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ON WEAKLY S-PRIME ELEMENTS OF LATTICES

Shahabaddin Ebrahimi Atani

Department of Pure Mathematics, Guilan University, Rasht, Iran
ebrahimi@guilan.ac.ir

Abstract. Let $\pounds$ be a bounded distributive lattice and S a join-subset of $\pounds$. In this paper, we introduce the concept of S-prime elements (resp. weakly S-prime elements) of $\pounds$. Let p be an element of $\pounds$ with $S \wedge p = 0$ (i.e. $s \wedge p = 0$ for all $s \in S$). We say that p is an S-prime element (resp. a weakly S-prime element) of $\pounds$ if there is an element $s \in S$ such that for all $x, y \in \pounds$ if $p \leq x \vee y$ (resp. $p \leq x \vee y \neq 1$), then $p \leq x \vee s$ or $p \leq y \vee s$. We extend the notion of S-prime property in commutative rings to S-prime property in lattices.

Key words and Phrases: lattice, S-prime element, weakly S-prime element.

1. INTRODUCTION

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are undoubtedly a natural choice of generalizations of rings. It is appropriate to ask which properties of rings can be extended to lattices. The lack of subtraction in lattices shows that many results in rings have no counterparts in lattices, hence, should be explored in the literature. The main aim of this article is that of extending some results obtained for ring theory to the theory of lattices.

The notion of prime ideals has a significant place in the theory of rings, and it is used to characterize certain classes of rings. For years, there have been many studies and generalizations on this issue. See, for example, [1–3, 5, 7–10]. Anderson and Smith generalized the concept of prime ideals in [1]. We recall from [1] that a nonzero proper ideal I of a commutative ring R is said to be a weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$, then either $a \in I$ or $b \in I$ (also see [8]). In 2019, Hamed and Malek [9] introduced the notion of an S-prime ideal. Let $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint from S. We say that I is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we

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have $sa \in I$ or $sb \in I$. Almahdi et. al. [3] introduced the notion of a weakly S-prime ideal as follows: We say that I is a weakly S-prime ideal of R if there is an element $s \in S$ such that for all $x, y \in R$ if $0 \neq xy \in I$, then $xs \in I$ or $ys \in I$. Let $\pounds$ be a bounded distributive lattice. Our objective in this paper is to extend the notion of S-prime property in commutative rings to S-prime property in the lattices, and to investigate the relations between S-prime elements, weakly S-prime elements, weakly prime elements and prime elements. We say that a subset $S \subseteq \pounds$ is *join-subset* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$. Clearly, if p is a nonzero prime element of $\pounds$, then $\pounds \setminus \{x \in \pounds : p \leq x\}$ is a join-subset of $\pounds$. Among the many results in this paper, the first, introduction section contains elementary observations needed later on.

In Section 2, we collect some basic properties of S-prime elements. At first, we give the definition of S-prime elements (Definition 2.1) and we provide an example (Example 2.2) of an S-prime element of $\pounds$ that is not a prime element. It is proved (Theorem 2.6) that if q is a nontrivial element of a complete lattice $\pounds$ and S a join-subset of $\pounds$ with $S \wedge q = 0$, then there exists a minimal S-prime element p that satisfies $p \leq q$ is constructible. It is shown (Theorem 2.9) that if S is a strongly join-subset of a complete lattice $\pounds$ and $\{p_i\}_{i \in \Lambda}$ is a chain of S-prime elements of $\pounds$, then $p = \bigvee_{i \in \Lambda} p_i$ is an S-prime element of $\pounds$. In the rest of this section, we investigate the properties of S-prime elements similar to prime elements. In particular, we investigate the behavior of S-prime elements under homomorphism and in cartesian products of lattices (see Theorem 2.10 and Theorem 2.12).

Section 3 is dedicated to the investigate the basic properties of weakly S-prime elements. At first, we define the concepts of weakly S-prime element and weakly prime element (Definition 3.1) and we provide an example (Example 3.2) of a weakly S-prime element of $\pounds$ that is not a S-prime element (so it is not a prime element of $\pounds$). It is proved (Theorem 3.4) that if S is a join-subset of $\pounds$ and p is a weakly S-prime element of $\pounds$ that is not S-prime, then $p = 1$. Theorem 3.6 proves that p is a weakly S-prime element of $\pounds$ if and only if there exists $s \in S$ such that for each $x \in \pounds$ with $p \not\leq x \vee s$ we have either $\{y \in \pounds : p \leq x \vee y\} \subseteq \{y \in \pounds : p \leq s \vee y\}$ or $\{y \in \pounds : p \leq x \vee y\} = \{y \in \pounds : x \vee y = 1\}$. Also, we show that if S is a join-subset of $\pounds$, then every weakly S-prime filter of $\pounds$ is prime if and only if $\pounds$ is a $\pounds$-domain and every S-prime element of $\pounds$ is prime (Proposition 3.8). In the rest of this section, we investigate the properties of weakly S-prime elements similar to weakly prime elements. In particular, we investigate the behavior of weakly S-prime elements under homomorphism and in cartesian products of lattices (see Theorem 3.11 and Theorem 3.12).

Let us recall some notions and notations. By a *lattice* we mean a *poset* $(\pounds, \leq)$ in which every couple elements x, y has a g.l.b. (greatest lower bound) (called the *meet* of x and y, and written $x \wedge y$) and a l.u.b. (called the *join* of x and y, and written $x \vee y$). A lattice $\pounds$ is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in $\pounds$. Setting $X = \pounds$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A lattice $\pounds$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

for all a, b, c in $\pounds$ (equivalently, $\pounds$ is *distributive* if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\pounds$). A non-empty subset F of a lattice $\pounds$ is called a *filter*, if for $a \in F$, $b \in \pounds$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$). A proper filter F of $\pounds$ is called *prime* (resp. *weakly prime*) if $x \vee y \in F$ (resp. $1 \neq x \vee y \in F$), then $x \in F$ or $y \in F$. A lattice $\pounds$ with 1 is called *$\pounds$-domain* if $a \vee b = 1$ $(a, b \in \pounds)$, then $a = 1$ or $b = 1$ (so $\pounds$ is $\pounds$-domain if and only if $\{1\}$ is a prime filter of $\pounds$). If $\pounds$ and $\pounds'$ are lattices, then a *lattice homomorphism* $f : \pounds \to \pounds'$ is a map from $\pounds$ to $\pounds'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \pounds$. A lattice homomorphism is said to be a *order lattice homomorphism* if $x \leq y$ if and only if $f(x) \leq f(y)$ for $x, y \in \pounds$. An element x of a lattice $\pounds$ is *nontrivial* (resp. *proper*) if $x \neq 0, 1$ (resp. $x \neq 1$). We say that an element x in a lattice L is an *atom* (resp. *coatom*) if there is no $y \in L$ such that $0 < y < x$ (resp. $x < y < 1$). Let A be subset of a lattice $\pounds$. Then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. It is clear that $T(A) = \{x \in L : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}$ (see [7]). For terminology and notation not defined here, the reader is referred to [4, 6].

2. Characterization of S-prime elements

Throughout this paper we shall assume, unless otherwise stated, that $\pounds$ is a *bounded distributive lattice*. In this section, we collect some basic properties concerning S-prime elements and remind the reader with the following definition.

Definition 2.1.

(1) A proper element p of a lattice $\pounds$ is called prime if $p \leq x \vee y$, then either $p \leq x$ or $p \leq y$.

(2) Let S be a join-subset of $\pounds$. An element p of $\pounds$ satisfying $S \wedge p = 0$ is said to be S-prime if there exists an element $s \in S$ such that, whenever $x, y \in \pounds$, $p \leq x \vee y$ implies $p \leq x \vee s$ or $p \leq y \vee s$.

Example 2.2.

(1) If $S = \{0\}$, then the prime and the S-prime elements of $\pounds$ are the same.

(2) If p is a prime element of $\pounds$ with $p \wedge S = 0$, then p is S-prime.

(3) Assume that $\pounds = \{0, a, b, c, d, 1\}$ is a lattice with the relations $0 < a < d < 1$, $0 < b < d < 1$, $0 < c < 1$ and $a \wedge b = 0 = a \wedge c = d \wedge c = b \wedge c$. Set $S = \{0, c\}$. Then S is a join-subset of $\pounds$ and d is clearly an S-prime element of $\pounds$. Since $d \leq a \vee b = d$, $d \nleq a$ and $d \nleq b$, we conclude that d is not a prime element. Thus an S-pime element need not be a prime element.

(4) Every atom element p of $\pounds$ with $S \wedge p = 0$ is an S-prime element. To see this, it suffices to show that p is prime by (2). On the contrary, assume that p is not prime. Then there are elements $x, y \in \pounds$ such that $p \leq x \vee y$, $p \nleq x$ and $p \nleq y$. Since p is an atom element, we conclude that $p \wedge x = 0 = p \wedge y$

which implies that $p = p \wedge (x \vee y) = (p \wedge x) \vee (p \wedge y) = 0$ *which is impossible. So* p *is prime.*

Compare the next proposition with Proposition 1 in [9].

Proposition 2.3. *Let* p *be an element of* $\pounds$ *and* S *a join-subset of* $\pounds$ *with* $p \wedge S = 0$. *The following assertions are equivalent:*

(1) p *is an* S*-prime element of* $\pounds$*;*
(2) $\mathcal{A}(p) = \{x \in \pounds : p \leq x \vee s\}$ *is a prime filter of* $\pounds$ *for some* $s \in S$.

Proof. (1) $\Rightarrow$ (2) Since p is an S-prime element, we conclude that there is an element $s \in S$ such that for all $x, y \in \pounds$ with $p \leq x \vee y$ we have $p \leq x \vee s$ or $p \leq y \vee s$. Now, we show that $\mathcal{A}(p)$ is a prime filter of $\pounds$. Let $x, y \in \mathcal{A}(p)$ and $z \in \pounds$. Then $p \leq (x \vee s) \wedge (y \vee s) = (x \wedge y) \vee s$ and $p \leq x \vee s \vee z$ gives $x \wedge y, x \vee z \in \mathcal{A}(p)$. Thus $\mathcal{A}(p)$ is a filter of $\pounds$. Let $x, y \in \pounds$ such that $x \vee y \in \mathcal{A}(p)$. Then $p \leq x \vee y \vee s$ gives $p \leq x \vee s \vee s = x \vee s$ or $p \leq y \vee s$ which means that $x \in \mathcal{A}(p)$ or $y \in \mathcal{A}(p)$. Thus $\mathcal{A}(p)$ is a prime filter of $\pounds$.

(2) $\Rightarrow$ (1) Suppose that s has the stated property in (2) and let $p \leq x \vee y$ (so $p \leq x \vee y \vee s$) for some $x, y \in \pounds$. Then $x \vee y \in \mathcal{A}(p)$ gives $x \in \mathcal{A}(p)$ or $y \in \mathcal{A}(p)$, as $\mathcal{A}(p)$ is a prime filter. This shows that $p \leq s \vee x$ or $p \leq s \vee y$, as needed.

Compare the next proposition with Proposition 4 in [9].

Proposition 2.4. *Let* p *be an element of* $\pounds$ *and* S *a join-subset of* $\pounds$ *with* $p \wedge S = 0$. *Then* p *is an* S*-prime element if and only if there exists* $s \in S$ *such that for all* $x_1, x_2, \cdots, x_n \in \pounds$*, if* $p \leq x_1 \vee \cdots \vee x_n$*, then* $p \leq s \vee x_i$ *for some* $i \in \{1, \cdots, n\}$.

Proof. Let p be an S-prime element of $\pounds$. Then there is an element $s \in S$ such that for all $x, y \in \pounds$, if $p \leq x \vee y$, then $p \leq s \vee x$ or $p \leq s \vee y$. We use induction on n. We can take $n = 2$ as a base case. Let $n \geq 3$, assume that the property holds up to the order $n-1$ and let $x_1, \cdots, x_n$ elements of $\pounds$ such that $p \leq x_1 \vee \cdots \vee x_n = (x_1 \vee \cdots \vee x_{n-1}) \vee x_n$. Then by definition, $p \leq s \vee x_n$ or $p \leq (s \vee x_1) \vee x_2 \vee \cdots \vee x_{n-1}$. Therefore $p \leq s \vee x_n$ or ($p \leq s \vee s \vee x_1 = s \vee x_1$ or $p \leq s \vee x_i$ for some $i \in \{2, \cdots, n-1\}$). In the same way we prove that $p \leq s \vee x_i$ for some $i \in \{1, 2, \cdots, n\}$. The other implication is clear.

Corollary 2.5. *Let* p *be a proper element of* $\pounds$. *Then* p *is a prime element if and only if for all* $x_1, \cdots, x_n$ *elements of* $\pounds$*, if* $p \leq x_1 \vee \cdots \vee x_n$*, then* $p \leq x_i$ *for some* $i \in \{1, \cdots, n\}$.

Proof. Take $S = \{0\}$ in Proposition 2.4.

The following theorem is a lattice counterpart of Proposition 5 in [9] describing the structure of S-prime elements.

Theorem 2.6. *Let* q *be a nontrivial element of a complete lattice* $\pounds$ *and* S *a join-subset of* $\pounds$ *with* $S \wedge q = 0$. *Then there exists a minimal* S*-prime element* p *that satisfies* $p \leq q$ *is constructible.*

Proof. If $\Omega = \{x \in £ : x \leq q \text{ and } S \wedge x = 0\}$, then $q \in \Omega$, and so $\Omega \neq \emptyset$. We have Ω is ordered by $\geq$. Moreover, Ω is inductive. Indeed, if $\{z_i\}_{i\in\Delta}$ is a chain of elements of Ω, then $p' = \bigwedge_{i\in\Delta} z_i \leq q$ and $p' \wedge S = 0$; hence $p' \in \Omega$ is an upper bound for the chain. Then, by Zorn's Lemma, Ω has a maximal element for $\geq$ and so there exists a minimal element (so an atom element) p such that $p \leq q$. Now the assertion follows from Example 2.2 (4).

Definition 2.7. *Let S be a join-subset of $£$. We say that S is a strongly join-subset if for each family $\{s_i\}_{i\in\Lambda}$ of elements of S we have $(\cap_{i\in\Lambda}T(\{s_i\})) \cap S \neq \emptyset$.*

Example 2.8. *Let $S = \{s_1, \cdots, s_k\}$ be a join-subset of $£$ and let $\{s_{i_1}, \cdots, s_{i_t}\} \subseteq S$. Set $s = s_{i_1} \vee \cdots \vee s_{i_t}$ (so $s \in S$). Then for each $j \in \{i_1, \cdots, i_t\} = S'$, $s \in T(s_{i_j})$; hence $s \in (\cap_{j\in S'}T(\{s_{i_j}\})) \cap S$. Thus every finite join-subset of $£$ is a strongly join-subset.*

Compare the next theorem with Theorem 4 in [9].

Theorem 2.9. *Assume that S is a strongly join-subset of a complete lattice $£$ and let $\{p_i\}_{i\in\Lambda}$ be a chain of S-prime elements of $£$. Then $p = \bigvee_{i\in\Lambda} p_i$ is an S-prime element of $£$.*

Proof. For each $i \in \Lambda$, there is an element $s_i \in S$ such that for all $x, y \in £$ with $p_i \leq x \vee y$ we have $p_i \leq s_i \vee x$ or $p_i \leq s_i \vee y$. Since S is a strongly join-subset, we conclude that $(\cap_{i\in\Lambda}T(\{s_i\})) \cap S \neq \emptyset$. Consider $t \in (\cap_{i\in\Lambda}T(\{s_i\})) \cap S$. Then for each $i \in \Lambda$, $t = s_i \vee a_i$, where $a_i \in £$. Now, we will show that p is S-prime. Let $a, b \in £$ such that $p \leq a \vee b$ and suppose that $p \not\leq t \vee a$. Then $p_j \not\leq t \vee a$ for some $j \in \Lambda$. Let $k \in \Lambda$. Then $p_k \leq p_j$ or $p_j \leq p_k$. We split the proof into two cases.
Case 1: $p_j \leq p_k$. Since $p_j \not\leq t \vee a$, we conclude that $p_k \not\leq t \vee a = s_k \vee a_k \vee a$; so $p_k \not\leq s_k \vee a$. This shows that $p_k \leq s_k \vee b$; hence $p_k \leq s_k \vee a_k \vee b = t \vee b$. Thus $p \leq t \vee b$.
Case 2: $p_k \leq p_j$. Since $p_j \not\leq t \vee a = s_j \vee a_j \vee a$, we get that $p_j \not\leq s_j \vee a$; so $p_k \leq p_j \leq s_j \vee b$ which gives $p_k \leq t \vee b = s_j \vee a_j \vee b$, and so $p \leq t \vee b$.

We continue this section by investigation the stability of S-prime elements under homomorphism and in cartesian products of lattices.

Theorem 2.10. *Let S be a join-subset of $£$, $f : £ \to £'$ an order lattice homomorphism such that $f(0) = 0$ and $f(u) \neq 1$ for all $1 \neq u \in £$. The following hold:*

(1) If f is an epimorphism and p is an S-prime element, then $f(p)$ is an $f(S)$-prime element of $£'$;
(2) If f is a monomorphism and $f(p)$ is an $f(S)$-prime element of $£'$, then p is an S-pime element of $£$.

Proof.

(1) Clearly, $f(S)$ is a join-subset of $£'$. If $s \in S$, then $f(s) \wedge f(p) = f(s \wedge p) = f(0) = 0$ gives $f(p) \wedge f(S) = 0$, as $p \wedge S = 0$. Let $x, y \in £'$ such that $f(p) \leq x \vee y$. Then there exist $a, b \in £$ such that $x = f(a)$, $y = f(b)$ and

$f(p) \leq f(a \vee b) = x \vee y$ which implies that $p \leq a \vee b$. This shows that $p \leq a \vee s$ or $p \leq s \vee b$ for some $s \in S$. It means that $f(p) \leq f(s) \vee x$ or $f(p) \leq f(s) \vee y$. Therefore, p is an $f(S)$-prime element of $\pounds'$.

(2) By assumption, there exists $s \in S$ such that for all $x, y \in \pounds'$ if $f(p) \leq x \vee y$, then $f(p) \leq f(s) \vee x$ or $f(p) \leq f(s) \vee y$. If $s \in S$, then $f(s \wedge p) = f(s) \wedge f(p) = 0 = f(0)$ gives $s \wedge p = 0$; hence $p \wedge S = 0$. Let $a, b \in \pounds$ such that $p \leq a \vee b$; so $f(p) \leq f(a \vee b) = f(a) \vee f(b)$ which gives $f(p) \leq f(s) \vee f(a) = f(s \vee a)$ or $f(p) \leq f(s) \vee f(b) = f(s \vee b)$. This implies that $p \leq s \vee a$ or $p \leq s \vee b$, as required.

Assume that $(\pounds_1, \leq_1), (\pounds_2, \leq_2), \cdots, (\pounds_n, \leq_n)$ are lattices ($n \geq 2$) and let $\pounds = \pounds_1 \times \pounds_2 \times \cdots \times \pounds_n$. We set up a partial order $\leq_c$ on $\pounds$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \pounds$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be used in this paper: It is straightforward to check that $(\pounds, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\pounds$ is a decomposable lattice.

Proposition 2.11. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where for each i, S_i is a join-subset of $\pounds_i$. Suppose that $p = (p_1, p_2)$ is an element of $\pounds$. The following statements are equivalent:*

(1) p is an S-prime element of $\pounds$;
(2) p_1 is an S_1-prime element of $\pounds_1$ and $p_2 = 0$ or p_2 is an S_2-prime element of $\pounds_2$ and $p_1 = 0$.

PROOF. (1) $\Rightarrow$ (2) Let p be an S-prime element of $\pounds$ and suppose that $s = (s_1, s_2) \in S$ satisfies the S-prime condition. Since $p \wedge S = 0$, we conclude that $p_1 \wedge S_1 = 0$ and $p_2 \wedge S_2 = 0$. As $p \leq (0, 1) \vee_c (1, 0) = (1, 1)$, we get $p \leq s \vee_c (0, 1) = (s_1, 1)$ or $p \leq s \vee_c (1, 0) = (1, s_2)$ and hence $p_1 \leq s_1$ (so $p_1 = p_1 \wedge s_1 = 0$) or $p_2 \leq s_2$ (so $p_2 = p_2 \wedge s_2 = 0$). Without loss of generality, we can assume that $p_1 = 0$. Let $p_2 \leq x \vee y$ for some $x, y \in \pounds_2$. As $p \leq (1, x) \vee_c (1, y) = (1, x \vee y)$ and p is an S-prime element, we obtain either $p \leq s \vee_c (1, x) = (1, s_2 \vee x)$ or $p \leq s \vee_c (1, y) = (1, s_2 \vee y)$ and this yields $p_2 \leq s_2 \vee x$ or $p_2 \leq s_2 \vee y$. Hence p_2 is an S_2-prime element of $\pounds_2$. In the other case, one can similarly show that p_1 is an S_1-prime element of $\pounds_1$.

(2) $\Rightarrow$ (1) Suppose that $p_1 = 0$, $s_1 \in S_1$ and p_2 is an S_2-prime element of $\pounds_2$. Then $p_1 = 0$ and $p_2 \wedge S_2 = 0$ gives $p \wedge S = 0$. Let $p \leq (a, b) \vee_c (c, d)$ for some $a, c \in \pounds_1$ and $b, d \in \pounds_2$. This shows that $p_2 \leq b \vee d$ and hence there exists $s_2 \in S_2$ such that $p_2 \leq s_2 \vee b$ or $p_2 \leq s_2 \vee d$. Set $s = (s_1, s_2) \in S$. Then we have $p \leq s \vee_c (a, b) = (s_1 \vee a, s_2 \vee b)$ or $p \leq s \vee_c (c, d) = (s_1 \vee c, s_2 \vee d)$. Thus p is an S-prime element of $\pounds$. In the other case, one can similarly prove that p is an S-prime element of $\pounds$.

Theorem 2.12. *Let $\pounds = \pounds_1 \times \cdots \times \pounds_n$ be a decomposable lattice and $S = S_1 \times \cdots \times S_n$, where for each i, S_i is a join-subset of $\pounds_i$. Suppose that $p = (p_1, p_2, \cdots, p_n)$ is an element of $\pounds$. The following statements are equivalent:*

(1) p is an S-prime element of $\pounds$;

(2) *p_i is an S_i-prime element of $\pounds_i$ for some $i \in \{1, \cdots, n\}$ and $p_j = 0$ for all $j \in \{1, \cdots, n\} \setminus \{i\}$.*

PROOF. We use induction on n. For $n = 1$, the result is true. If $n = 2$, then (1) and (2) are equivalent by Proposition 2.11. Assume that (1) and (2) are equivalent when $k < n$. Set $p' = (p_1, \cdots, p_{n-1})$, $S' = S_1 \times \cdots \times S_{n-1}$ and $\pounds' = \pounds_1 \times \cdots \times \pounds_{n-1}$. Then by Proposition 2.11, $p = (p', p_n)$ is an S-prime element of $\pounds$ if and only if p' is an S'-prime element of $\pounds'$ and $p_n = 0$ or $p' \leq_c (t_1, \cdots, t_{n-1}) = t'$ for some $t' \in S'$ and p_n is a S_n-prime element of $\pounds_n$. Now the assertion follows from the induction hypothesis.

3. Characterization of weakly S-prime elements

In this section, the concept of weakly S-prime elements is introduced and investigated. We remind the reader with the following definition.

Definition 3.1.

(1) *A proper element p of a lattice is called weakly prime if $p \leq x \vee y \neq 1$, then either $p \leq x$ or $p \leq y$.*
(2) *Let S be a join-subset of $\pounds$. An element p of $\pounds$ satisfying $S \wedge p = 0$ is said to be weakly S-prime if there exists an element $s \in S$ such that, whenever $x, y \in \pounds$, $p \leq x \vee y \neq 1$ implies $p \leq x \vee s$ or $p \leq y \vee s$.*

Example 3.2.

(1) *A weakly prime element p of $\pounds$ is weakly S-prime for each join-subset S of $\pounds$ such that $S \wedge p = 0$.*
(2) *Assume that $\pounds = \{0, a, b, c, 1\}$ is a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$ and let $S = \{0, a\}$. Then c is a weakly S-prime element of $\pounds$. Also, c is not a weakly prime element of $\pounds$ because $c \leq a \vee b = c \neq 1$, $c \nleq a$ and $c \nleq b$. Thus a weakly S-prime element need not be a weakly prime element.*
(3) *If $S = \{0\}$, then the weakly prime and the weakly S-prime elements of $\pounds$ are the same.*
(4) *Every S-prime element is a weakly S-prime element. To see this, Assume that $s \in S$ satisfies S-prime condition and let $p \leq a \vee b \neq 1$ for some $a, b \in \pounds$. Then p is S-prime gives $p \leq s \vee a$ or $p \leq s \vee b$, as required.*
(5) *Let $D = \{a, b, c\}$. Then $\pounds = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and the least element $\emptyset$ (note that if $x, y \in \pounds$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $p = 1$ and $S = \{\{a\}, \emptyset\}$. Then S is a join-subset of $\pounds$ and p is clearly a weakly S-prime element of $\pounds$. Since $p \leq \{a, b\} \vee \{c\}$, $p \nleq \{a\} \vee \{a, b\}$ and $p \nleq \{a\} \vee \{c\}$, it follows that p is not a S-prime element of $\pounds$. Thus a weakly S-prime element need not be an S-prime element.*

An element x of $\pounds$ is called identity join of a lattice $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. An element $x \in \pounds$ is said to be regular, whenever $x \vee y = 1$

implies $y = 1$ (i.e. it is an element that is not an identity join). Compare this proposition with Proposition 14 in [3].

Proposition 3.3. *Let p be an element of $\pounds$ and S be a join-subset of $\pounds$ consisting of regular elements with $p \wedge S = 0$. The following assertions are equivalent:*

(1) p is a weakly S-prime element of $\pounds$;
(2) $\mathcal{A}(p) = \{x \in \pounds : p \leq x \vee s\}$ is a weakly prime filter of $\pounds$ for some $s \in S$.

Proof. (1) $\Rightarrow$ (2) Since p is a weakly S-prime, then there is an element $s \in S$ such that for all $x, y \in \pounds$ with $p \leq x \vee y \neq 1$, we have $p \leq x \vee s$ or $p \leq y \vee s$. Now, we show that $\mathcal{A}(p)$ is a weakly prime filter of $\pounds$. Let $x, y \in \pounds$ such that $1 \neq x \vee y \in \mathcal{A}(p)$. Then $p \leq x \vee y \vee s \neq 1$ gives $p \leq x \vee s \vee s = x \vee s$ or $p \leq y \vee s$ which means that $x \in \mathcal{A}(p)$ or $y \in \mathcal{A}(p)$. Thus $\mathcal{A}(p)$ is a weakly prime filter of $\pounds$.

(2) $\Rightarrow$ (1) Suppose that s has the stated property in (2) and let $p \leq x \vee y \neq 1$ (so $p \leq x \vee y \vee s \neq 1$) for some $x, y \in \pounds$. Then $1 \neq x \vee y \in \mathcal{A}(p)$ gives $x \in \mathcal{A}(p)$ or $y \in \mathcal{A}(p)$, as $\mathcal{A}(p)$ is a weakly prime filter. This shows that $p \leq s \vee x$ or $p \leq s \vee y$, as needed.

Compare the next Theorem with Proposition 4 in [3].

Theorem 3.4. *Let S be a join-subset of $\pounds$ and let p a weakly S-prime element of $\pounds$. If p is not S-prime, then $p = 1$.*

Proof. By hypothesis, there exists $s \in S$ such that, whenever $x, y \in \pounds$, $p \leq x \vee y \neq 1$ implies $p \leq s \vee x$ or $p \leq s \vee y$. On the contrary, assume that $p \neq 1$. We show that p is S-prime. Let $a, b \in \pounds$ such that $p \leq a \vee b$. If $p \leq a \vee b \neq 1$, then p being a weakly S-prime gives $p \leq s \vee a$ or $p \leq s \vee b$. Now, suppose that $a \vee b = 1$. Since $p \leq (a \wedge p) \vee (b \wedge p) = p \neq 1$, we have $p \leq s \vee (a \wedge p) = (s \vee a) \wedge (s \vee p)$ or $p \leq (s \vee b) \wedge (s \vee p)$. Therefore, $p \leq s \vee a$ or $p \leq s \vee b$. This shows that p is an S-prime element, as required.

Compare the next corollary with Corollary 5 in [3].

Corollary 3.5. *If p is a weakly prime element that is not prime, then $p = 1$.*

Proof. Take $S = \{0\}$ in Theorem 3.4.

Compare the next theorem with Theorem 7 in [3].

Theorem 3.6. *Assume that p is an element of $\pounds$ and let S be a join-subset of $\pounds$ with $p \wedge S = 0$. The following assertions are equivalent:*

(1) p is a weakly S-prime element of $\pounds$;
(2) There exists $s \in S$ such that for each $x \in \pounds$ with $p \not\leq x \vee s$ we have either $\{y \in \pounds : p \leq x \vee y\} \subseteq \{y \in \pounds : p \leq s \vee y\}$ or $\{y \in \pounds : p \leq x \vee y\} = \{y \in \pounds : x \vee y = 1\}$.

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists $s \in S$ such that, whenever $x, y \in \pounds$, $p \leq x \vee y$ implies $p \leq s \vee x$ or $p \leq s \vee y$. Let $p \not\leq x \vee s$ and suppose that $\{y \in \pounds : p \leq x \vee y\} \neq \{y \in \pounds : x \vee y = 1\}$; so we conclude that there exists $a \in \pounds$ such that $p \leq a \vee x$ and $a \vee x \neq 1$. Thus $p \leq a \vee x \neq 1$ gives $p \leq a \vee s$, as p is weakly S-prime. Let $p \leq z \vee x$ $(z \in \pounds)$. If $z \vee x \neq 1$, then $p \leq z \vee s$. Now, suppose that $x \vee z = 1$. Then $p \leq a \vee x = (a \vee x) \wedge (x \vee z) = x \vee (a \wedge z) \neq 1$ implies that $p \leq s \vee (a \wedge z) = (s \vee a) \wedge (s \vee z)$; hence $p \leq s \vee z$, i.e. for each $z \in \pounds$, $p \leq x \vee z$ implies $p \leq z \vee s$. Thus $\{y \in \pounds : p \leq x \vee y\} \subseteq \{y \in \pounds : p \leq s \vee y\}$.

(2) $\Rightarrow$ (1) Let $x, y \in \pounds$ such that $p \leq x \vee y \neq 1$ with $p \not\leq s \vee x$. Since $p \leq x \vee y$ and $x \vee y \neq 1$, we conclude that $p \leq y \vee s$ by (2). This shows that p is weakly S-prime.

Compare the next corollary with Corollary 8 in [3].

Corollary 3.7. *For a proper element p of $\pounds$, the following are equivalent:*

(1) p is a weakly prime element of $\pounds$;
(2) For each $x \in \pounds$ with $p \not\leq x$ we have either $\{y \in \pounds : p \leq x \vee y\} = \{y \in \pounds : p \leq y\}$ or $\{y \in \pounds : p \leq x \vee y\} = \{y \in \pounds : x \vee y = 1\}$.

Proof. Take $S = \{0\}$ in Theorem 3.6.

Compare the next proposition with Proposition 17 in [3].

Proposition 3.8. *Suppose that S is a join-subset of $\pounds$. The following assertions are equivalent:*

(1) Every weakly S-prime element of $\pounds$ is prime;
(2) $\pounds$ is a $\pounds$-domain and every S-prime element of $\pounds$ is prime.

Proof. (1) $\Rightarrow$ (2) Since 1 is a weakly S-prime element, we conclude that it is a prime element by (1) which implies that $\pounds$ is a $\pounds$-domain. Finally, since every S-prime element p of $\pounds$ is weakly S-prime, we get p is prime by (1).

(2) $\Rightarrow$ (1) Suppose that p is a weakly S-prime element; we show that p is S-prime. Let $a, b \in \pounds$ such that $p \leq a \vee b$. If $a \vee b \neq 1$, then there exists $s \in S$ such that $p \leq s \vee a$ or $p \leq s \vee b$. If $a \vee b = 1$, then $a = 1$ or $b = 1$; so $p \leq s \vee a = 1 = a$ or $p \leq s \vee b = 1 = b$ for every $s \in S$. Consequently, every weakly S-prime element of $\pounds$ is prime by (2).

We close this section by investigation the stability of weakly S-prime elements in various lattice-theoretic constructions.

Example 3.9. *Let $S' \subseteq S$ be join-subsets of $\pounds$ and p an element of $\pounds$ with $p \wedge S = 0$. It is clear that if p is a weakly S'-prime element of $\pounds$, then p is a weakly S-prime element. However, the converse is not true in general. Indeed, assume that $\pounds$ is the lattice as in Example 2.2 (3) and let $S' = \{0\} \subseteq S = \{0, a\}$. Then $p = c$ is a weakly S-prime element of $\pounds$ but not a weakly S'-prime element of $\pounds$.*

Compare the next proposition with Proposition 20 in [3].

Proposition 3.10. *Let $S' \subseteq S$ be join-subsets of $\pounds$ such that for any $s \in S$, there exists $t \in S$ satisfying $s \vee t \in S'$. If p is a weakly S-prime element of $\pounds$, then p is a weakly S'-prime element of $\pounds$.*

Proof. Let $x, y \in \pounds$ such that $p \leq x \vee y \neq 1$. Then there exists $s \in S$ such that $p \leq s \vee x$ or $p \leq s \vee y$. By the hypothesis, there is $t \in S$ such that $s \vee t \in S'$

and then $p \leq s \vee t \vee x$ or $p \leq s \vee t \vee y$. This shows that p is a weakly S'-prime element.

Compare the next theorem with Proposition 22 in [3].

Theorem 3.11. *Let S be a join-subset of $\pounds$, $f : \pounds \to \pounds'$ a order lattice homomorphism such that $f(0) = 0$, $f(1) = 1$ and $f(u) \neq 1$ for all $1 \neq u \in \pounds$. The following hold:*

(1) If f is an epimorphism and p is a weakly S-prime element, then $f(p)$ is a weakly $f(S)$-prime element of $\pounds'$;

(2) If f is a monomorphism and $f(p)$ is a weakly $f(S)$-prime element of $\pounds'$, then p is a weakly S-prime element of $\pounds$.

Proof. (1) If $s \in S$, then $f(p) \wedge f(s) = f(s \wedge p) = f(0) = 0$ gives $f(p) \wedge f(S) = 0$. Let $x, y \in \pounds'$ such that $f(p) \leq x \vee y \neq 1$. Then there exist $a, b \in \pounds$ such that $x = f(a)$, $y = f(b)$ and $f(p) \leq f(a \vee b) = x \vee y$ (so $a \vee b \neq 1$) which implies that $p \leq a \vee b \neq 1$. By assumption, $p \leq s \vee a$ or $p \leq s \vee b$ for some $s \in S$. It means that $f(p) \leq f(s) \vee x$ or $f(p) \leq f(s) \vee y$. Therefore, $f(p)$ is a weakly $f(S)$-prime element of $\pounds'$.

(2) By assumption, there exists $s \in S$ such that for all $x, y \in \pounds'$, $f(p) \leq x \vee y$ implies $f(p) \leq f(s) \vee x$ or $f(p) \leq f(s) \vee y$. If $s \in S$, then $f(s \wedge p) = f(s) \wedge f(p) = 0 = f(0)$ giving $s \wedge p = 0$; hence $p \wedge S = 0$. Let $a, b \in \pounds$ such that $p \leq a \vee b \neq 1$. Since f is injective, we conclude that $f(p) \leq f(a \vee b) = f(a) \vee f(b) \neq 1$; so $f(p) \leq f(s) \vee f(a) = f(s \vee a)$ or $f(p) \leq f(s) \vee f(b) = f(s \vee b)$. Hence, $p \leq s \vee a$ or $p \leq s \vee b$, and so p is a weakly S-prime filter of $\pounds$.

Compare the next Theorem with Proposition 24 in [3].

Theorem 3.12. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where for each i, S_i is a join-subset of $\pounds_i$. Suppose that $p = (p_1, p_2)$ is an element of $\pounds$, where $p_1 \neq 1$ and $p_2 \neq 1$. The following statements are equivalent:*

(1) p is a weakly S-prime filter of $\pounds$;

(2) p_1 is an S_1-prime element of $\pounds_1$ and $p_2 = 0$ or p_2 is an S_2-prime element of $\pounds_2$ and $p_1 = 0$;

(3) p is an S-prime element of $\pounds$.

Proof. (1) $\Rightarrow$ (2) Since $p \wedge S = 0$, we conclude that $p_1 \wedge S_1 = 0$ and $p_2 \wedge S_2 = 0$. As $p \leq (p_1, 0) \vee_c (0, p_2) \neq (1, 1)$, we conclude that there exists $s = (s_1, s_2) \in S$ such that $p \leq s \vee_c (p_1, 0) = (s_1 \vee p_1, s_2)$ or $p \leq s \vee_c (0, p_2) = (s_1, s_2 \vee p_2)$ which implies that $p_1 \leq s_1$ (so $p_1 = 0$) or $p_2 \leq s_2$ (so $p_2 = 0$). Suppose that $p_2 = 0$. Now, we claim that p_1 is an S_1-prime element of $\pounds_1$. Let $x, y \in \pounds_1$ such that $p_1 \leq x \vee y$. Then $p \leq (x, 0) \vee_c (y, 0) = (x \vee y, 0) \neq (1, 1)$ which gives $p \leq s \vee_c (x, 0) = (s_1 \vee x, s_2)$ or $p \leq s \vee_c (y, 0) = (s_1 \vee y, s_2)$ implying that $p_1 \leq s_1 \vee x$ or $p_1 \leq s_1 \vee y$. In the other case, one can similarly show that p_2 is an S_1-prime element of $\pounds_2$.

(2) $\Rightarrow$ (3) Follows directly from Proposition 2.11.

(3) $\Rightarrow$ (1) Follows from Example 3.2 (4) since every S-prime is a weakly S-prime element.

Corollary 3.13. *Let £ be a decomposable lattice and S a join-subset of £. A proper element p of £ with $S \wedge p = 0$ is a weakly S-prime if and only if $p = 1$ or p is S-prime.*

Proof. This follows from Theorem 3.12.

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