In column eight, the iterations does not seems to be periodic at the beginning. After ten iterations it becomes periodic. This type of behavior is called: **eventually periodic**. This behavior is displayed for the choice of α and β that satisfy: $\beta - \alpha = 1$.

α	0.2500	1.6500	−1.5000	1.2500	1.2470	0.3000	−0.7500
β	0.5000	−0.6806	−0.5625	−1.000	−1.000	0.7000	0.2500
1	4.0000	4.0000	4.0000	4.0000	**4.0000**	4.0000	4.0000
2	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
3	2.7500	2.2275	−6.7500	−0.2500	−0.2591	3.7000	−1.2500
4	2.1875	1.6335	8.4375	−3.3125	−3.3230	3.2100	1.6875
5	1.9219	1.1792	−8.8594	−3.8906	−3.8847	3.5530	−1.5781
6	1.5742	0.8339	8.5430	−1.5508	−1.5211	3.3129	1.6055
7	1.3545	0.5733	−7.8311	1.9521	1.9879	3.4810	−1.5986
8	1.1257	0.3784	6.9412	3.9910	**4.0000**	3.3633	1.6003
9	0.9587	0.2341	−6.0068	3.0366	3.0000	3.4457	−1.5999
10	0.8025	0.1288	5.1058	−0.1953	−0.2591	3.3880	1.6000
11	0.6800	0.0531	−4.2798	−3.2806	−3.3230	3.4284	−1.6000
12	0.5713	0.0000	3.5478	−3.9055	−3.8847	3.4001	1.6000
13	0.4828	−0.0362	−2.9142	−1.6013	−1.5211	3.4199	−1.6000
14	0.4063	−0.0596	2.3757	1.9039	1.9879	3.4061	1.6000
15	0.3430	−0.0738	−1.9243	3.9812	**4.0000**	3.4158	−1.6000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
64	0.0001	−0.0001	0.0000	3.8926	**4.0000**	3.4118	1.6000
65	0.0001	−0.0001	0.0000	3.3085	3.0000	3.4118	−1.6000
66	0.0001	−0.0001	0.0000	0.2430	−0.2591	3.4118	1.6000
67	0.0000	−0.0001	0.0000	−3.0047	−3.3230	3.4118	−1.6000
68	0.0000	−0.0001	0.0000	−3.9989	−3.8847	3.4118	1.6000
69	0.0000	0.0000	0.0000	−1.9939	−1.5211	3.4118	−1.6000
70	0.0000	0.0000	0.0000	1.5065	1.9879	3.4118	1.6000
71	0.0000	0.0000	0.0000	3.8770	**4.0000**	3.4118	−1.6000

TABLE 1. A few samples of iteration (1) for various combination of (α, β). The first column indicates the iteration number n. The second column is for (α, β) that satisfies: $\alpha^2 + 4\beta > 0$, and both $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The third and fourth columns are for $\alpha^2 + 4\beta = 0$. The fifth column is an example of quasi-periodic behavior, while the sixth column is 7-periodic behavior. These are for $\alpha^2 + 4\beta < 0$. The seventh column is for $\alpha + \beta = 1$, while the eighth is for $\beta - \alpha = 1$.

Convergence to a nontrivial limit. As the last example to be discussed, we have choose: $\alpha = 0.3000$ and $\beta = 0.7000$. Clearly, the choice of α and β in this example, satisfy: $\alpha + \beta = 1$. In this case, a nontrivial limit exists, i.e.

$$x_1 + \frac{x_2 - x_1}{2 - \alpha} = 4 + \frac{3 - 4}{2 - 0.3} \approx 3.4118.$$

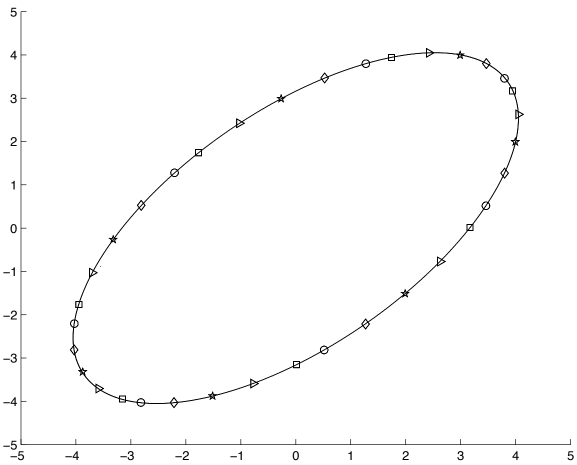

FIGURE 4. In this figure we have plotted an ellipse which is formed by 5000 iterations of (1) for $\alpha = 1.25$ and $\beta = -1$. The initial conditions are $x_1 = 4$ and $x_2 = 3$. This iterations densely fill up an ellipse. The horizontal axis is x_n while the vertical axis is x_{n+1}. On the ellipse, there are five 7-periodic sequences (see Definition 2.1) which are obtained by 150 iterations of (1) for $\alpha = 2\cos\frac{2\pi}{7}$ and $\beta = -1$, but five different sets of initial conditions: $\{x_1 = 3.4589, x_2 = 0.5159\}$ ($\circ$), $\{x_1 = -2.2150, x_2 = -4.0321\}$ ($\diamond$), $\{x_1 = -3.3212, x_2 = -0.2652\}$ ($\star$), $\{x_1 = 2.4215, x_2 = 4.0486\}$ ($\triangleright$), and $\{x_1 = 3.1707, x_2 = 3.1707\}$ ($\square$).

The eigenvalues of $A(\alpha, \beta)$ are 1 and -0.7; thus the convergence is similar to the first three examples.

5. SEQUENCE OF RATIOS OF THE GENERALIZED FIBONACI SEQUENCE

Let us now look at the ratios of the Generalized Fibonaci sequence (1). Dividing (1) by x_{n+1} we have:

$$\frac{x_{n+2}}{x_{n+1}} = \alpha + \beta\frac{x_n}{x_{n+1}},$$

assuming $x_n \neq 0$ for all n. Writing: $\rho_n = \frac{x_{n+1}}{x_n}$, we have the following recursive formula.

$$\rho_{n+1} = \alpha + \frac{\beta}{\rho_n}, n \in \mathbb{N}. \tag{6}$$

Note that if (6) converges, then the limiting point satisfies:

$$\rho^2 - \alpha\rho - \beta = 0,$$

which is also the equation for eigenvalues of $A(\alpha, \beta)$. Thus, if (6) converges, it converges to an eigenvalue of $A(\alpha, \beta)$. This is not surprising, since the ratio: ρ_n "measures" the growth of x_n, while the eigenvalue measures the growth of $\boldsymbol{v}_n$.

As a sequence of real numbers, it might be interesting to discuss the convergence of (6) for general value of α and β. However, we are interested on using (6) as an alternative way to construct rational approximation for radicals. Thus, our interest is restricted to the cases where: $\alpha^2 + 4\beta > 0$. In particular, for $\alpha > 0$, $\beta > 0$, $x_0 > 0$ and $y_0 > 0$, then (1) defines a monotonic increasing sequence. As a consequence, $\rho_n > 1$ for all n. Then we have, the sequence (6) converges if $\beta < 1$. Let $\beta < 1$, then the sequence (6) converges to a positive root of: $\rho^2 - \alpha\rho - \beta$, i.e.:

$$\rho = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2},$$

for every $\alpha > 0$, $\rho_1 > 1$ as long as $0 < \beta < 1$.

The sequence (6) can be used for constructing an approximation for an algebraic number: $\sqrt{r}$, $1 < r \in \mathbb{Q}$, by setting: $r = \alpha^2 + 4\beta$, where α and β are rationals. To do this, we translate the sequence (6) by setting

$$r_n = 2\rho_n - \alpha.$$

Then (6) becomes:

$$r_{n+1} = \alpha + \frac{4\beta}{r_n + \alpha}, \tag{7}$$

which converges to: $\sqrt{\alpha^2 + 4\beta}$.

For example, we want to construct an approximation for $\sqrt{7}$. We need to find a rational solution for: $7 = \alpha^2 + 4\beta$. Then we can set:

$$\beta = \frac{3}{4} \text{ and } \alpha = 2.$$

This solution is clearly not unique. By remembering that: 7 can be written as

$$\frac{28}{2^2}, \frac{63}{3^2}, \text{ or } \frac{175}{5^2},$$

we can choose α to be

$$\frac{5}{2}, \frac{7}{3} \text{ or } \frac{13}{5},$$

while β is:

$$\frac{3}{16}, \frac{7}{18}, \text{ or } \frac{3}{50},$$

respectively. The result is presented in Table 2.

The results in Table 2 is an agreement with our analysis. The convergence of (6) depends on the value of β. When β is closer to 0.75 then it takes 18 iteration to converge, while when $\beta = 0.06$ then it takes only 8 iteration. Furthermore, the convergence seems to be independent with the distance of the initial condition to the actual value of the radical.

	α	2	$\frac{5}{2}$	$\frac{7}{3}$	$\frac{13}{5}$
	β	$\frac{3}{4}=0.75$	$\frac{3}{16}=0.1875$	$\frac{7}{18}\approx 0.389$	$\frac{3}{50}=0.06$
$r_1=8$	Iteration	18	10	13	8
	Error	10^{-15}	10^{-15}	10^{-16}	10^{-17}
$r_2=1.2$	Iteration	18	10	13	8
	Error	10^{-15}	10^{-15}	10^{-16}	10^{-17}

TABLE 2. In this table we present some results on approximating $\sqrt{7}$ by (7).

The rate of convergence. Let $f(x)=x^2-\alpha x-\beta$, then

$$f'(x)=2x-\alpha, \text{ and } f''(x)=2.$$

Suppose that ρ is the limit of (6) and let $e_n=\rho-\rho_n$, $n\in\mathbb{N}$. Then

$$\begin{aligned} 0 &= f(\rho) \\ &= f(e_n+\rho_n) \\ &= f(\rho_n)+f'(\rho_n)e_n+\frac{1}{2}f''(\rho_n){e_n}^2+O({e_n}^3) \end{aligned}$$

Thus,

$$\frac{f(\rho_n)}{\rho_n}=-\frac{2\rho_n-\alpha}{\rho_n}e_n-\frac{1}{\rho_n}{e_n}^2+O({e_n}^3)=-\left(2-\frac{\alpha}{\rho_n}\right)e_n+O({e_n}^2).$$

Since:

$$e_{n+1}=\rho-\rho_{n+1}=\rho-\alpha-\frac{\beta}{\rho_n}=\rho-\rho_n+\frac{f(\rho_n)}{\rho_n}=e_n+\frac{f(\rho_n)}{\rho_n},$$

then

$$e_{n+1}=\left(\frac{\alpha}{\rho_n}-1\right)e_n+O({e_n}^2).$$

Thus, in contrast with the quadratic convergence of Newton iteration, the convergence of (6) is linear.

Generalization to higher dimension. Interesting generalization to higher dimension is as the following. Let us consider:

$$x_{n+3}=\alpha x_{n+2}+\beta x_{n+1}+\gamma x_n, \tag{8}$$

where $n \in \mathbb{N}$. Let us assume that the initial conditions are positive, i.e.: $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$. Furthermore: α, β and γ are also positive. If at least one of them is greater than one, then (8) is monotonically increasing. Dividing (8) by x_{n+2}, gives:

$$\frac{x_{n+3}}{x_{n+2}} = \alpha + \beta\frac{x_{n+1}}{x_{n+2}} + \gamma\frac{x_n}{x_{n+2}}.$$

$\rho_n = \frac{x_{n+1}}{x_n}$, we have:

$$\rho_{n+2} = \alpha + \frac{\beta}{\rho_{n+1}} + \frac{\gamma}{\rho_n\rho_{n+1}} \tag{9}$$

Thus, the limit of (9), if exists, satisfies:

$$\rho^3 - \alpha\rho^2 - \beta\rho - \gamma = 0. \tag{10}$$

Let

$$p = -\beta - \frac{\alpha^2}{3}, \text{ and } q = -\frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} - \gamma,$$

and furthermore:

$$P = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \text{ and } Q = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

Then, the solution for (10) are:

$$\begin{gathered} P + Q + \tfrac{\alpha}{3}, \\ -\frac{P+Q}{2} + \frac{\alpha}{3} - \mathrm{i}\frac{1}{2}\sqrt{3}\,(P+Q)\,, \text{ and} \\ -\frac{P+Q}{2} + \frac{\alpha}{3} + \mathrm{i}\frac{1}{2}\sqrt{3}\,(P+Q)\,. \end{gathered}$$

This is known as Cardano's solution for cubic equation. If

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4} > 0,$$

then the equation (10) has a unique real solution.

Let us look at an example, where: $\alpha = 3$, $\beta = \frac{1}{3}$, and $\gamma = \frac{1}{9}$. For these choices, we have: $p = -\frac{2}{3}$ and $q = -\frac{8}{27}$, while $\Delta = \frac{8}{729}$. Then, the Real solution for (10), is

$$r = \sqrt[3]{4 + 2\sqrt{2}} + \sqrt[3]{4 - 2\sqrt{2}} + \frac{1}{3} \approx 1.317124345.$$

By generating the sequence (8), and then computing the ratio, after 22 iterations the same accuracy is achieved for approximating r.

6. Concluding remarks

We have studied the convergence of the *two terms recurrent* sequence, also known as the generalized Fibonacci sequence. We have derived an open domain Ω around the origin of (α, β)-plane, where the sequence converges to 0. On the boundary of that domain, interesting behavior, such as: convergent to a nontrivial limit, period behavior or quasi-periodic behavior have been analyzed. There are still possibilities that for (α, β) which are not in Ω the sequence converges. However, we have to choose carefully the initial conditions.

Interesting application that we have proposed is an alternative for constructing a rational approximation to algebraic numbers: $\sqrt{r}$, where $r \in \mathbb{Q}$, i.e. the formula (7). A slight generalization is by considering (8) to approximate algebraic numbers of the form:

$$\sqrt[3]{a + b\sqrt{c}} + \sqrt[3]{a - b\sqrt{c}} + d$$

In contrast with the previous formula (i.e. (7)), the computation is less cumbersome if one construct the sequence using (8), and then directly construct the ratios.

The proof for the convergence of iteration (9) is still open. In the next study, our attention will be devoted on proving the convergence of (9), for some value of α, β and γ.

References

[1] Atkins, J. and Geist, R., "Fibonacci numbers and computer algorithms", *College Math. J.*, **18**, (1987), 328–337.

[2] Cooper, C., "Classroom capsules: Application of a generalized Fibonacci sequence", *College Math. J.*, **15** (1984), 145–148.

[3] Edson, M. and Yayenie, O., "A new generalization of Fibonacci Sequence & Extended Binet's formula", *Integers* **9** (2009), 639-654.

[4] Kalman, D. and Mena, R., "The Fibonacci Numbers—Exposed", *Mathematics Magazine*, **76:3** (June 2003), 167–181.

[5] Katz, V. J., *A History of Mathematics: an introduction*, 2nd ed., Addison-Wesley, Reading, Massachusetts, 1998.

[6] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York, 2001

[7] Li, H-C., "Complete and reduced residue systems of second-order recurrences modulo p", *Fibonacci Quart.*, **38** (2000), 272–281.

[8] Lucas, E., "Théorie des fonctions numériques simplement périodiques", *Amer. J. Math.*, **1** (1878), 184–240, 289–321.

[9] Sun, Z-H. and Sun, Z-W., "Fibonacci numbers and Fermats last theorem", *Acta Arith.*, **60** (1992), 371–388.

[10] Tuwankotta, J.M., "Contractive Sequence", *Majalah Ilmiah Matematika dan Ilmu Pengetahuan Alam: INTEGRAL*, **2:2** (Oktober 1997), FMIPA Universitas Parahyangan.

[11] Walton, J.E. and Horadam, A.F., "Some Aspects of Generalized Fibonacci Sequence", *The Fibonacci Quaterly*, 12 (3) (1974), 241–250,

[12] Wall, D.D., "Fibonacci series modulo m", *Amer. Math. Monthly*, **67** (1960), 525–532.