J. Indones. Math. Soc.
Vol. 30, No. 03 (2024), pp. 438–446.

ON X-SUB-LINEARLY INDEPENDENT OF ROUGH GROUPS

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Abstract. In 1982, Pawlak introduced a concept, namely, rough set theory. Several studies have been conducted regarding applying rough set theory to algebraic structures such as rough groups, rough rings, and rough modules. Furthermore, X-sub-linearly independent is a generalization of linearly independent concepts. In this paper, we investigate the X-sub-linearly independent of rough groups.

Key words and Phrases: Rough set, modules, X-sub-linearly independent, rough groups.

1. INTRODUCTION

The concept of linear independence is important in module theory. For instance, this concept is needed to determine the basis of a module over a ring. In 2015, Suprapto introduced the generalization of relative linear independence to a module M over ring R, as shown in the following: Let M be a module over a ring R. The family $N = \{N_\lambda\}_\Lambda$ is said linearly independent to M if there exist a monomorphism $f : \bigsqcup_\Lambda N_\lambda \to M$. We can say that $N = \{N_\lambda\}_\Lambda$ is linearly independent to M if $0 \to \bigsqcup_\Lambda N_\lambda \xrightarrow{f} M$ is exact sequence at $\bigsqcup_\Lambda N_\lambda$ [4]. The family of R-modules $N = \{N_\lambda\}_\Lambda$ is an X-sub-linearly independent to M if the triple $(0, \bigsqcup_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\bigsqcup_\Lambda N_\lambda$). Therefore, linearly independent modules are generalized into sub-linearly independent modules [1].

In addition, rough set theory developed rapidly. Pawlak first introduced this theory in 1982 [15]. Rough set theory has been successfully applied in several aspects involving machine learning, data mining, decision analysis, and algebraic structures. There are several studies regarding the application of rough set theory in the field of algebraic structures, such as the implementation of a rough set on a group structure [5], homomorphisms and isomorphisms [11], rough set of projective

2020 Mathematics Subject Classification: 16D10, 16D60, 16P40.
Received: 18-01-2024, accepted: 10-11-2024.

module [10], the properties of rough V-coexact sequence in rough groups [7], sub-exact sequence of rough groups [12], and sub-exact sequence of rough modules [2]. Motivated by the generalization of linearly independent modules, this paper aims to introduce $X-$sub-linearly independent rough groups, which is a generalization of linearly independent rough groups.

Let N be a rough group. The family of groups $K = \{K_i\}_{i\in I}$ is said to be rough $X-$sub-linearly independent to N if the triple $(0, \bigsqcup_{i\in I} K_i, N)$ is rough X-subexact (where X is a rough subgroup of $\bigsqcup_{i\in I} K_i$). Then, we collect all subgroup rough X of $\bigsqcup_{i\in I} K_i$ such that K is rough $X-$sub-linearly independent to N, we denoted it by $\sigma(0, \bigsqcup_{i\in I} K_i, N)$. We give some properties rough $X-$sub-linearly independent groups and $\sigma(0, \bigsqcup_{i\in I} K_i, N)$. Furthermore, we will show that $\sigma(0, \bigsqcup_{i\in I} K_i, N)$ close under rough subgroups and intersections.

2. PRELIMINARIES

In this part, we recall some definitions in algebraic structures and rough set theory used in this research.

Definition 2.1. *A relation R on a set P is called an equivalence relation if R has reflexive, symmetric, and transitive properties.*

(1) If xRx for every $x \in P$, then R is reflexive.
(2) If xRy results in yRx for every $x, y \in P$, then R is symmetric.
(3) If xRy and yRz result in xRz for every $x, y, z \in P$, then R is transitive [3].

Definition 2.2. *Let U be a universal set, where $U \neq \emptyset$, and R be an equivalence relation on U. The approximation space (U, R), denoted by $K = (U, R)$* [15].

Definition 2.3. *Let (U, R) be an approximation space and X be a subset of U. The lower approximation and upper approximation of X are defined as follows:*

$$\underline{Apr}(X) = \{x \mid [x]_R \subseteq X\},$$
$$\overline{Apr}(X) = \{x \mid [x]_R \cap X \neq \emptyset\}.$$

$\underline{Apr}(X)$ is lower approximation of X and $\overline{Apr}(X)$ is upper approximation of X on approximation space (U, R) [8].

Next, we will give definition of the rough set.

Definition 2.4. *Let R be an equivalence relation in universal set U. A subset $X \subseteq U$ is rough set if $\overline{Apr}(X) - \underline{Apr}(X) \neq \emptyset$* [15].

We give an illustration of Definition 2.4.

Example 2.5. *Given an approximation space (U, R), where $U = \{x_1, x_2, ..., x_{10}\}$ and R is equivalence relation on U with equivalence classes as follows:*

$$E_1 = \{x_1, x_3\}, E_2 = \{x_2, x_4\}, E_3 = \{x_5, x_6, x_8\}, E_4 = \{x_7, x_9\}, E_5 = \{x_{10}\}.$$

If $X = \{x_1, x_2, x_3, x_4, x_5\}$*, then upper approximation and lower approximation of* X *are* $\overline{Apr}(X) = \{x_1, x_3\} \cup \{x_2, x_4\} \cup \{x_5, x_6, x_8\}$ *and* $\underline{Apr}(X) = \{x_1, x_3\} \cup \{x_2, x_4\}$*. Hence* $\overline{Apr}(X) - \underline{Apr}(X) \neq \emptyset$*. Therefore,* X *is a rough set in an approximation space* (U, R)*.*

Definition 2.6. *Given an approximation space* $K = (U, R)$ *and binary operation* $*$ *defined on* U*. A subset* G *of universe* U *is called a rough group if the following properties are satisfied:*

(1) $x * y \in \overline{Apr}(G)$*, for every* $x, y \in G$*;*
(2) *the associative property applies in* $\overline{Apr}(G)$;
(3) *there exists* $e \in \overline{Apr}(G)$ *such that* $x * e = e * x = x$*, for every* $x \in G$*;* e *is called a rough identity element of rough group* G;
(4) *for every* $x \in G$*, there exists* $y \in G$ *such that* $x * y = y * x = e; y$ *is called a rough inverse element of* x *in* G [8].

Next, we give an example of a rough group.

Example 2.7. *Given an approximation space* (U, R)*, where* $U = \mathbb{Z}_{16}$*. We define* $xRy = 4m$*, for some* $m \in \mathbb{Z}$*. We have four equivalence classes are* $E_1 = \{\overline{1}, \overline{5}, \overline{9}, \overline{13}\}, E_2 = \{\overline{2}, \overline{6}, \overline{10}, \overline{14}\}, E_3 = \{\overline{3}, \overline{7}, \overline{11}, \overline{15}\}, E_4 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}\}$*. Let* $X = \{\overline{4}, \overline{6}, \overline{7}, \overline{9}, \overline{10}, \overline{12}\}$*. We have* $\overline{Apr}(X) = E_1 \cup E_2 \cup E_3 \cup E_4 = \mathbb{Z}_{16}$*. Based on Definition 2.6,* X *is a rough group.*

Definition 2.8. *Given approximation space* (U_1, R_1) *and* (U_2, R_2) *and binary operation* $*, *'$ *on* U_1 *and* U_2*, respectively. Let* $G_1 \subseteq U_1$ *and* $G_2 \subseteq U_2$ *be two rough groups. If the mapping* $f : \overline{Apr}(G_1) \to \overline{Apr}(G_2)$ *satisfies* $f(x * y) = f(x) *' f(y)$*, for all* $x, y \in \overline{Apr}(G_1)$*, then* f *is called a rough homomorphism* [6].

Definition 2.9. *Given two rough groups* G_1 *and* G_2*. A rough homomorphism* $f : \overline{Apr}(G_1) \to \overline{Apr}(G_2)$ *is said to be:*

(1) *a rough epimorphism (or surjective) if* f *is onto;*
(2) *a rough monomorphism if* f *is one-to-one;*
(3) *a rough isomorphism if* f *is both onto and one-to-one* [11].

Definition 2.10. *Given an approximation space* (S, θ)*, rough group* K, L, M *in* (S, θ)*, and rough subgroup* U *of* M*. A sequence*

$$\overline{Apr}(K) \xrightarrow{f} \overline{Apr}(L) \xrightarrow{g} \overline{Apr}(M)$$

is called rough $U-$*exact in* M *if* $im(f) = g^{-1}(U)$ [9].

Definition 2.11. *Given rough groups* U', U, U''*, and rough subgroup* X *of* U*. Then the triple* (U', U, U'') *is said to be rough* $X-$*sub-exact at* U *if there exist rough homomorphisms* f *and* g *such that the sequence of rough groups and rough group homomorphisms:*

$$\overline{Apr}(U') \xrightarrow{f} \overline{Apr}(U) \xrightarrow{g} \overline{Apr}(U'')$$

is exact [12].

Definition 2.12. *Given module M over ring R. The family of modules over ring R $N = \{N_\lambda\}_\Lambda$ is said to be $X-$sub-linearly independent to M if the triple $(0, \bigsqcup_\Lambda N_\lambda, M)$ is $X-$sub-exact (where X is a sub-module of $\bigsqcup_\Lambda N_\lambda$), i.e. the sequence*

$$0 \to X \to M$$

is exact [1].

3. MAIN RESULTS

We define the rough X-subexact sequence of rough groups as follows.

Definition 3.1. *Given rough group N. The family of groups $K = \{K_i\}_{i \in I}$ is said to be rough X-sub-linearly independent to N if the triple $(0, \bigsqcup_i K_i, N)$ is rough X-subexact (where X is a rough subgroup of $\bigsqcup_i K_i$), i.e., the sequence*

$$0 \to \overline{Apr}(X) \to \overline{Apr}(N)$$

is exact.

Example 3.2. *Given an approximation space (U, R), where $U = \mathbb{Z}_8$ and a relation R on U, where $R = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{1}), (\overline{1}, \overline{3}), (\overline{1}, \overline{5}), (\overline{1}, \overline{7}), (\overline{2}, \overline{2}), (\overline{2}, \overline{4}), (\overline{2}, \overline{6}), (\overline{3}, \overline{1}), (\overline{3}, \overline{3}),$ $(\overline{3}, \overline{5}), (\overline{3}, \overline{7}), (\overline{4}, \overline{2}), (\overline{4}, \overline{4}), (\overline{4}, \overline{6}), (\overline{5}, \overline{1}), (\overline{5}, \overline{3}), (\overline{5}, \overline{5}), (\overline{5}, \overline{7}), (\overline{6}, \overline{2}), (\overline{6}, \overline{4}), (\overline{6}, \overline{6}), (\overline{7}, \overline{1}), (\overline{7}, \overline{3}),$ $(\overline{7}, \overline{5}), (\overline{7}, \overline{7})\}$. We can show that R is an equivalence relation on U. We have three equivalence classes in the table from the equivalence relation.*

TABLE 1. The Equivalence Classes of U

No	*The Equivalence Classes*	*The Elements*
1	E_1	$\{\overline{0}\}$
2	E_2	$\{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$
3	E_3	$\{\overline{2}, \overline{4}, \overline{6}\}$

Furthermore, we give $K_1, K_2, N \subseteq U$ with $K_1 = \{\overline{0}, \overline{1}, \overline{2}, \overline{6}, \overline{7}\}, K_2 = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}\}$, and $N = \{\overline{0}, \overline{1}, \overline{4}, \overline{7}\}$. Next, we will prove that $K_1 = \{\{\overline{0}, \overline{1}, \overline{2}, \overline{6}, \overline{7}\}$ is a rough group. We obtain $\overline{Apr}(K_1) = E_1 \cup E_2 \cup E_3 = \mathbb{Z}_8$ and $\underline{Apr}(K_1) = E_1 = \{0\}$. Since $\overline{Apr}(K_1) - \underline{Apr}(K_1) \neq \emptyset$, we can conclude that K_1 is a rough set. Then, we define an operation $+_8$ in K_1.

TABLE 2. Operation on Element K_1 with operation $+_8$

$+_8$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{6}$	$\overline{7}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{6}$	$\overline{7}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{7}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{0}$	$\overline{1}$
$\overline{6}$	$\overline{6}$	$\overline{7}$	$\overline{0}$	$\overline{4}$	$\overline{5}$
$\overline{7}$	$\overline{7}$	$\overline{0}$	$\overline{1}$	$\overline{5}$	$\overline{6}$

We will show that $(K_1, +_8)$ *is a rough group.*

(1) $a +_8 b \in \overline{Apr}(K_1)$, *for every* $a, b \in K_1$;

(2) $a +_8 (b +_8 c) = (a +_8 b) +_8 c$ *in* $\overline{Apr}(K_1)$, *for every* $a, b, c \in \overline{Apr}(K_1)$;

(3) *there exists a rough identity element* $e = \overline{0} \in \overline{Apr}(K_1)$, *such that* $e +_8 a = a +_8 e = a$, *for every* $a \in K_1$;

TABLE 3. Inverse Element of K_1

Element	*Invers*
$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{7}$
$\overline{2}$	$\overline{6}$
$\overline{6}$	$\overline{2}$
$\overline{7}$	$\overline{1}$

(4) *based on Table 3, for every* $a \in K_1$, *there exists* $b = a^{-1} \in K_1$, *such that* $a +_8 b = e$.

Hence, it proves that K_1 *is a rough group on* $\mathbb{Z}_8$. *Next, we will prove that* $K_2 = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}\}$ *is rough group. We obtain* $\overline{Apr}(K_2) = E_1 \cup E_2 \cup E_3 = \mathbb{Z}_8$ *and* $\underline{Apr}(K_2) = E_1 = \{0\}$. *Since* $\overline{Apr}(K_2) - \underline{Apr}(K_2) \neq \emptyset$, *we can conclude that* K_2 *is a rough set. Then, we define an operation* $+_8$ *in* K_2.

TABLE 4. Operation on Element K_2 with Operation $+_8$

$+_8$	$\overline{0}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{3}$	$\overline{3}$	$\overline{6}$	$\overline{7}$	$\overline{0}$
$\overline{4}$	$\overline{4}$	$\overline{7}$	$\overline{0}$	$\overline{1}$
$\overline{5}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$

We will show that $(K_2, +_8)$ *is a rough group.*

(1) $a +_8 b \in \overline{Apr}(K_2)$, *for every* $a, b \in K_2$;

(2) $a +_8 (b +_8 c) = (a +_8 b) +_8 c$, *for every* $a, b, c \in \overline{Apr}(K_2)$,

(3) *There is a rough identity element, that is* $e = \overline{0} \in \overline{Apr}(K_2)$ *such that* $e +_8 a = a +_8 e = a$, *for every* $a \in K_2$;

TABLE 5. Inverse Element of K_2

Element	*Invers*
$\overline{0}$	$\overline{0}$
$\overline{3}$	$\overline{5}$
$\overline{4}$	$\overline{4}$
$\overline{5}$	$\overline{3}$

(4) Based on Table 5, for every $a \in K_2$, there is $b = a^{-1} \in K_2$ such that $a +_8 b = e$.

Therefore, it proves that K_2 is a rough group on $\mathbb{Z}_8$. Next, we will prove that $N = \{\overline{0}, \overline{1}, \overline{4}, \overline{7}\}$ is rough group. We obtain $\overline{Apr}(N) = E_1 \cup E_2 \cup E_3 = \mathbb{Z}_8$ and $\underline{Apr}(N) = E_1 = \{0\}$. Since $\overline{Apr}(N) - \underline{Apr}(N) \neq \emptyset$, we can conclusion that N is a rough set. Then, we define an operation $+_8$ in N.

TABLE 6. Operation on Element N with Operation $+_8$

$+_8$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{7}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{7}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{5}$	$\overline{0}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{3}$
$\overline{7}$	$\overline{7}$	$\overline{0}$	$\overline{3}$	$\overline{6}$

We will show that $(N, +_8)$ is a rough group.

(1) $a +_8 b \in \overline{Apr}(N)$, for every $a, b \in N$;
(2) $a +_8 (b +_8 c) = (a +_8 b) +_8 c$, for every $a, b, c \in \overline{Apr}(N)$;
(3) There is a rough identity element $e = \overline{0} \in \overline{Apr}(N)$ such that $e +_8 a = a +_8 e = a$, for every $a \in N$;

TABLE 7. Inverse Element of N

Element	*Invers*
$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{7}$
$\overline{4}$	$\overline{4}$
$\overline{7}$	$\overline{1}$

(4) Based on Table 7, for every $a \in N$, there is $b = a^{-1} \in N$ such that $a +_8 b = e$.

Therefore, it proves that N is a rough group on $\mathbb{Z}_8$. Let $K = \{K_1, K_2\}$ be the family of the rough group, and N be a rough group on $\mathbb{Z}_8$. We define

$$f : \overline{Apr}(K_1) \to \overline{Apr}(N),$$

where $f(a) = 3a$, for every $a \in \overline{Apr}(K_1)$. So, f is a rough monomorphism. Hence, the sequence $0 \to \overline{Apr}(K_1) \xrightarrow{f} \overline{Apr}(N)$ is rough exact. Therefore, the triple $(0, K_1 \bigoplus K_2, N)$ is rough K_1-sub-exact. Hence, K is rough K_1- sub-linearly independent to N.

Proposition 3.3. *Let (U, θ) be an approximation space and X_1, X_2 be rough groups of U, such that $\overline{Apr}(X_1) = \overline{Apr}(X_2)$. Then the family of rough groups $K = \{K_i\}_{i \in I}$ is rough X_2-sub-linearly independent to N.*

Proof. We assume that the family of rough groups $K = \{K_i\}_{i\in I}$ is rough X_1- sub-linearly independent to N, such that the triplet $(0, \bigsqcup_{i\in I} K_i, N)$ is rough X_1- sub-exact. Based on the Definition 3.1, the sequence:

$$0 \xrightarrow{\alpha} \overline{Apr}(X_1) \xrightarrow{\beta} \overline{Apr}(N)$$

is rough exact. Since $\overline{Apr}(X_1) = \overline{Apr}(X_2)$, the triple $(0, \bigsqcup_{i\in I} K_i, N)$ is rough X_2- sub-exact. Hence, the family of rough groups $K = \{K_i\}_{i\in I}$ is rough X_2-sub-linearly independent to N. □

To prove the family of rough groups $K = \{K_i\}_{i\in I}$ is rough X_2- sub-linearly independent to N implies that the family of rough groups $K = \{K_i\}_{i\in I}$ is rough X_1-sub-linearly independent to N can be shown similarly.

Proposition 3.4. *Let $K = \{K_i\}_{i\in I}$ be a family of rough groups. Then K is rough 0-sub-linearly independent to N, for any rough group N.*

Proof. Since $0 \to 0 \to \overline{Apr}(N)$ is a rough exact sequence, the triple $(0, \bigsqcup_{i\in I} K_i, N)$ is rough 0-sub-exact at $\bigsqcup_{i\in I} K_i$. Hence, K is rough 0-sub-linearly independent to N. □

For every rough group N, we can define a rough monomorphism to itself. Therefore, any rough group N is rough N-sub-linearly independent to N. We know that any subset of a rough linearly independent set is a rough linearly independent. Then, we will show that N is rough X-sub-linearly independent to N, for every rough subgroup X of N.

Proposition 3.5. *Let N be a rough group. Then N is a rough $X-$sub-linearly independent to N, for every rough subgroup X of N.*

Proof. Let N be a rough group and X be a rough subgroup of N. There exist inclusion $j : \overline{Apr}(X) \to \overline{Apr}(N)$ such that the sequence $0 \to \overline{Apr}(X) \xrightarrow{j} \overline{Apr}(N)$ is rough exact. Hence, the triple $(0, N, N)$ is rough $X-$sub-exact. Therefore M is rough $X-$sub-linearly independent to N. □

Proposition 3.6. *Let $K = \{K_i\}_{i\in I}$ be a family of rough groups. Then K is rough $X-$sub-linearly independent to K_i, for any subgroup rough X of K_i.*

Proof. Let X be a subgroup rough of $K_i \subset \bigsqcup_{i\in I} K_i$. We have the inclusion $i : X \to K_i$ such that $0 \to \overline{Apr}(X) \to \overline{Apr}(K_i)$ is a rough exact sequence. This implies the triple $(0, \bigsqcup_{i\in I} K_i, K_i)$ is rough $X-$sub-exact sequence at $\bigsqcup_{i\in I} K_i$. Therefore K is rough $X-$sub-linearly independent to K_i. □

Let S, T, U be rough groups. We define

$$\sigma(S, T, U) = \{X \leq T | (S, T, U) \text{ rough } X - \text{subexact at } T\}.$$

Since $0 \in \sigma(S,T,U)$, $\sigma(S,T,U) \neq \emptyset$. Now, let K be the family of rough groups. If we take $S = 0, T = \bigsqcup_{i\in I} K_i$, then

$$\begin{aligned}\sigma(0,T,U) &= \{X \leq T | (0,T,U) \text{ is rough } X-\text{sub-exact at } T\} \\ &= \{X \leq T | K \text{ is rough X-sub-linearly independent to } U\}.\end{aligned}$$

In the following propositions, we give the properties of $\sigma(0, \sqcup_{i\in I} K_i, U)$.

Proposition 3.7. *Let $K = \{K_i\}_{i\in I}$ be a family of rough groups and N be a rough group. If K is rough $X_\lambda-$sub-linearly independent to N, for every $\lambda \in \Lambda$, then K is rough $\bigcap_{\lambda\in\Lambda} X_\lambda-$sub-linearly independent to N. In particularly, $\sigma(0, \sqcup_{i\in I} K_i, N)$ is closed under intersection.*

Proof. If K is rough $X_\lambda-$sub-linearly independent to N, for every $\lambda \in \Lambda$, then triple $(0, \sqcup_{i\in I} K_i, N)$ is rough $X_\lambda-$sub-exact. Therefore, $X_\lambda \in \sigma(0, \sqcup_I K_i, N)$, for every $\lambda \in \Lambda$. We have $\cap_{\lambda\in\Lambda} X_\lambda \in \sigma(0, \sqcup_{i\in I} K_i, N)$. So, K is rough $\cap_{\lambda\in\Lambda} X_\lambda-$ sub-linearly independent to N. □

Now, we will prove that $\sigma(0, \sqcup_I K_i, N)$ is closed under rough subgroups.

Proposition 3.8. *Given the family of rough groups $K = \{K_i\}_{i\in I}$. If K is rough $X-$sub-linearly independent to N, then K is rough $X'-$sub-linearly independent to N, for every subgroup X' of X. In particularly, $\sigma(0, \sqcup_I K_i, N)$ is closed under rough subgroups.*

Proof. Since K is rough $X-$sub-linearly independent to N, there is rough monomorphism $f : X \to N$. Let X' be a rough subgroup of X. The inclusion is defined by $i : X' \to X$. So $f \circ i : X' \to N$ is a rough monomorphism. Thus, K is rough $X'-$sub-linearly independent to N. Therefore, if $X \in \sigma(0, \sqcup_I K_i, N)$, then $X' \in \sigma(0, \sqcup_I K_i, N)$, for every rough subgroup X' of X. □

4. CONCLUSION

The family of rough groups $K = \{K_i\}_{i\in I}$ is a rough $X-$sub-linearly independent to N if the triple $(0, \sqcup_{i\in I} K_i, N)$ is rough $X-$sub-exact (where X is rough subgroup of $\sqcup_{i\in I} K_i$). If we take $X = \sqcup_{i\in I} K_i$, then $K = \{K_i\}_{i\in I}$ is rough linearly independent. Hence, rough linearly independent groups are generalized into rough sub-linearly independent groups. We denote the set of all subgroup X of $\sqcup_{i\in I} K_i$ such that K is rough $X-$sub-linearly independent to N by $\sigma(0, \sqcup_{i\in I} K_i, N)$. We have $\sigma(0, \sqcup_{i\in I} K_i, N)$ is closed under rough subgroups and intersections.

Acknowledgements

We are very grateful to experts for their appropriate and constructive suggestions to improve this manuscript. Furthermore, we wish to thank the The Directorate General of Higher Education, Research, and Technology of The Ministry of Education, Culture, Research, and Technology of the Republic of Indonesia for this research support and funding under the Research Contract Number: 131/E5/PG.02.00.PL/2023.

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