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ON STRONG AND WEAK CONVERGENCE IN n-HILBERT SPACES

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Abstract. We discuss the concepts of strong and weak convergence in n-Hilbert spaces and study their properties. Some examples are given to illustrate the concepts. In particular, we prove an analogue of Banach-Saks-Mazur theorem and Radon-Riesz property in the case of n-Hilbert space.

Key words: Strong and weak convergence, n-Hilbert space.

Abstrak. Makalah ini menjelaskan konsep konvergensi kuat dan lemah dalam ruang n-Hilbert dan mempelajari sifat-sifatnya. Beberapa contoh diberikan untuk menjelaskan konsep-konsep tersebut. Khususnya, teorema analog dari Banach-Saks-Mazur dan sifat Radon-Riesz dibuktikan untuk kasus ruang n-Hilbert..

Kata kunci: Konvergensi kuat dan lemah, ruang n-Hilbert.

1. INTRODUCTION AND PRELIMINARIES

The notion of n-normed spaces was introduced by Gähler ([4]) as a generalization of normed spaces. It was initially suggested by the area function of a triangle determined by a triple in Euclidean space. The corresponding theory of n-inner product spaces was then established by Misiak ([10]). Since then, various aspects of the theory have been studied, for instance the study of Mazur-Ulam theorem and Aleksandrov problem in n-normed spaces are done in [1, 2], the study of operators in n-Banach space is done in [5, 11], and many others.

In this paper, we will generalize the notion of weak convergence in Hilbert space to the case of n-Hilbert space and study its properties. In particular, we will expand on the results in [6]. We will also give an analogue of Radon-Riesz property (on conditions relating strong and weak convergence) in the case of n-Hilbert space. Furthermore, an analogue of the well-known Banach-Saks-Mazur theorem (on the

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strong convergence of a convex combination of a weakly convergent sequence) will be given.

We begin with some preliminary results. Let X be a real vector space with $\dim(X) \geq n$, where n is a positive integer. We allow $\dim(X)$ to be infinite. A real-valued function $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ is called an *n-norm* on X^n if the following conditions hold:

(1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
(2) $\|x_1, \dots, x_n\|$ is invariant under permutations of $x_1, \dots, x_n$;
(3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in \mathbb{R}$ and $x_1, \dots, x_n \in X$;
(4) $\|x_0 + x_1, x_2, \dots, x_n\| \leq \|x_0, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x_n\|$, for all $x_0, x_1, \dots, x_n \in X$.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is then called an *n-normed space*. It also follows from the definition that an n-norm is always non-negative.

Let X be a real vector space with $\dim(X) \geq n$, where n is a positive integer. A real-valued function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ is called an *n-inner product* on X if the following conditions hold:

(1) $\langle z_1, z_1 | z_2, \dots, z_n \rangle \geq 0$, with equality if and only if $z_1, z_2, \dots, z_n$ are linearly dependent;
(2) $\langle z_1, z_1 | z_2, \dots, z_n \rangle = \langle z_{i_1}, z_{i_1} | z_{i_2}, \dots, z_{i_n} \rangle$ for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$;
(3) $\langle x, y | z_2, \dots, z_n \rangle = \langle y, x | z_2, \dots, z_n \rangle$;
(4) $\langle \alpha x, y | z_2, \dots, z_n \rangle = \alpha \langle x, y | z_2, \dots, z_n \rangle$ for every $\alpha \in \mathbb{R}$;
(5) $\langle x + x', y | z_2, \dots, z_n \rangle = \langle x, y | z_2, \dots, z_n \rangle + \langle x', y | z_2, \dots, z_n \rangle$.

The pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is then called an *n-inner product space*.

Observe that any inner product space $(X, \langle \cdot, \cdot \rangle)$ can be equipped with the *standard n-inner product*:

$$\langle x, y | z_2, \dots, z_n \rangle := \left| \begin{pmatrix} \langle x, y \rangle & \langle x, z_2 \rangle & \cdots & \langle x, z_n \rangle \\ \langle z_2, y \rangle & \langle z_2, z_2 \rangle & \cdots & \langle z_2, z_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y \rangle & \langle z_n, z_2 \rangle & \cdots & \langle z_n, z_n \rangle \end{pmatrix} \right| \tag{1}$$

where $|A|$ denotes the determinant of A.

In that case, the induced *standard n-norm* on X is given by

$$\|x_1, \dots, x_n\|_S := \sqrt{\det[\langle x_i, x_j \rangle]} \tag{2}$$

Note that the value of $\|x_1, \dots, x_n\|_S$ is just the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$.

Further examples and results on n-normed space can be found in [8, 9]. In particular, for the standard case, completeness in the norm is equivalent to that in the induced standard n-norm.

Every n-inner product space is an n-normed space with the induced n-norm:

$$\|x_1, \dots, x_n\| := \langle x_1, x_1 | x_2, \dots, x_n \rangle^{1/2} \tag{3}$$

An analogue of Cauchy-Schwarz inequality also holds for n-inner product space, i.e. for all $x, y, z_2, \ldots, z_n \in X$, we have

$$|\langle x, y|z_2, \ldots, z_n\rangle| \leq \|x, z_2, \ldots, z_n\|\|y, z_2, \ldots, z_n\| \tag{4}$$

The following definitions are taken and inspired from [12].

Definition 1.1. *A sequence $\{x_k\}$ in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to* converge *to $x \in X$ if $\|x_k - x, z_2, \ldots, z_n\| \to 0$ as $k \to \infty$ for all $z_2, \ldots, z_n \in X$.*

Definition 1.2. *A sequence $\{x_k\}$ in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is a* Cauchy sequence *if $\|x_k - x_l, z_2, \ldots, z_n\| \to 0$ as $k, l \to \infty$ for all $z_2, \ldots, z_n \in X$.*

Definition 1.3. *If every Cauchy sequence in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ converges to an $x \in X$, then X is said to be* complete. *A complete n-inner product space is called an* n-Banach space. *A complete n-inner product space is called an* n-Hilbert space.

2. STRONG AND WEAK CONVERGENCE

In this section, we will consider the notions of strong and weak convergence in n-Hilbert space. The notion of (strong) convergence in 2-normed space has been studied extensively in [12]. Here, we will focus more on the weak convergence and the relationships between the two concepts. Let $(X, \langle\cdot, \cdot|\cdot, \ldots, \cdot\rangle)$ be an n-Hilbert space and $\|\cdot, \ldots, \cdot\|$ be the induced n-norm.

Definition 2.1 (Strong convergence)**.** *A sequence (x_k) in X is said to* converge strongly *to a point $x \in X$ if $\|x_k - x, z_2, \ldots, z_n\| \to 0$ as $k \to \infty$ for every $z_2, \ldots, z_n \in X$. In this case, we write $x_k \to x$.*

Definition 2.2 (Weak convergence)**.** *A sequence (x_k) in X is said to* converge weakly *to a point $x \in X$ if $\langle x_k - x, y|z_2, \ldots, z_n\rangle \to 0$ as $k \to \infty$ for every $y, z_2, \ldots, z_n \in X$. In this case, we write $x_k \rightharpoonup x$.*

The following proposition is immediate from the definition.

Proposition 2.3. *If (x_k) and (y_k) converges strongly (resp. weakly) to x and y respectively, then $(\alpha x_k + \beta y_k)$ converges strongly (resp. weakly) to $\alpha x + \beta y$.*

Here we mention some of the basic properties, the proofs of which can be found in [6].

Proposition 2.4 (Continuity)**.** *The following results hold:*

(1) *The n-norm is continuous in each variable.*
(2) *The n-inner product is continuous in the first two variables.*

Proposition 2.5. *If (x_k) converges strongly (resp. weakly) to x and x', then $x = x'$.*

Note that strong convergence implies weak convergence.

Proposition 2.6. *If (x_k) converges strongly to x, then it converges weakly to x.*

Proof. Refer to [6]. ■

However, the converse is not true in general. The following highlights some of the way a sequence can fail to converge strongly.

Example 2.7. *Let $X = L^2[0,1]$ which is a Hilbert space with the usual inner product. Equip X with the standard 2-inner product. Define a sequence (f_n) by $f_n(x) = \sin n\pi x$. Then for all $g, h \in X$,*

$$\begin{aligned}\langle f_n, g|h\rangle &= \langle f_n, g\rangle\langle h, h\rangle - \langle f_n, h\rangle\langle h, g\rangle \\ &\leq \left|\left(\int_0^1 g(x)\sin n\pi x\ dx\right)\|h\|_2^2\right| + \left|\left(\int_0^1 h(x)\sin n\pi x\ dx\right)\|h\|_2\|g\|_2\right|\end{aligned}$$

so that $f_n \rightharpoonup 0$, where we used the Riemann-Lebesgue lemma. However,

$$\|f_n, h\| = \left(\|f_n\|_2^2\|h\|_2^2 - \langle f_n, h\rangle^2\right)^{1/2}$$

As $n \to \infty$, $\|f_n, h\| \to \frac{1}{\sqrt{2}}\|h\|_2$, which is not zero as long as $h \neq 0$ a.e., showing that f_n does not converge strongly to the zero function.

Example 2.8. *Let $(X, \langle\cdot,\cdot\rangle)$ be a separable infinite-dimensional Hilbert space with $(e_k)_{k=1}^{\infty}$ as an orthonormal basis. Equip X with the standard n-inner product. In* [6]*, it is proven that (e_k) converges weakly, but not strongly to 0.*

Remark 2.9. More generally, if X is a separable Hilbert space and $\{\phi_k\}$ is an orthonormal sequence in X. Then $\phi_k \rightharpoonup 0$ in the induced standard n-inner product.

Example 2.10. *Let $X = L^2(\mathbb{R})$, equipped with the standard 2-inner product. Define a sequence (f_n) by $f_n(x) = \chi_{(n,n+1)}(x)$, where χ is the characteristic function. Then one can check that (f_n) converges weakly, but not strongly, to zero in X.*

Remark 2.11. In [6], it is observed that in standard, finite-dimensional n-Hilbert spaces, the notions of strong and weak convergence are equivalent.

We will now give an extension of Radon-Riesz property for n-Hilbert space.

Theorem 2.12. *If $x_k \rightharpoonup x$, then*

$$\|x, z_2, \ldots, z_n\| \leq \liminf_{k\to\infty} \|x_k, z_2, \ldots, z_n\| \tag{5}$$

If, in addition,

$$\lim_{k\to\infty} \|x_k, z_2, \ldots, z_n\| = \|x, z_2, \ldots, z_n\|$$

for all $z_2, \ldots, z_n \in X$, then $x_k \to x$.

Proof. Using weak convergence of (x_k) and Cauchy-Schwarz inequality,

$$\begin{aligned}\|x, z_2, \ldots, z_n\|^2 = \langle x, x|z_2, \ldots, z_n\rangle &= \lim_{k\to\infty}\langle x, x_k|z_2, \ldots, z_n\rangle \\ &\leq \|x, z_2, \ldots, z_n\| \liminf_{k\to\infty}\|x_k, z_2, \ldots, z_n\|\end{aligned}$$

proving (5). Next, by expanding the n-norm,

$$\|x_k - x, z_2, \ldots, z_n\|^2 = \|x_k, z_2, \ldots, z_n\|^2 - 2\langle x_k, x|z_2, \ldots, z_n\rangle + \|x, z_2, \ldots, z_n\|^2 \to 0$$

using the assumptions given. Hence $x_k \to x$. ■

Next, we give an analogue of Banach-Saks-Mazur theorem for the case of n-Hilbert spaces.

Theorem 2.13. *If $x_k \rightharpoonup x$ in X and*

$$\lim_{m\to\infty} \frac{1}{m^2} \sum_{i=1}^{m} \|x_i - x, z_2, \dots, z_n\|^2 = 0 \tag{6}$$

for all $z_2, \dots, z_n \in X$, then there exists a sequence (y_k) of finite convex combinations of (x_k) such that $y_k \to x$ (strongly).

Proof. Replacing x_k by $x_k - x$, we may assume $x_k \rightharpoonup 0$. Pick $k_1 = 1$ and choose $k_2 > k_1$ such that $\langle x_{k_1}, x_{k_2} | z_2, \dots, z_n\rangle \le 1$ for all $z_2, \dots, z_n$. Inductively, given $k_1, \dots, k_m$, pick $k_{m+1} > k_m$ such that

$$|\langle x_{k_1}, x_{k_{m+1}} | z_2, \dots, z_n\rangle \le \frac{1}{k}, \ \dots\ , |\langle x_{k_m}, x_{k_{m+1}} | z_2, \dots, z_n\rangle \le \frac{1}{k}$$

which is possible since by the weak convergence of (x_k), $\langle x_{k_i}, x_k | z_2, \dots, z_n\rangle \to 0$ as $k \to \infty$ for $1 \le i \le m$. Let

$$y_m := \frac{1}{m}(x_{k_1} + \dots + x_{k_m})$$

Then we have

$$\begin{aligned}\|y_m, z_2, \dots, z_n\|^2 &= \frac{1}{m^2}\sum_{i=1}^{m} \|x_{n_i}, z_2, \dots, z_n\|^2 + \frac{2}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{j-1} \langle x_{n_i}, x_{n_j} | z_2, \dots, z_n\rangle \\ &\le \frac{1}{m^2}\sum_{i=1}^{m} \|x_{n_i}, z_2, \dots, z_n\|^2 + \frac{2}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{j-1} \frac{1}{j-1} \\ &= \frac{1}{m^2}\sum_{i=1}^{m} \|x_{n_i}, z_2, \dots, z_n\|^2 + \frac{2}{m}\end{aligned}$$

so that $y_m \to 0$ strongly as $m \to \infty$ as required. ■

Corollary 2.14. *Let X be an Hilbert space equipped with the standard n-inner product. Suppose $x_k \rightharpoonup x$ in X and $\|x_i\| < M$ for all i, where M is a constant and $\|\cdot\|$ is the norm induced by the inner product on X. Then there exists a sequence (y_k) of finite convex combinations of (x_k) such that $y_k \to x$ (strongly).*

Proof. It suffices to check that (6) holds in this case. Clearly, $\|x_i - x\|^2$ is also bounded in norm, say $\|x_i - x\|^2 < M'$. By Hadamard's inequality,

$$\frac{1}{m^2}\sum_{i=1}^{m} \|x_i - x, z_2, \dots, z_n\|^2 \le \frac{M'\|z_2\|^2 \dots \|z_n\|^2}{m} \to 0$$

as $m \to \infty$ for all $z_2, \dots, z_n \in X$, hence the statement is proven. ■

3. APPLICATIONS

In this section, we apply the theorems deduced earlier to L^2-space, $L^2(X,\mu)$, where (X,μ) is a measure space, equipped with the usual inner product

$$\langle f, g\rangle = \int_X f(x)g(x)\; d\mu(x)$$

We then equip $L^2(X)$ with the standard n-inner product. Note that when $n=1$, the following reduce to the familiar cases. Subsequently, $|A| = \det(A)$.

Proposition 3.1. *Let $f_k \in L^2(X,\mu)$, $k = 1,2,\ldots$, be such that*

$$\lim_{k\to\infty} \left| \begin{pmatrix} \int_X f_k^2\; d\mu & \int_X f_k h_2\; d\mu & \cdots & \int_X f_k h_n\; d\mu \\ \int_X h_2 f_k\; d\mu & \int_X h_2^2\; d\mu & \cdots & \int_X h_2 h_n\; d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f_k\; d\mu & \int_X h_n h_2\; d\mu & \cdots & \int_X h_n^2\; d\mu \end{pmatrix} \right| = 0$$

for all $h_2,\ldots,h_n \in L^2(X,\mu)$. Then

$$\lim_{k\to\infty} \left| \begin{pmatrix} \int_X f_k g\; d\mu & \int_X f_k h_2\; d\mu & \cdots & \int_X f_k h_n\; d\mu \\ \int_X h_2 g\; d\mu & \int_X h_2^2\; d\mu & \cdots & \int_X h_2 h_n\; d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n g\; d\mu & \int_X h_n h_2\; d\mu & \cdots & \int_X h_n^2\; d\mu \end{pmatrix} \right| = 0$$

for all $g, h_2,\ldots,h_n \in L^2(X,\mu)$.

Proof. This follows from Proposition 2.6. ■

Proposition 3.2. *Let $f_k \in L^2(X,\mu)$, $k = 1,2,\ldots$, be such that*

$$\lim_{k\to\infty} \left| \begin{pmatrix} \int_X (f_k - f) g\; d\mu & \int_X (f_k - f) h_2\; d\mu & \cdots & \int_X (f_k - f) h_n\; d\mu \\ \int_X h_2 g\; d\mu & \int_X h_2^2\; d\mu & \cdots & \int_X h_2 h_n\; d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n g\; d\mu & \int_X h_n h_2\; d\mu & \cdots & \int_X h_n^2\; d\mu \end{pmatrix} \right| = 0$$

for all $g, h_2,\ldots,h_n \in L^2(X,\mu)$. Then

$$\left| \begin{pmatrix} \int_X f^2\; d\mu & \int_X f h_2\; d\mu & \cdots & \int_X f h_n\; d\mu \\ \int_X h_2 f\; d\mu & \int_X h_2^2\; d\mu & \cdots & \int_X h_2 h_n\; d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f\; d\mu & \int_X h_n h_2\; d\mu & \cdots & \int_X h_n^2\; d\mu \end{pmatrix} \right| \le \liminf_{k\to\infty} \left| \begin{pmatrix} \int_X f_k^2\; d\mu & \int_X f_k h_2\; d\mu & \cdots & \int_X f_k h_n\; d\mu \\ \int_X h_2 f_k\; d\mu & \int_X h_2^2\; d\mu & \cdots & \int_X h_2 h_n\; d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f_k\; d\mu & \int_X h_n h_2\; d\mu & \cdots & \int_X h_n^2\; d\mu \end{pmatrix} \right|$$

for all $h_2,\ldots,h_n \in L^2(X,\mu)$.

If, in addition,

$$\lim_{k\to\infty}\left|\begin{pmatrix} \int_X f_k^2\,d\mu & \int_X f_k h_2\,d\mu & \cdots & \int_X f_k h_n\,d\mu \\ \int_X h_2 f_k\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f_k\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right| = \left|\begin{pmatrix} \int_X f^2\,d\mu & \int_X f h_2\,d\mu & \cdots & \int_X f h_n\,d\mu \\ \int_X h_2 f\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right|$$

for all $h_2,\dots,h_n \in L^2(X,\mu)$, *then*

$$\lim_{k\to\infty}\left|\begin{pmatrix} \int_X (f_k-f)^2\,d\mu & \int_X (f_k-f) h_2\,d\mu & \cdots & \int_X (f_k-f) h_n\,d\mu \\ \int_X h_2 (f_k-f)\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n (f_k-f)\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right| = 0$$

for all $h_2,\dots,h_n \in L^2(X,\mu)$.

Proof. This follows from Theorem 2.12. ■

Proposition 3.3. *Let* $f_k \in L^2(X,\mu)$, $k = 1,2,\dots$, *be such that*

$$\lim_{k\to\infty}\left|\begin{pmatrix} \int_X f_k g\,d\mu & \int_X f_k h_2\,d\mu & \cdots & \int_X f_k h_n\,d\mu \\ \int_X h_2 g\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n g\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right| = 0$$

for all $g, h_2,\dots,h_n \in L^2(X,\mu)$, *and there exists a constant* M *such that*

$$\lim_{m\to\infty}\frac{1}{m^2}\sum_{k=1}^{m}\left|\begin{pmatrix} \int_X f_k^2\,d\mu & \int_X f_k h_2\,d\mu & \cdots & \int_X f_k h_n\,d\mu \\ \int_X h_2 f_k\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n f_k\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right| = 0$$

for all $h_2,\dots,h_n \in L^2(X,\mu)$. *Then there exists a sequence* (g_k) *of finite convex combinations of* (f_k), *such that*

$$\lim_{k\to\infty}\left|\begin{pmatrix} \int_X g_k^2\,d\mu & \int_X g_k h_2\,d\mu & \cdots & \int_X g_k h_n\,d\mu \\ \int_X h_2 g_k\,d\mu & \int_X h_2^2\,d\mu & \cdots & \int_X h_2 h_n\,d\mu \\ \vdots & \vdots & \ddots & \vdots \\ \int_X h_n g_k\,d\mu & \int_X h_n h_2\,d\mu & \cdots & \int_X h_n^2\,d\mu \end{pmatrix}\right| = 0$$

for all $h_2,\dots,h_n \in L^2(X,\mu)$.

Proof. This follows from Theorem 2.13. ■

Remark 3.4. Note that we also have another equivalent formula for the standard n-inner product and n-norm in $L^2(X)$ as follows (see [3, 9])

$$\langle f, g|h_2, \dots, h_n\rangle = \frac{1}{n!} \underbrace{\int_X \int_X \dots \int_X}_{n \text{ times}} \det(F) \det(G) \; d\mu(x_1) \dots d\mu(x_n)$$

where

$$\det(F) = \left| \begin{pmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ h_2(x_1) & h_2(x_2) & \cdots & h_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x_1) & h_n(x_2) & \cdots & h_n(x_n) \end{pmatrix} \right|$$

and

$$\det(G) = \left| \begin{pmatrix} g(x_1) & g(x_2) & \cdots & g(x_n) \\ h_2(x_1) & h_2(x_2) & \cdots & h_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x_1) & h_n(x_2) & \cdots & h_n(x_n) \end{pmatrix} \right|$$

The standard n-norm is therefore

$$\|f, h_2, \dots, h_n\| = \left(\frac{1}{n!} \underbrace{\int_X \int_X \dots \int_X}_{n \text{ times}} [\det(F)]^2 \; d\mu(x_1) \dots d\mu(x_n) \right)^{1/2}$$

The above propositions hold accordingly using this form of n-inner product and n-norm.

Similarly, we can get other results concerning weak and strong convergence in other n-Hilbert spaces. For instance, we mention the Sobolev space $W^{s,2}(\Omega) = H^s(\Omega)$, which is a Hilbert space with inner product

$$\langle f, g\rangle = \int_\Omega f(x)g(x) \; dx + \sum_{i=1}^{s} \int_\Omega D^i f(x) \cdot D^i g(x) \; dx$$

We can equip $H^s(\Omega)$ with the standard n-inner product (1), and all the above convergence results will hold accordingly.

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