J. Indones. Math. Soc.
Vol. 31, No. 01 (2025), pp. 1–11.

On Energy of Prime Ideal Graph of A Commutative Ring Associated with Seidel-Based Matrices

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Abstract. Some energies of the prime ideal graph are found for a commutative ring associated with Seidel-based matrices including Seidel, Seidel Laplacian, and Seidel signless Laplacian matrices.

Key words and Phrases: Prime ideal graph, the energy of a graph, commutative ring, Seidel-based matrices

1. INTRODUCTION

In the study of mathematical chemistry, the discussion focuses on chemical graph theory [1]. The graph energy is described as *pi*-electron energy by considering the molecule as a graph [2]. The graph energy applications can be found in several research including the study of protein sequences [3], pattern and facial recognition [4], and object identification [5].

Initially in the graph, the energy is constructed from its adjacency matrix. Nowadays, the discussion of graph matrices is extended to degree or distance-based and Seidel-based matrices. This research focuses on the Seidel-based matrices, including Seidel [6], Seidel Laplacian [7], and Seidel signless Laplacian matrices [8].

Furthermore, graphs defined on group and ring are interesting topics in the last few decades, for instance, the prime ideal graph [9]. The topology properties of this graph have been done by Syarifudin, et al. [10] with the following definition.

Definition 1.1. [9] *The prime ideal graph is represented by $\Omega(R, I)$, where R is any commutative ring as the vertex set excluding $\{0\}$, I is its prime ideal which two distinct vertices u and v are linking with an edge whenever $uv \in I$.*

Through the years, discussion on the Seidel energy of a graph has developed. In 2021, Sarmin, et al. [11] presented the Seidel energy of the Cayley graph,

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2020 Mathematics Subject Classification: 05C25, 15A18
Received: 27-05-2024, accepted: 06-11-2024.

and Romdhini, et al. [12] investigated the Seidel-based energies of the commuting graph. Both research are constructed from the dihedral groups. Moreover, the Seidel matrix is also applied to the commuting graph for U_{6n} group [13].

The above background motivated us to extend the study to $\Omega(R, I)$ where R is associated with the Seidel-based matrices. In Section 2, we write several results from the previous literature. Then the formulation of energies is presented in Section 3.

2. PRELIMINARIES

In this part, we shall discuss the fundamental properties of $\Omega(R, I)$. First, we show the result from the previous literature from [10].

Let the cardinality of R is η, with $R\backslash\{0\} = \{p_1, p_2, \ldots, p_\mu, r_1, r_2, \ldots, r_{\eta-\mu-1}\}$ and $I = \{p_1, p_2, \ldots, p_\mu\}$. Let d_{v_p} be the degree of vertex v_p in $\Omega(R, I)$ which is the number of vertices adjacent to v_p. The distance between v_p and v_q in $\Omega(R, I)$ is the number of edges in the shortest path from v_p to v_q and is denoted by d_{pq}.

Theorem 2.1. *(Lemma 2.1 in* [10]*) The vertex degree of v_p in $\Omega(R, I)$ is*

$$d_{v_p} = \begin{cases} \eta - 2, & \text{for every } v_p \in I\backslash\{0\} \\ \mu - 1, & \text{for every } v_p \in R\backslash I. \end{cases}$$

Afterward, the distance between two vertices was explored in [10].

Theorem 2.2. *(Lemma 2.2 in* [10]*) The distance between v_p and v_q in $\Omega(R, I)$ is given by*

$$d_{pq} = \begin{cases} 1, & \text{for every } v_p \in I\backslash\{0\} \text{and } v_q \in R \\ 2, & \text{for every } v_p, v_q \in R\backslash I. \end{cases}$$

For the construction of the Seidel matrix of $\Omega(R, I)$, we need the following definition.

Definition 2.3. [6] *An $n \times n$ Seidel matrix of $\Omega(R, I)$ is $S(\Omega(R, I)) = [s_{pq}]$ in which (p, q)-th entry is*

$$s_{pq} = \begin{cases} -1, & \text{if } v_p \neq v_q \text{ are adjacent} \\ 1, & \text{if } v_p \neq v_q \text{ are not adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Let $S(\Omega(R, I))$ be the degree matrix of $\Omega(R, I)$ as $Diag(d_{v_1}, d_{v_2}, \ldots, d_{v_n})$. Now the Seidel-based definitions of $\Omega(R, I)$ are given below.

Definition 2.4. [7] *An $n \times n$ Seidel Laplacian matrix of $\Omega(R, I)$ is*

$$SL(\Omega(R, I)) = D(\Omega(R, I)) - S(\Omega(R, I)).$$

Definition 2.5. [8] *An $n \times n$ Seidel signless Laplacian matrix of $\Omega(R, I)$ is*

$$SSL(\Omega(R, I)) = D(\Omega(R, I)) + S(\Omega(R, I)).$$

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n numbers that stated as eigenvalues of $S(\Omega(R, I))$, hence the spectrum of $S(\Omega(R, I))$ is

$$Spec_S(\Omega(R, I)) = \left\{ \lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m} \right\},$$

where $k_1, k_2, \ldots, k_m$ are the respective multiplicities, and $m \leq n$. The Seidel energy of $\Omega(R, I)$ [2] is written by

$$E_S(\Omega(R, I)) = \sum_{i=1}^{m} k_i \left| \lambda_i \right|, \quad (1)$$

and the Seidel-spectral radius of $\Omega(R, I)$ [14] is

$$\rho_S(\Omega(R, I)) = max\{|\lambda| : \lambda \in Spec_s(\Omega(R, I))\}.$$

The above definitions also apply to $SL(\Omega(R, I))$ as well as $SSL(\Omega(R, I))$. Suppose I_n is the identity matrix of order n and J_n is $n \times n$ square matrix with all entries are 1. Furthermore, for calculating the eigenvalues of $\Omega(R, I)$, we need the following formulation of the determinant.

Lemma 2.6. *(Lemma 2.2 in* [15]*) For real numbers $\alpha, \beta, \gamma, \delta$, the determinant of*

$$\begin{vmatrix} (\lambda + \alpha)I_{n_1} - \alpha J_{n_1} & -\gamma J_{n_1 \times n_2} \\ -\delta J_{n_2 \times n_1} & (\lambda + \beta)I_{n_2} - \beta J_{n_2} \end{vmatrix}$$

can be declared in simple formula as

$$(\lambda + \alpha)^{n_1 - 1}(\lambda + \beta)^{n_2 - 1}\left((\lambda - (n_1 - 1)\alpha)(\lambda - (n_2 - 1)\beta) - n_1 n_2 \gamma\delta\right).$$

3. MAIN RESULTS

This section presents the $\Omega(R, I)$ energy corresponding with Seidel-based matrices. Firstly, we need to show the simplification of the following determinants.

Theorem 3.1. *For real numbers α, β, γ, the determinant of*

$$|M| = \begin{vmatrix} (\lambda - \alpha + \beta)I_\mu - \beta J_\mu & -\beta J_{\mu \times (\eta - \mu - 1)} \\ -\beta J_{(\eta - \mu - 1) \times \mu} & (\lambda - \gamma + \beta)I_{\eta - \mu - 1} + \beta J_{\eta - \mu - 1} \end{vmatrix}$$

can be declared in simple form as

$$(\lambda - \alpha + \beta)^{\mu - 1}(\lambda - \beta - \gamma)^{\eta - \mu - 2}\left((\lambda - \alpha - \beta(\mu - 1))(\lambda - \gamma + (\eta - \mu - 2)\beta) - \beta^2 \mu(\eta - \mu - 1)\right).$$

Proof. We apply some row and column operations. Let R_i be the i-th row and we denote C_i as the i-column of $|M|$. The first step, substitute R_{1+i} with $R'_{1+i} = R_{1+i} - R_i$ where $1 \leq i \leq \mu - 1$ and substitute $R_{\mu+1+i}$ with $R'_{\mu+1+i} = R_{\mu+1+i} - R_{\mu+1}$ in which $1 \leq i \leq \eta - \mu - 2$, thus

$$|M| = \begin{vmatrix} \lambda - \alpha & -\beta J_{1 \times (\mu - 2)} & -\beta & -\beta J_{1 \times (\eta - \mu - 2)} \\ -(\lambda - \alpha + \beta)J_{(\mu - 2) \times 1} & (\lambda - \alpha + \beta)I_{\mu - 2} & 0_{(\mu - 2) \times 1} & 0_{\eta - \mu - 2} \\ -\beta & -\beta J_{1 \times (\mu - 2)} & \lambda - \delta & \beta J_{1 \times (\eta - \mu - 2)} \\ 0_{(\eta - \mu - 2) \times 1} & 0_{\mu - 2} & (\lambda - \delta - \beta)J_{(\eta - \mu - 2) \times 1} & (\lambda - \delta - \beta)I_{\eta - \mu - 2} \end{vmatrix}.$$

Furthermore, we replace C_1 with $C_1^{'} = C_1 + C_2 + \ldots + C_{\mu-1}$ and replace C_μ with $C_\mu^{'} = C_\mu + C_{\mu+1} + \ldots + C_{\eta-2}$, then we have

$$|M| = \begin{vmatrix} \lambda-\alpha-\beta(\mu-1) & -\beta J_{1\times(\mu-2)} & -\beta(\eta-\mu-1) & -\beta J_{1\times(\eta-\mu-2)} \\ 0_{(\mu-2)\times 1} & (\lambda-\alpha+\beta)I_{\mu-2} & 0_{(\mu-2)\times 1} & 0_{\eta-\mu-2} \\ -\beta\mu & -\beta J_{1\times(\mu-2)} & \lambda-\delta+(\eta-\mu-2)\beta & \beta J_{1\times(\eta-\mu-2)} \\ 0_{(\eta-\mu-2)\times 1} & 0_{\mu-2} & 0_{(\eta-\mu-2)\times 1} & (\lambda-\delta-\beta)I_{\eta-\mu-2} \end{vmatrix}.$$

We replace R_μ with $R_\mu^{'} = R_\mu + \frac{\beta}{\lambda-\alpha+\beta}R_2 + \frac{\beta}{\lambda-\alpha+\beta}R_3 + \ldots + \frac{\beta}{\lambda-\alpha+\beta}R_{\mu-1}$ and following by replacing C_1 with $C_1^{'} = C_1 + \frac{\beta\mu}{\lambda-\delta+(\eta-\mu-1)\beta}C_\mu$, then we have

$$|M| = \begin{vmatrix} u & -\beta J_{1\times(\mu-2)} & -\beta(\eta-\mu-1) & -\beta J_{1\times(\eta-\mu-2)} \\ 0_{(\mu-2)\times 1} & (\lambda-\alpha+\beta)I_{\mu-2} & 0_{(\mu-2)\times 1} & 0_{\eta-\mu-2} \\ 0 & 0_{1\times(\mu-2)} & \lambda-\delta+(\eta-\mu-2)\beta & \beta J_{1\times(\eta-\mu-2)} \\ 0_{(\eta-\mu-2)\times 1} & 0_{\mu-2} & 0_{(\eta-\mu-2)\times 1} & (\lambda-\delta-\beta)I_{\eta-\mu-2} \end{vmatrix}, \tag{2}$$

with $u = \lambda-\alpha-\beta(\mu-1) + \frac{\beta\mu}{\lambda-\delta+(\eta-\mu-2)\beta}(-\beta(\eta-\mu-1))$. Equation 2 is an upper diagonal matrix, hence

$$|M| = (\lambda-\alpha+\beta)^{\mu-1}(\lambda-\beta-\delta)^{\eta-\mu-2}\left((\lambda-\alpha-\beta(\mu-1))(\lambda-\delta+(\eta-\mu-2)\beta) - \beta^2\mu(\eta-\mu-1)\right).$$

□

3.1. **Seidel Energy.** This part demonstrates the Seidel energy of $\Omega(R, I)$.

Theorem 3.2. *The characteristic formula of* $S(\Omega(R, I))$ *is*
$P_{S(\Omega(R,I))}(\lambda) = (\lambda-1)^{\mu-1}(\lambda+1)^{\eta-\mu-2}\left(\lambda^2 + (2\mu-\eta+1)\lambda + 2\mu(\mu-\eta+1) + \eta - 2\right).$

Proof. Let $R\backslash\{0\} = \{p_1, p_2, \ldots, p_\mu, r_1, r_2, \ldots, r_{\eta-\mu-1}\}$ and $I = \{p_1, p_2, \ldots, p_\mu\}$. We have $\eta - 1$ vertices for $\Omega(R, I)$. By Definition 2.3 and Theorem 2.1, we obtain the Seidel matrix of $\Omega(R, I)$ as $(\eta-1)\times(\eta-1)$ matrix as follows:

$$S(\Omega(R,I)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_\mu \\ r_1 \\ r_2 \\ \vdots \\ r_{\eta-\mu-1} \end{array} \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & \ldots & p_\mu & r_1 & r_2 & \ldots & r_{\eta-\mu-1} \end{array} \\ \begin{pmatrix} 0 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 \\ -1 & 0 & \ldots & -1 & -1 & -1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & 0 & -1 & -1 & \ldots & -1 \\ -1 & -1 & \ldots & -1 & 0 & 1 & \ldots & 1 \\ -1 & -1 & \ldots & -1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & -1 & 1 & 1 & \ldots & 0 \end{pmatrix} \end{array}. \tag{3}$$

We can choose the partition such that

$$S(\Omega(R,I)) = \begin{pmatrix} (I-J)_\mu & -J_{\mu\times(\eta-\mu-1)} \\ -J_{(\eta-\mu-1)\times\mu} & (J-I)_{\eta-\mu-1} \end{pmatrix}.$$

The formulation of characteristic polynomial of $S(\Omega(R,I))$ is presented below:

$$P_{S(\Omega(R,I))}(\lambda) = \begin{vmatrix} (\lambda-1)I_\mu + J_\mu & J_{\mu\times(\eta-\mu-1)} \\ J_{(\eta-\mu-1)\times\mu} & (\lambda+1)I_{\eta-\mu-1} - J_{\eta-\mu-1} \end{vmatrix}.$$

By Lemma 2.6 with $\alpha=\gamma=\delta=-1$, $\beta=1$, $n_1=\mu$, $n_2=\eta-\mu-1$, then we get $P_{S(\Omega(R,I))}(\lambda) = (\lambda-1)^{\mu-1}(\lambda+1)^{\eta-\mu-2}\left(\lambda^2+(2\mu-\eta+1)\lambda+2\mu(\mu-\eta+1)+\eta-2\right).$

□

As a consequence of the above fact, we present our results as follows:

Theorem 3.3. *The spectral radius of $\Omega(R,I)$ associated with the Seidel matrix is*

$$\rho_S(\Omega(R,I)) = \frac{\eta-2\mu-1+\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2)}}{2}.$$

Proof. Based on Theorem 3.2, the roots of $P_{S(\Omega(R,I))}(\lambda)=0$ are eigenvalues of $S(\Omega(R,I))$ or in other words,

$$(\lambda-1)^{\mu-1}(\lambda+1)^{\eta-\mu-2}\left(\lambda^2+(2\mu-\eta+1)\lambda+2\mu(\mu-\eta+1)+\eta-2\right)=0. \quad (4)$$

Equation 4 holds if and only if

$$(\lambda-1)^{\mu-1}=0,\ (\lambda+1)^{\eta-\mu-2}=0, \quad (5)$$

and

$$\lambda^2+(2\mu-\eta+1)\lambda+2\mu(\mu-\eta+1)+\eta-2=0. \quad (6)$$

Therefore, we obtain $\lambda_1=1$ with multiplicity $\mu-1$ and $\lambda_2=-1$ with multiplicity $\eta-\mu-2$ conforming Equation 5. The quadratic formula in Equation 6 gives 2 eigenvalues $\lambda_{3,4}=\frac{\eta-2\mu-1\pm\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}}{2}$. According to this fact, we get the spectrum of $\Omega(R,I)$, $Spec_S(\Omega(R,I))$ as follows:

$$\begin{bmatrix} \frac{\eta-2\mu-1+\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}}{2} & 1 & -1 & \frac{\eta-2\mu-1-\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}}{2} \\ 1 & \mu-1 & \eta-\mu-2 & 1 \end{bmatrix}.$$

This leads to the Seidel spectral radius of $\Omega(R,I)$ as

$$\rho_S(\Omega(R,I)) = \frac{\eta-2\mu-1+\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}}{2},$$

and we complete the proof. □

Using the fact from Theorem 3.3, we now compute the Seidel energy of $\Omega(R,I)$.

Theorem 3.4. *The Seidel energy of $\Omega(R,I)$ is*

$$E_S(\Omega(R,I)) = \eta-3+\sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}.$$

Proof. According to the spectrum of $\Omega(R,I)$ in the proofing part of Theorem 3.3, the Seidel energy of $\Omega(R,I)$ can be obtained as

$$\begin{aligned} E_S(\Omega(R,I)) =& (\mu-1)|1| + (\eta-\mu-2)|-1| + \left|\frac{\eta-2\mu-1 \pm \sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}}{2}\right| \\ =& \eta - 3 + \sqrt{4\mu(\eta-1-\mu)+(\eta-3)^2}. \end{aligned}$$

□

3.2. **Seidel Laplacian Energy.** This subsection derives the Seidel Laplacian energy of $\Omega(R,I)$.

Theorem 3.5. *The characteristic polynomial of $SL(\Omega(R,I))$ is*

$$P_{SL(\Omega(R,I))}(\lambda) = (\lambda-\mu+2)^{\mu-1}(\lambda-\eta+1)^{\eta-\mu-2}\left(\lambda^2-(3\mu-2)\lambda+\mu(3\mu-\eta-1)\right).$$

Proof. The construction of the Seidel Laplacian matrix of $\Omega(R,I)$ is dependent on the degree matrix of $\Omega(R,I)$, $D(\Omega(R,I))$. By Theorem 2.1, we provide $D(\Omega(R,I))$ as $(\eta-1)\times(\eta-1)$ matrix:

$$D(\Omega(R,I)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_\mu \\ r_1 \\ r_2 \\ \vdots \\ r_{\eta-\mu-1} \end{array} \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & \dots & p_\mu & r_1 & r_2 & \dots & r_{\eta-\mu-1} \end{array} \\ \begin{pmatrix} \mu-1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \mu-1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu-1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \eta-2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \eta-2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \eta-2 \end{pmatrix} \end{array} \tag{7}$$

By Definition 2.4, Equations 3 and 7, we provide the Seidel Laplacian matrix of $\Omega(R,I)$ as given below:

$$SL(\Omega(R,I)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_\mu \\ r_1 \\ r_2 \\ \vdots \\ r_{\eta-\mu-1} \end{array} \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & \dots & p_\mu & r_1 & r_2 & \dots & r_{\eta-\mu-1} \end{array} \\ \begin{pmatrix} \mu-1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & \mu-1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \mu-1 & 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & \eta-2 & -1 & \dots & -1 \\ 1 & 1 & \dots & 1 & -1 & \eta-2 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & -1 & -1 & \dots & \eta-2 \end{pmatrix} \end{array}. \tag{8}$$

From Equation 8, we obtain:

$$SL(\Omega(R,I)) = \begin{pmatrix} (\mu-2)I_\mu + J_\mu & J_{\mu\times(\eta-\mu-1)} \\ J_{(\eta-\mu-1)\times\mu} & (\eta-1)I_{\eta-\mu-1} - J_{\eta-\mu-1} \end{pmatrix}.$$

Therefore, we get

$$P_{SL(\Omega(R,I))}(\lambda) = \begin{vmatrix} (\lambda-\mu+2)I_\mu - J_\mu & -J_{\mu\times(\eta-\mu-1)} \\ -J_{(\eta-\mu-1)\times\mu} & (\lambda-\eta+1)I_{\eta-\mu-1} + J_{\eta-\mu-1} \end{vmatrix}.$$

By Lemma 3.1 with $\alpha = \mu-1$, $\beta = 1$, and $\gamma = \eta-2$, then we obtain

$$P_{SL(\Omega(R,I))}(\lambda) = (\lambda-\mu+2)^{\mu-1}(\lambda-\eta+1)^{\eta-\mu-2}\left(\lambda^2 - (3\mu-2)\lambda + \mu(3\mu-\eta-1)\right).$$

□

Theorem 3.6. *The spectral radius of $\Omega(R,I)$ associated with the Seidel Laplacian matrix is*

$$\rho_{SL}(\Omega(R,I)) = \frac{3\mu-2+\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2}.$$

Proof. We already observed $P_{SL(\Omega(R,I))}(\lambda)$ in Theorem 3.5. By a similar argument, the roots of $P_{SL(\Omega(R,I))}(\lambda) = 0$ provides the eigenvalues of $\Omega(R,I)$. Now we have

$$(\lambda-\mu+2)^{\mu-1}(\lambda-\eta+1)^{\eta-\mu-2}\left(\lambda^2 - (3\mu-2)\lambda + \mu(3\mu-\eta-1)\right) = 0. \quad (9)$$

With a similar idea of Equation 4, then Equation 9 holds whenever

$$(\lambda-\mu+2)^{\mu-1} = 0,\ (\lambda-\eta+1)^{\eta-\mu-2} = 0, \quad (10)$$

and

$$\left(\lambda^2 - (3\mu-2)\lambda + \mu(3\mu-\eta-1)\right) = 0. \quad (11)$$

Equation 10 derives $\lambda_1 = \mu-2$ with multiplicity $\mu-1$ and $\lambda_2 = \eta-1$ of multiplicity $\eta-\mu-2$. The quadratic formula of of Equation 11 result $\lambda_{3,4} = \frac{3\mu-2\pm\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2}$. That means the spectrum of $\Omega(R,I)$ as given by

$$\begin{bmatrix} \frac{3\mu-2+\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2} & \mu-2 & \eta-1 & \frac{3\mu-2-\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2} \\ 1 & \mu-1 & \eta-\mu-2 & 1 \end{bmatrix}.$$

The Seidel Laplacian spectral radius of $\Omega(R,I)$ is

$$\rho_{SL}(\Omega(R,I)) = \frac{3\mu-2+\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2},$$

and this states that the proof of the theorem is equipped. □

As a result of the above fact, we derive here is a straightforward result that counts the SL-energy of $\Omega(R,I)$.

Theorem 3.7. *The SL-energy of $\Omega(R,I)$ is*

$$E_{SL}(\Omega(R,I)) = (\eta-2)(\eta-1) + \mu(\mu-\eta+1).$$

Proof. In accordance with the spectrum of $\Omega(R,I)$ in the proofing part of Theorem 3.6, the Seidel Laplacian energy of $\Omega(R,I)$ can be obtained as

$$\begin{aligned}E_{SL}(\Omega(R,I)) =&(\mu-1)|\mu-2| + (\eta-\mu-2)|\eta-\mu-1| + \\ &\left|\frac{3\mu-2\pm\sqrt{(3\mu-3)^2-4\mu(3\mu-\eta-1)}}{2}\right| \\ =&(\eta-2)(\eta-1)+\mu(\mu-\eta+1).\end{aligned}$$

□

3.3. **Seidel Signless Laplacian Energy.** This section focuses on the Seidel signless Laplacian matrix of $\Omega(R,I)$.

Theorem 3.8. *The characteristic formula of $SSL(\Omega(R,I))$ is*
$P_{SSL(\Omega(R,I))}(\lambda) = (\lambda-\mu)^{\mu-1}(\lambda-\eta+3)^{\eta-\mu-2}\left(\lambda^2+(\mu-2\eta+4)\lambda-\mu(\eta-\mu-1)\right).$

Proof. By reasoning similar to the proof of Theorem 3.5, by Definition 2.5, Equations 3 and 7, we construct the Seidel signless Laplacian matrix of $\Omega(R,I)$ as $(\eta-1)\times(\eta-1)$ matrix as follows:

$$SSL(\Omega(R,I)) = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_\mu \\ r_1 \\ r_2 \\ \vdots \\ r_{\eta-\mu-1} \end{array} \begin{array}{c} \begin{array}{cccccccc} p_1 & p_2 & \dots & p_\mu & r_1 & r_2 & \dots & r_{\eta-\mu-1} \end{array} \\ \begin{pmatrix} \mu-1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ -1 & \mu-1 & \dots & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & \mu-1 & -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & \eta-2 & 1 & \dots & 1 \\ -1 & -1 & \dots & -1 & 1 & \eta-2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 & 1 & 1 & \dots & \eta-2 \end{pmatrix}\end{array}.$$

It follows that

$$SSL(\Omega(R,I)) = \begin{pmatrix} \mu I_\mu - J_\mu & -J_{\mu\times(\eta-\mu-1)} \\ -J_{(\eta-\mu-1)\times\mu} & (\eta-3)I_{\eta-\mu-1}+J_{\eta-\mu-1} \end{pmatrix}.$$

This implies

$$P_{SSL(\Omega(R,I))}(\lambda) = \begin{vmatrix} (\lambda-\mu)I_\mu + J_{mu} & J_{\mu\times(\eta-\mu-1)} \\ J_{(\eta-\mu-1)\times\mu} & (\lambda-\eta+3)I_{\eta-\mu-1}-J_{\eta-\mu-1} \end{vmatrix}.$$

By Lemma 3.1 with $\alpha=\mu-1$, $\beta=-1$, and $\gamma=\eta-2$, then we get
$P_{SSL(\Omega(R,I))}(\lambda) = (\lambda-\mu)^{\mu-1}(\lambda-\eta+3)^{\eta-\mu-2}\left(\lambda^2+(\mu-2\eta+4)\lambda-\mu(\eta-\mu-1)\right).$

□

Theorem 3.9. *The spectral radius of $\Omega(R,I)$ associated with the Seidel Laplacian matrix is*

$$\rho_{SSL}(\Omega(R,I)) = \frac{2\eta-\mu-4+\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2}.$$

Proof. According to Theorem 3.8, the roots of $P_{SSL(\Omega(R,I))}(\lambda)=0$ are eigenvalues of $SL(\Omega(R,I))$, or in other words

$$(\lambda-\mu)^{\mu-1}(\lambda-\eta+3)^{\eta-\mu-2}\left(\lambda^2+(\mu-2\eta+4)\lambda-\mu(\eta-\mu-1)\right)=0.$$

We consider the above equation and it holds if and only if

$$(\lambda-\mu)^{\mu-1},\ (\lambda-\eta+3)^{\eta-\mu-2}=0, \tag{12}$$

and

$$\left(\lambda^2+(\mu-2\eta+4)\lambda-\mu(\eta-\mu-1)\right)=0. \tag{13}$$

Therefore, from Equation 12 we obtain $\lambda_1=\mu$ with multiplicity $\mu-1$, and $\lambda_2=\eta-3$ of multiplicity $\eta-\mu-2$. The solution of quadratic formula in Equation 13 are $\lambda_{3,4}=\frac{2\eta-\mu-4\pm\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2}$. According to this fact, we get the spectrum of $\Omega(R,I)$, $Spec_S(\Omega(R,I))$ as follows:

$$\begin{bmatrix} \frac{2\eta-\mu-4+\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2} & \mu & \eta-3 & \frac{2\eta-\mu-4-\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2} \\ 1 & \mu-1 & \eta-\mu-2 & 1 \end{bmatrix}.$$

This leads to the Seidel Laplacian spectral radius of $\Omega(R,I)$ as

$$\rho_{SSL}(\Gamma(R,P))=\frac{2\eta-\mu-4+\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2}.$$

□

The next theorem presents the energy of $\Omega(R,I)$ with respect to the Seidel signless Laplacian matrix.

Theorem 3.10. *The SSL-energy of* $\Omega(R,I)$ *is*

$$E_{SSL}(\Omega(R,I))=(\eta-3)(\eta-2)+\mu(\mu-\eta+2)+\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}.$$

Proof. By using the spectrum of $\Gamma(R,P)$, the Seidel signless Laplacian energy of $\Omega(R,I)$ can be calculated as

$$\begin{aligned} E_{SSL}(\Omega(R,I)) =&(\mu-1)|\mu-2|+(\eta-\mu-2)|\eta-\mu-1|+ \\ &\left|\frac{3\mu-2\pm\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}}{2}\right| \\ =&(\eta-3)(\eta-2)+\mu(\mu-\eta+2)+\sqrt{(\mu-2\eta+4)^2+4\mu(\eta-\mu-1)}. \end{aligned}$$

□

4. CONCLUDING REMARKS

In this research, we derive the spectrum, spectral radius, and energy of the prime ideal graph. It is associated with the commutative ring and corresponds to the Seidel-based matrices.

Acknowledgement. We would like to thank the University of Mataram, Indonesia, for the partial funding assistance that has been provided.

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