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Some Topological Indices of Order Divisor Graphs of Cyclic Groups

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Abstract. This study investigates order divisor graphs' structural and topological properties derived from cyclic groups. Focusing on the relationship between group order and graph topology, we explore key indices, including the Wiener index, the Harary index, the first Zagreb index, and the second Zagreb index. We use a case-based approach to analyze graphs for cyclic groups of varying orders, from prime powers to more general composite structures. This work extends the theoretical framework of order divisor graphs and provides explicit formulations for their topological indices, highlighting the interplay between algebraic and graph-theoretic properties. These findings contribute to the broader understanding of algebraic graph theory and its applications.

Key words and phrases: order divisor graphs, cyclic groups, Wiener index, Harary index, Zagreb index

1. INTRODUCTION

In 1878, Arthur Cayley introduced the concept of representing groups using a graph known as a Cayley graph [1]. Visualizing groups in graphs has evolved and is now commonly called algebraic graphs. The authors first introduced Prime graphs of finite groups in [2]. In the case of non-abelian groups, a non-commuting graph is established based on the commutative property, as described in [3]. In 2016, Mansoori et al. proposed the idea of a non-coprime graph for finite groups, which considers the order of elements within the group [4]. Another algebraic graph that considers the order of group elements is the order divisor graph of finite groups, which was discussed by Rehman et al. in [5]. The order divisor graph of finite groups, denoted as $OD(G)$, comprises vertices from the group G. In this graph, two distinct vertices, a and b, each with different orders, are connected by an edge if and only if the order of a (represented as $|a|$) divides the order of b, or vice versa. An edge exists between two vertices if one vertex's order divides the other's or vice

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versa. Furthermore, in [6], the signed total domination number of order divisor graphs for specific finite groups was discussed. Additionally, a generalization of order divisor graphs was presented in [7].

There are many studies on graph theory, one of which is related to topological indices (e.g., see [8], [9], [10], [11]). Commonly acknowledged topological indices encompass the Wiener and Harary indexes as the first and second Zagreb indices. The Wiener index $W(\gamma)$ of a connected graph γ, is the sum of the form

$$W(\gamma) = \sum_{\{x,y\}\subseteq V(\gamma)} d(x,y),$$

where $d(x,y)$ is the distance of x and y [9]. Analogously, the Harary index $H(\gamma)$ of a connected graph γ is the sum

$$H(\gamma) = \sum_{\{x,y\}\subseteq V(\gamma)} \frac{1}{d(x,y)}$$

[8]. The first Zagreb index of a graph γ, denoted by $M_1(\gamma)$, is defined as

$$M_1(\gamma) = \sum_{x\in V(\gamma)} (deg(x))^2,$$

where $deg(x)$ is the degree of x, i.e. the number of edges that incident to x [10]. The second Zagreb index of a graph γ is defined as the sum of all the multiplications between the degrees of the two adjacent vertices, i.e.,

$$M_2(\gamma) = \sum_{xy\in E(\gamma)} deg(x)deg(y)$$

[10].

So far, no investigation has been conducted into the Wiener index, Harary index, first Zagreb index, and second Zagreb index of the order divisor graph. Therefore, we are interested in examining these indices, specifically within order divisor graphs about cyclic groups. In this paper, all concepts and notations related to groups are referred to [12]. In this paper, all groups discussed are finite and the identity element is represented by e.

2. MAIN RESULTS

This section will discuss several topological indices of the order divisor graph of a cyclic group of order n. The discussion is divided into several cases of n, i.e. for $n = p$, $n = p^k$, $n = p_1p_2$, and $n = p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ where $p, p_1, p_2, \dots, p_m$ are prime.

For prime p, the structure of the order divisor graph of group $\mathbb{Z}_p$ is given by [5] as follows.

Recall that a star graph is a connected graph consisting of some vertices of degree one that are incident to a common vertex.

Theorem 2.1. [5] *The order divisor graph $OD(\mathbb{Z}_p)$ is a star graph if and only if p is a prime.*

The corollary presented in the following is a direct consequence of Theorem 2.1.

Corollary 2.2. *Let G be a cyclic group of order p. The order divisor graph $OD(G)$ is a star graph if and only if p is a prime.*

Below are the theorems that outline the Wiener index, Harary index, and the first and second Zagreb indices for the order divisor graph of cyclic groups with prime order p.

Theorem 2.3. *Let p be a prime number. If G is a cyclic group of order p, then the Wiener index of $OD(G)$ is $W(OD(G)) = (p-1)^2$.*

Proof. Consider a cyclic group G of prime order p. The vertex e is connected to all other vertices; thus, for any pair of non-adjacent vertices $x, y \in V(OD(G)) \setminus \{e\}$, the distance $d(x, y)$ equals 2. By Corollary 2.2, $OD(G)$ is a star graph, then we have

$$\begin{aligned}
W(OD(G)) &= \sum_{\{x,y\} \subseteq V(OD(G))} d(x,y) \\
&= \sum_{x \in V(OD(G)) \setminus \{e\}} d(e,x) + \sum_{\{x,y\} \subseteq V(OD(G)) \setminus \{e\}} d(x,y) \\
&= (p-1).1 + \binom{p-1}{2}.2 \\
&= (p-1) + (p-1)(p-2) \\
&= (p-1)^2.
\end{aligned}$$

□

Theorem 2.4. *Let p be a prime number. If G is a cyclic group of order p, then the Harary index of $OD(G)$ is $H(OD(G)) = \frac{(p-1)(p+2)}{4}$.*

Proof. Consider a cyclic group G of prime order p. The vertex e is adjacent to all other vertices, so for any two vertices that are not adjacent $x, y \in V(OD(G)) \setminus \{e\}$ the distance $d(x, y)$ equals 2. By Corollary 2.2, $OD(G)$ is a star graph, then we have

$$\begin{aligned}
H(OD(G)) &= \sum_{\{x,y\} \subseteq V(OD(G))} \frac{1}{d(x,y)} \\
&= \sum_{x \in V(OD(G)) \setminus \{e\}} \frac{1}{d(e,x)} + \sum_{\{x,y\} \subseteq V(OD(G)) \setminus \{e\}} \frac{1}{d(x,y)} \\
&= (p-1).1 + \binom{p-1}{2}.\frac{1}{2}
\end{aligned}$$

$$= (p-1) + \frac{(p-1)(p-2)}{4}$$
$$= \frac{(p-1)(p+2)}{4}.$$

□

Theorem 2.5. *Let p be a prime number. If G is a cyclic group of order p, then the first Zagreb index of $OD(G)$ is $M_1(OD(G)) = p^2 - p$.*

Proof. Consider a cyclic group G with a prime order p. Since the vertex e is connected to all other vertices, we have $deg(e) = p-1$. According to Corollary 2.2, $OD(G)$ forms a star graph, implying that the degree of each vertex $x \in V(OD(G)) \setminus \{e\}$ is 1. Hence,

$$\begin{aligned} M_1(OD(G)) &= \sum_{x \in V(OD(G))} (deg(x))^2 \\ &= (deg(e))^2 + \sum_{x \in V(OD(G)) \setminus \{e\}} (deg(x))^2 \\ &= (p-1)^2 + (p-1).1^2 \\ &= p^2 - p. \end{aligned}$$

□

Theorem 2.6. *Let p be a prime number. If G is a cyclic group of order p, then the second Zagreb index of $OD(G)$ is $M_2(OD(G)) = (p-1)^2$.*

Proof. Consider a cyclic group G of prime order p. Since the vertex e is connected to all other vertices, we have $deg(e) = p-1$. According to Corollary 2.2, $OD(G)$ forms a star graph, implying that the degree of each vertex $x \in V(OD(G)) \setminus \{e\}$ is 1. Hence,

$$\begin{aligned} M_2(OD(G)) &= \sum_{xy \in E(OD(G))} deg(x)deg(y) \\ &= \sum_{x \in V(OD(G)) \setminus \{e\}} deg(e)deg(x) \\ &= \sum_{x \in V(OD(G)) \setminus \{e\}} (p-1).1 \\ &= (p-1)^2. \end{aligned}$$

□

For some prime number p and $k \in \mathbb{N}$, the structure of the order divisor graph of a cyclic group of order p^k is given by [5] as follows.

Theorem 2.7. [5] *Let p be a prime number and $k \in \mathbb{N}$. If G is a cyclic group of order p^k, then order divisor graph $OD(G)$ is a complete $(k+1)-$partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}$.*

The following theorems explain the Wiener index, Harary index, and the first and second Zagreb indices of an order divisor graph of a cyclic group of order p^k, whose structure is described in Theorem 2.7.

Theorem 2.8. *Let p be a prime number and $k \in \mathbb{N}$. If G is a cyclic group of order p^k, then the Wiener index of $OD(G)$ is*

$$W(OD(G)) = (p^k - 1) + \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1)-1) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.$$

Proof. Consider G as a cyclic group of order p^k, where p is a prime number and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)-$partite graph

$$K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}.$$

Let the corresponding partition on $V(OD(G))$ be $\{P_0, P_1, \ldots, P_k\}$ such that

$$|P_0| = 1, |P_1| = p-1, |P_2| = p(p-1), \ldots, |P_k| = p^{k-1}(p-1).$$

Clearly, for every distinct vertices $x \in P_i$ and $y \in P_j$ we have

$$d(x,y) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

for all $i, j = 0, 1, 2, \ldots, k$. Therefore,

$$\begin{aligned}
W(OD(G)) = & \sum_{\{x,y\} \subseteq V(OD(G))} d(x,y) \\
= & \sum_{i=1}^{k} \sum_{\{x,y\} \subseteq P_i} d(x,y) + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \sum_{x \in P_i} \sum_{y \in P_j} d(x,y) \\
= & \sum_{i=1}^{k} \binom{p^{i-1}(p-1)}{2} .2 + \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i|.|P_j|.1 \\
= & \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1)-1) + \sum_{j=1}^{k} |P_0|.|P_j| \\
& + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (p^{i-1}(p-1))(p^{j-1}(p-1)) \\
= & (p^k - 1) + \sum_{i=1}^{k} (p^{i-1}(p-1))(p^{i-1}(p-1)-1) \\
& + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.
\end{aligned}$$

□

Theorem 2.9. *Let p be a prime number and $k \in \mathbb{N}$. If G is a cyclic group of order p^k, then the Harary index of $OD(G)$ is*

$$H(OD(G)) = (p^k-1)+\sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4}+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2.$$

PROOF. Let G be a cyclic group of order p^k, where p is a prime number and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)-$partite graph

$$K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{k-1}(p-1)}.$$

Again, let the corresponding partition on $V(OD(G))$ be $\{P_0, P_1, \ldots, P_k\}$ such that

$$|P_0| = 1, |P_1| = p-1, |P_2| = p(p-1), \ldots, |P_k| = p^{k-1}(p-1).$$

Hence, for every distinct vertices $x \in P_i$ and $y \in P_j$ we have

$$d(x,y) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

for all $i, j = 0, 1, 2, \ldots, k$. Therefore,

$$\begin{aligned} H(OD(G)) =& \sum_{\{x,y\}\subseteq V(OD(G))} \frac{1}{d(x,y)} \\ =& \sum_{i=1}^{k} \sum_{\{x,y\}\subseteq P_i} \frac{1}{d(x,y)} + \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}\sum_{x\in P_i}\sum_{y\in P_j} \frac{1}{d(x,y)} \\ =& \sum_{i=1}^{k} \binom{p^{i-1}(p-1)}{2} . \frac{1}{2} + \sum_{i=0}^{k-1}\sum_{j=i+1}^{k} |P_i|.|P_j|.1 \\ =& \sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4} + \sum_{j=1}^{k} |P_0|.|P_j| \\ &+ \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} (p^{i-1}(p-1))(p^{j-1}(p-1)) \\ =& (p^k-1) + \sum_{i=1}^{k} \frac{(p^{i-1}(p-1))(p^{i-1}(p-1)-1)}{4} \\ &+ \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} p^{i+j-2}(p-1)^2. \end{aligned}$$

□

Theorem 2.10. *Let p be a prime number and $k \in \mathbb{N}$. If G is a cyclic group of order p^k, then the first Zagreb index of $OD(G)$ is*

$$M_1(OD(G)) = (p^k - 1)^2 + \sum_{i=0}^{k-1} p^i(p-1)(p^k - p^i(p-1))^2.$$

Proof. Let us consider G as a cyclic group of order p^k, where p is a prime number and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)-$partite graph

$$K_{1,p-1,p(p-1),p^2(p-1),\dots,p^{k-1}(p-1)}.$$

Again, let the corresponding partition on $V(OD(G))$ be $\{P_0, P_1, \dots, P_k\}$ such that

$$|P_0| = 1, |P_1| = p-1, |P_2| = p(p-1), \dots, |P_k| = p^{k-1}(p-1).$$

Thus, for all $i = 0, 1, 2, \dots, k$, we have $deg(x) = p^k - |P_i|$, for every $x \in P_i$. Therefore,

$$\begin{aligned}
M_1(OD(G)) &= \sum_{x \in V(OD(G))} (deg(x))^2 \\
&= \sum_{x \in P_0} (deg(x))^2 + \sum_{x \in P_1} (deg(x))^2 + \dots + \sum_{x \in P_k} (deg(x))^2 \\
&= 1.(p^k - 1)^2 + (p-1)(p^k - (p-1))^2 + \dots \\
&\quad + p^{k-1}(p-1)(p^k - p^{k-1}(p-1))^2 \\
&= (p^k - 1)^2 + \sum_{i=0}^{k-1} p^i(p-1)(p^k - p^i(p-1))^2.
\end{aligned}$$

□

Theorem 2.11. *Let p be a prime number and $k \in \mathbb{N}$. If G is a cyclic group of order p^k, then the second Zagreb index of $OD(G)$ is*

$$M_2(OD(G)) = \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i||P_j|(p^k - |P_i|)(p^k - |P_j|).$$

Proof. Consider G as a cyclic group of order p^k, where p is a prime number and $k \in \mathbb{N}$. By Theorem 2.7, $OD(G)$ is a complete $(k+1)-$partite graph

$$K_{1,p-1,p(p-1),p^2(p-1),\dots,p^{k-1}(p-1)}.$$

Again, let the corresponding partition on $V(OD(G))$ be $\{P_0, P_1, \dots, P_k\}$ such that

$$|P_0| = 1, |P_1| = p-1, |P_2| = p(p-1), \dots, |P_k| = p^{k-1}(p-1).$$

Hence, for all $i = 0, 1, 2, \dots, k$, we have $deg(x) = p^k - |P_i|$, for every $x \in P_i$. Therefore,

$$M_2(OD(G)) = \sum_{xy \in E(OD(G))} deg(x)deg(y)$$

$$
\begin{aligned}
&= \sum_{x \in P_0} \sum_{y \in V(OD(G)) \setminus P_0} deg(x)deg(y) \\
&\quad + \sum_{x \in P_1} \sum_{y \in V(OD(G)) \setminus (P_0 \cup P_1)} deg(x)deg(y) + \dots \\
&\quad + \sum_{x \in P_{k-1}} \sum_{y \in P_k} deg(x)deg(y) \\
&= \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} |P_i||P_j|(p^k - |P_i|)(p^k - |P_j|).
\end{aligned}
$$

□

The following theorem explains the structure of the order divisor graph of a cyclic group of order p_1p_2, where p_1 and p_2 are distinct primes.

Theorem 2.12. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then $OD(G)$ is complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$.*

Proof. Suppose G is a cyclic group of order p_1p_2, where the primes p_1 and p_2 are different. We take a partition $\{Q_0, Q_1, Q_2\}$ on $V(OD(G))$ with

$$
\begin{aligned}
Q_0 &= \{a \in G \mid |a| = 1\}, \\
Q_1 &= \{a \in G \mid |a| = p_1 \text{ atau } |a| = p_2\}, \\
Q_2 &= \{a \in G \mid |a| = p_1p_2\}.
\end{aligned}
$$

Taking note that G is a cyclic group, we find that $\phi(d)$ (i.e., the Euler phi function of d) is the number of elements of order d where $d \mid p_1p_2$. Hence $|Q_0| = 1$, $|Q_1| = \phi(p_1) + \phi(p_2) = p_1 + p_2 - 2$, and $|Q_2| = \phi(p_1p_2) = (p_1 - 1)(p_2 - 1)$.

Furthermore, take arbitrary $x \in Q_i$ and $y \in Q_j$ with $i, j \in \{0, 1, 2\}$. If $i = j$, then $|x| = |y|$ or $|x| = p_1 \neq p_2 = |y|$, so $xy \notin E(OD(G))$. If $i \neq j$, then $|x| \neq |y|$. Without loss of generality, suppose $i < j$. Thus $|x| \mid |y|$, so $xy \in E(OD(G))$. Therefore $OD(G)$ is complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$. □

The following theorems provide the order divisor graph's Wiener and Harary indices for cyclic groups of order p_1p_2.

Theorem 2.13. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then the Wiener index of $OD(G)$ is*

$$W(OD(G)) = (p_1+p_2-2)(p_1p_2-2)+(p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2)+(p_1p_2-1).$$

Proof. Given that p_1 and p_2 are two distinct primes, let G be a cyclic group of order p_1p_2. $OD(G)$ is a complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$ according to Theorem 2.12. Thus, for every distinct vertices $x \in Q_i$ and $y \in Q_j$ we have

$$d(x, y) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

for all $i, j = 0, 1, 2$. Therefore,

$$\begin{aligned}
W(OD(G)) =& \sum_{\{x,y\}\subseteq V(OD(G))} d(x,y) \\
=& \sum_{\{x,y\}\subseteq Q_1} d(x,y) + \sum_{\{x,y\}\subseteq Q_2} d(x,y) + \sum_{x\in Q_1\cup Q_2} d(e,x) + \sum_{x\in Q_1}\sum_{y\in Q_2} d(x,y) \\
=& \binom{p_1+p_2-2}{2}.2 + \binom{(p_1-1)(p_2-1)}{2}.2 + (p_1p_2-1) \\
&+ (p_1+p_2-2)(p_1-1)(p_2-1) \\
=& (p_1+p_2-2)(p_1p_2-2) + (p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2) \\
&+ (p_1p_2-1). \quad \square
\end{aligned}$$

Theorem 2.14. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then the Harary index of $OD(G)$ is*

$$\begin{aligned}
H(OD(G)) =& \frac{(p_1+p_2-2)(4p_1p_2-3p_1-3p_2+1)}{4} \\
&+ \frac{(p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2)}{4} + (p_1p_2-1).
\end{aligned}$$

Proof. Given that p_1 and p_2 are two distinct primes, let G be a cyclic group of order $n = p_1p_2$. $OD(G)$ is a complete tripartite graph $K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$ according to Theorem 2.12. Thus, for every distinct vertices $x \in Q_i$ and $y \in Q_j$ we have

$$d(x,y) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

for all $i, j = 0, 1, 2$. Therefore,

$$\begin{aligned}
H(OD(G)) =& \sum_{\{x,y\}\subseteq V(OD(G))} \frac{1}{d(x,y)} \\
=& \sum_{\{x,y\}\subseteq Q_1} \frac{1}{d(x,y)} + \sum_{\{x,y\}\subseteq Q_2} \frac{1}{d(x,y)} + \sum_{x\in Q_1\cup Q_2} \frac{1}{d(e,x)} + \sum_{x\in Q_1}\sum_{y\in Q_2} \frac{1}{d(x,y)} \\
=& \binom{p_1+p_2-2}{2}.\frac{1}{2} + \binom{(p_1-1)(p_2-1)}{2}.\frac{1}{2} + (p_1p_2-1) \\
&+ (p_1+p_2-2)(p_1-1)(p_2-1) \\
=& \frac{(p_1+p_2-2)(4p_1p_2-3p_1-3p_2+1)}{4} \\
&+ \frac{(p_1p_2-p_1-p_2+1)(p_1p_2-p_1-p_2)}{4} + (p_1p_2-1).
\end{aligned}$$

$\square$

Before discussing the first Zagreb and the second Zagreb indices of order divisor graph of the cyclic group of order p_1p_2, we present the following lemma as a consequence of Theorem 2.12.

Lemma 2.15. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then*

$$deg(x) = \begin{cases} p_1p_2 - 1, & x \in Q_0 \\ 1 + (p_1 - 1)(p_2 - 1), & x \in Q_1 \\ p_1 + p_2 - 1, & x \in Q_2, \end{cases}$$

with $\{Q_0, Q_1, Q_3\}$ is the partition of the vertices in graph $OD(G)$ as given in Theorem 2.12.

PROOF. Given that p_1 and p_2 are two distinct primes, let G be a cyclic group of order p_1p_2. $OD(G)$ is a complete tripartite graph

$$K_{1,p_1+p_2-2,(p_1-1)(p_2-1)}$$

according to Theorem 2.12. There are some conditions as follows.

(1) For partition Q_0, we have $Q_0 = \{e\}$, so $deg(e) = n - 1 = p_1p_2 - 1$.
(2) For all $x \in Q_1$, $xy \in E(OD(G))$ for every $y \in Q_0 \cup Q_2$, so $deg(x) = 1 + (p_1 - 1)(p_2 - 1)$.
(3) For all $x \in Q_2$, $xy \in E(OD(G))$ for every $y \in Q_0 \cup Q_1$, so $deg(x) = p_1 + p_2 - 1$.

□

The following theorems explain the first Zagreb and the second Zagreb indices of an order divisor graph of a p_1p_2-order cyclic group.

Theorem 2.16. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then the first Zagreb index of $OD(G)$ is*

$$\begin{aligned} M_1(OD(G)) =&(p_1p_2 - 1)^2 + (p_1 + p_2 - 2)(1 + (p_1 - 1)(p_2 - 1))^2 \\ &(p_1 - 1)(p_2 - 1)(p_1 + p_2 - 1)^2. \end{aligned}$$

PROOF. Given that p_1 and p_2 are two distinct primes, let G be a cyclic group of order p_1p_2. By Lemma 2.15, we get

$$\begin{aligned} M_1(OD(G)) &= \sum_{x \in V(OD(G))} (deg(x))^2 \\ &= \sum_{x \in Q_0} (deg(x))^2 + \sum_{x \in Q_1} (deg(x))^2 + \sum_{x \in Q_2} (deg(x))^2 \\ &=(p_1p_2 - 1)^2 + (p_1 + p_2 - 2)(1 + (p_1 - 1)(p_2 - 1))^2 \\ &\quad + (p_1 - 1)(p_2 - 1)(p_1 + p_2 - 1)^2. \end{aligned}$$

□

Theorem 2.17. *Let p_1 and p_2 be two distinct primes. If G is a cyclic group of order p_1p_2, then the second Zagreb index of $OD(G)$ is*

$$\begin{aligned} M_2(OD(G)) =&(p_1+p_2-2)(p_1p_2-1)(1+(p_1-1)(p_2-1)) \\ &+(p_1-1)(p_2-1)(p_1p_2-1)(p_1+p_2-1) \\ &+(p_1+p_2-2)(p_1-1)(p_2-1)(1+(p_1-1)(p_2-1))(p_1+p_2-1). \end{aligned}$$

Proof. Given that p_1 and p_2 are two distinct primes, let G be a cyclic group of order p_1p_2. By Lemma 2.15, we get

$$\begin{aligned} M_2(OD(G)) =& \sum_{xy\in E(OD(G))} deg(x)deg(y) \\ =& \sum_{x\in Q_1} deg(e)deg(x) + \sum_{x\in Q_2} deg(e)deg(x) + \sum_{x\in Q_1}\sum_{y\in Q_2} deg(x)deg(y) \\ =&(p_1+p_2-2)(p_1p_2-1)(1+(p_1-1)(p_2-1)) \\ &+(p_1-1)(p_2-1)(p_1p_2-1)(p_1+p_2-1) \\ &+(p_1+p_2-2)(p_1-1)(p_2-1)(1+(p_1-1)(p_2-1))(p_1+p_2-1). \end{aligned}$$

□

For all $i=1,2,\ldots,m$, p_i are distinct primes and $k_i\in\mathbb{N}$. The following theorem describes the structure of the order divisor graph of a cyclic group of order $p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$.

Theorem 2.18. *Let p_i be distinct primes and $k_i\in\mathbb{N}$ for all $i=1,2,\ldots,m$. If G is a cyclic group of order $p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$, then the order divisor graph $OD(G)$ is $(k_1+k_2+\cdots+k_m+1)-$partite graph.*

Proof. Consider G as a cyclic group of order $p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$, where p_i are distinct primes and $k_i\in\mathbb{N}$ for all $i=1,2,\ldots,m$. For each $i=0,1,2,\ldots,k_1+k_2+\cdots+k_m$, consider the partition $\{R_0,R_1,R_2,\ldots,\ R_{k_1+k_2+\cdots+k_m}\}$ on $V(OD(G))$, where the set R_i comprises every element in G whose order is a multiplication of i prime numbers (which may be the same). For each $i=0,1,2,\ldots,k_1+k_2+\cdots+k_m$, take any $x,y\in R_i$. Then $o(x)$ and $o(y)$ is a multiplication of i prime numbers. Therefore $o(x)\nmid o(y)$ and $o(y)\nmid o(x)$, so $xy\notin E(OD(G))$. Hence order divisor graph $OD(G)$ is $(k_1+k_2+\cdots+k_m+1)-$partite graph. □

Given a cyclic group G of order $n=p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$. Define $D(n)$ as the set of all positive factors of n. Take a partition $\{A_d|d\in D(n)\}$ on $V(OD(G))$, with

$$A_d=\{v\in V(OD(G)) \mid |v|=d\}.$$

Note that as G is a cyclic group, $|A_d|=\phi(d)$ where $\phi(d)$ is Euler phi function of d. The following two theorems describe the Wiener and Harary indices of the order divisor graph of a cyclic group of order $p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$.

Theorem 2.19. *Let p_i be distinct primes and $k_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, m$. If G is a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, then the Wiener index of $OD(G)$ is*

$$W(OD(G)) = \sum_{d \in D(n)} 2\binom{\phi(d)}{2} + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s|t} \phi(s)\phi(t) + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s<t, s \nmid t} 2\phi(s)\phi(t).$$

PROOF. Let G be a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, where p_i are distinct primes and $k_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, m$. For arbitrary $s, t \in D(n)$, for every $x \in A_s$ and $y \in A_t$, there are three conditions as the following:

(1) if $s = t$, then $d(x, y) = 2$;
(2) if $s \neq t$ and $s \mid t$ or $t \mid s$, then $d(x, y) = 1$;
(3) if $s \neq t$, $s \nmid t$, and $t \nmid s$, then $d(x, y) = 2$.

Therefore,

$$\begin{aligned} W(OD(G)) &= \sum_{\{x,y\} \subseteq V(OD(G))} d(x, y) \\ &= \sum_{d \in D(n)} \sum_{x,y \in A_d} d(x, y) + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s|t} \sum_{x \in A_s} \sum_{y \in A_t} d(x, y) \\ &\quad + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s<t, s \nmid t} \sum_{x \in A_s} \sum_{y \in A_t} d(x, y) \\ &= \sum_{d \in D(n)} 2\binom{\phi(d)}{2} + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s|t} \phi(s)\phi(t) \\ &\quad + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s<t, s \nmid t} 2\phi(s)\phi(t). \end{aligned}$$

□

Theorem 2.20. *Let p_i be distinct primes and $k_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, m$. If G is a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, then the Harary index of $OD(G)$ is*

$$H(OD(G)) = \sum_{d \in D(n)} \frac{1}{2}\binom{\phi(d)}{2} + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s|t} \phi(s)\phi(t) + \sum_{s \in D(n)} \sum_{t \in D(n) \setminus \{s\}, s<t, s \nmid t} \frac{1}{2}\phi(s)\phi(t).$$

PROOF. Let G be a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, where p_i are distinct primes and $k_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, m$. Analogous to the Theorem 2.19, we have

$$H(OD(G)) = \sum_{\{x,y\} \subseteq V(OD(G))} \frac{1}{d(x, y)}$$

$$= \sum_{d\in D(n)} \sum_{x,y\in A_d} \frac{1}{d(x,y)} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \sum_{x\in A_s} \sum_{y\in A_t} \frac{1}{d(x,y)}$$
$$+ \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \sum_{x\in A_s} \sum_{y\in A_t} \frac{1}{d(x,y)}$$
$$= \sum_{d\in D(n)} \frac{1}{2}\binom{\phi(d)}{2} + \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s|t} \phi(s)\phi(t)$$
$$+ \sum_{s\in D(n)} \sum_{t\in D(n)\setminus\{s\},s<t,s\nmid t} \frac{1}{2}\phi(s)\phi(t).$$

□

The following lemma is useful in the theorem that characterizes the first Zagreb and the second Zagreb indices of an order divisor graph of a cyclic group of order $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$.

Lemma 2.21. *Let p_i be distinct primes and $k_i \in \mathbb{N}$ for all $i = 1,2,\dots,m$. If G is a cyclic group of order $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$, then*

$$deg(x) = \sum_{s\in D(n)\setminus\{d\},s|d} \phi(s) + \sum_{t\in D(n)\setminus\{d\},d|t} \phi(t)$$

for every $x \in A_d$.

The following theorems explain the first Zagreb and the second Zagreb indices of an order divisor graph of a cyclic group of order $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$.

Theorem 2.22. *Let p_i be distinct primes and $k_i \in \mathbb{N}$ for all $i = 1,2,\dots,m$. If G is a cyclic group of order $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$, then the first Zagreb index of $OD(G)$ is*

$$M_1(OD(G)) = \sum_{d\in D(n)} \phi(d)\left(\sum_{s\in D(n)\setminus\{d\},s|d} \phi(s) + \sum_{t\in D(n)\setminus\{d\},d|t} \phi(t)\right)^2.$$

Proof. For all $i = 1,2,\dots,m$, p_i are distinct primes, and $k_i \in \mathbb{N}$, let G be a cyclic group of order $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$. By Lemma 2.21 we have,

$$M_1(OD(G)) = \sum_{x\in V(OD(G))} (deg(x))^2$$
$$= \sum_{d\in D(n)} \sum_{x\in A_d} (deg(x))^2$$
$$= \sum_{d\in D(n)} \phi(d)\left(\sum_{s\in D(n)\setminus\{d\},s|d} \phi(s) + \sum_{t\in D(n)\setminus\{d\},d|t} \phi(t)\right)^2.$$

□

Theorem 2.23. *Let p_i be distinct primes and $k_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, m$. If G is a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$, then the second Zagreb index of $OD(G)$ is*

$$M_2(OD(G)) = \sum_{c \in D(n) \setminus \{n\}} \left(\phi(c) deg(A_c) \sum_{d \in D(n) \setminus \{c\}, c|d} \phi(d) deg(A_d) \right)$$

where

$$deg(A_c) = \sum_{s \in D(n) \setminus \{c\}, s|c} \phi(s) + \sum_{t \in D(n) \setminus \{c\}, c|t} \phi(t).$$

Proof. For all $i = 1, 2, \ldots, m$, p_i are distinct primes, and $k_i \in \mathbb{N}$, let G be a cyclic group of order $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$. By Lemma 2.21 we have,

$$\begin{aligned} M_2(OD(G)) &= \sum_{xy \in E(OD(G))} deg(x) deg(y) \\ &= \sum_{c \in D(n) \setminus \{n\}} \sum_{x \in A_c} \sum_{d \in D(n) \setminus \{c\}, c|d} \sum_{y \in A_d} deg(x) deg(y) \\ &= \sum_{c \in D(n) \setminus \{n\}} \sum_{x \in A_c} \left(deg(x) \sum_{d \in D(n) \setminus \{c\}, c|d} \sum_{y \in A_d} deg(y) \right) \\ &= \sum_{c \in D(n) \setminus \{n\}} \left(\phi(c) deg(A_c) \sum_{d \in D(n) \setminus \{c\}, c|d} \phi(d) deg(A_d) \right) \end{aligned}$$

where

$$deg(A_c) = \sum_{s \in D(n) \setminus \{c\}, s|c} \phi(s) + \sum_{t \in D(n) \setminus \{c\}, c|t} \phi(t).$$

□

3. Conclusion

This study comprehensively examined the structure of order divisor graphs (ODGs) associated with cyclic groups, focusing on how the group's order influences their graphical representation. Additionally, we derived and analyzed several key topological indices of these graphs, including the Wiener index, Harary index, first Zagreb index, and second Zagreb index. Through a detailed case-by-case exploration, we provided explicit formulations and insights into the relationship between the algebraic properties of cyclic groups and the corresponding graphical parameters, enriching the theoretical understanding of algebraic graph theory.

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