J. Indones. Math. Soc.
Special Edition (2011), pp. 27–38.

# ON THE EDGE-BALANCE INDEX SETS OF $L$-PRODUCT OF CYCLES

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**Abstract.** Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. Any edge labeling $f$ induces a partial vertex labeling $f^+ : V(G) \to \mathbb{Z}_2$ assigning 0 or 1 to $f^+(v)$, $v$ being an element of $V(G)$, depending on whether there are more 0-edges or 1-edges incident with $v$, and no label is given to $f^+(v)$ otherwise. For each $i \in \mathbb{Z}_2$, let $v_f(i) = \left|\{v \in V(G) : f^+(v) = i\}\right|$ and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. An edge-labeling $f$ of $G$ is said to be edge-friendly if $\left\{|e_f(0) - e_f(1)| \le 1\right\}$. The edge-balance index set of the graph G is defined as $EBI(G) = \left\{|v_f(0) - v_f(1)| : f \text{ is edge-friendly.}\right\}$. In this paper, exact values of the edge-balance index sets of $L$-product of cycles with cycles, $C_n \times_L C_m$ are presented.

*Key words*: Edge labeling, edge-friendly labeling, cordiality, edge-balance index set, $L$-products, cycles.

**Abstrak.** Misalkan $G$ adalah graf sederhana dengan himpunan titik $V(G)$ dan himpunan sisi $E(G)$, dan misalkan $\mathbb{Z}_2 = \{0,1\}$. Setiap pelabelan sisi $f$ menginduksi pelabelan titik parsial $f^+ : V(G) \to \mathbb{Z}_2$ memberikan 0 atau 1 untuk $f^+(v)$, dengan $v$ titik di $V(G)$, tergantung pada apakah terdapat lebih sisi-sisi-0 atau sisi-sisi-1 terkait dengan $v$, dan yang lain adalah tidak ada label diberikan untuk $f^+(v)$. Untuk setiap $i \in \mathbb{Z}_2$, misalkan $v_f(i) = \left|\{v \in V(G) : f^+(v) = i\}\right|$ dan misalkan $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. Sebuah pelabelan sisi $f$ dari $G$ dikatakan ramah-sisi jika $\left\{|e_f(0) - e_f(1)| \le 1\right\}$. Himpunan indeks seimbang-sisi dari graf G didefinisikan sebagai $EBI(G) = \left\{|v_f(0) - v_f(1)| : f \text{ adalah ramah-sisi}\right\}$. Dalam paper ini disajikan nilai-nilai eksak himpunan-himpunan indeks seimbang-sisi hasil-kali-$L$ dari lingkaran dengan lingkaran, $C_n \times_L C_m$.

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*2000 Mathematics Subject Classification*: 05C78, 05C25.
Received: 09-08-2011, revised: 09-09-2011, accepted: 04-12-2011.

*Kata kunci*: Pelabelan sisi, pelabelan ramah-sisi, kordialitas, himpunan indeks seimbang-sisi, hasil-kali-$L$, lingkaran.

## 1. Introduction

In [8], Kong and Lee considered a new labeling problem of graph theory. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the edges labeled 0 incident on $v$ is more than the number of edges labeled 1 incident on $v$, and $f^+(v) = 1$ if the edges labeled 1 incident on $v$ is more than the number of edges labeled 0 incident on $v$. $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$.

With these notations, we now introduce the notion of an edge-balanced graph.

**Definition 1.1.** *An edge labeling $f$ of a graph $G$ is said to be* **edge-friendly** *if $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is said to be an* **edge-balanced** *graph if there is an edge-friendly labeling $f$ of $G$ satisfying $|v_f(0) - v_f(1)| \leq 1$.*

Chen, Lee, et al in [2] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

**Definition 1.2.** *The* **edge-balance index set** *of the graph $G$, EBI($G$), is defined as $\{|v_f(0) - v_f(1)| :$ the edge labeling $f$ is edge-friendly.$\}$.*

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

**Example 1.** EBI $(nK_2)$ is $\{0\}$ if $n$ is even and $\{2\}$ if $n$ is odd. □

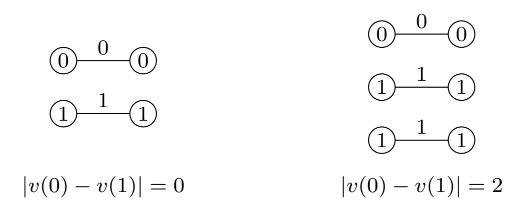


Figure 1. The edge-balance index sets of $2K_2$ and $3K_2$.

For any $n \geq 1$, we denote the tree with $n + 1$ vertices of diameter two by St($n$). The star has a center $c$ and $n$ appended edges from $c$.

**Example 2.** The edge-balance index set of the star St($n$) is

$$\text{EBI}\,(\text{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

□

**Example 3**. In [16], Lee, Lo and Tao showed that

$$\mathrm{EBI}\,(P_n) = \begin{cases} \{2\} & \text{if } n \text{ is } 2; \\ \{0\} & \text{if } n \text{ is } 3; \\ \{1,2\} & \text{if } n \text{ is } 4; \\ \{0,1\} & \text{if } n \text{ is odd and greater than } 3; \\ \{0,1,2\} & \text{if } n \text{ is even and greater than } 4. \end{cases}$$

Figure 2 shows the edge-balance index sets of $P_3$ and $P_4$. □

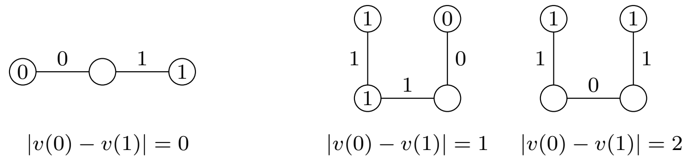


FIGURE 2. The edge-balance index set of $P_3$ and $P_4$.

**Example 4**. Figure 3 shows that the edge-balance index set of a tree with six vertices is $\{0,1,2\}$. □

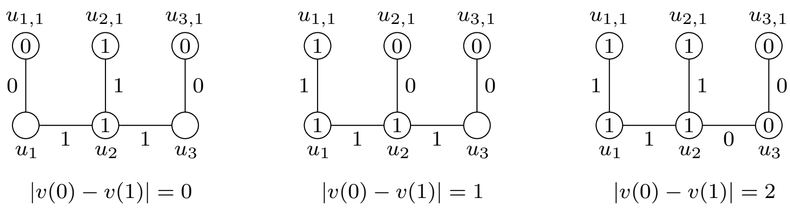


FIGURE 3. The edge-balance index set of a tree with six vertices.

Edge-balance index sets of trees, flower graphs and $(p, p+1)$-graphs were considered in [10, 16, 18].

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [9, 11, 12, 13, 14, 15, 17, 19, 20]. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$ for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling $f$ of a graph $G$ is said to be ***friendly*** if $|v_f(0) - v_f(1)| \le 1$. If $|e_f(0) - e_f(1)| \le 1$ then $G$ is said to be ***balanced***. The ***balance index set*** of the graph $G$, $\mathrm{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

**Definition 1.3.** *Let $H$ be a connected graph with a distinguished vertex $s$. Construct a new graph $G \times_L (H, s)$ as follows: take $|V(G)|$ copies of $(H, s)$ and identify each vertex of $G$ with $s$ of a single copy of $H$. We call the resulting graph the* **$L$-product** *of $G$ and $(H, s)$.*

Note here that, in [7], Koh, Rogers and Tan defined the same graph operation and used the notation $G \triangle H$, instead of $G \times_L (H, s)$.

In a series of papers [1, 5], Chou et al. investigated the edge-balance index sets of $L$-product of cycles with stars, $C_n \times_L (\mathrm{St}(m), c))$. The edge-balance index of $L$-product of graphs are also considered in [6]. In this paper, exact values of the edge-balance index sets of $L$-product of cycles with cycles, $C_n \times_L C_m$, are presented.

## 2. The Highest Edge-balance Index

In order to find the full set of the edge-balance indexes, we begin by finding the highest edge-balance index first. The following notations and propositions are borrowed from [3]. You can also find them in [4].

**Notation 1**. Let $C_n$ be a cycle with a vertex set $\{c_1, c_2, \ldots, c_n\}$. Let $f : E(C_n) \to \mathbb{Z}_2$ be an edge labeling on $C_n$ (not necessarily edge-friendly), where $E(C_n)$ is the edge set of $C_n$. We denote the numbers of edges labeled 0 or 1 by $f$ by $e_C(0)$ or $e_C(1)$, respectively. We also denote the number of vertices labeled 0, 1, or not labeled by $f^+$ by $v_C(0)$, $v_C(1)$, or $v_C(\times)$, respectively.

**Proposition 2.1.** *In a cycle $C_n$ with an edge labeling $f$ (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(1) = e_C(1) - k.$$

**Proposition 2.2.** *In a cycle $C_n$ with an edge labeling $f$ (not necessarily edge-friendly), assume that $v_C(\times) = 2k > 0$. Then*

$$v_C(0) = n - e_C(1) - k.$$

We note here that when $v_C(\times) = 0$, i.e., either $e_C(0) = n$ or $e_C(1) = n$, the above propositions are still true.

For a finite disjoint union of cycles, by using the same technique from [3], we can calculate $v_C(0)$ and $v_C(1)$ for each cycle $C$ and then add all up to get

**Theorem 2.3.** *In a finite disjoint union of cycles $\cup_i C^i_{n_i}$ (for notational convenience, we still call it $C$) with an edge labeling $f$ (not necessarily edge-friendly), we have*

$$v_C(0) - v_C(1) = \sum_i n_i - 2e_C(1).$$

We note here that, for a $C_n \times_L C_m$ graph, since there are exactly $n$ outer cycles with $m$ vertices each, it follows that $\sum_i n_i = nm$. In general, it is also true that $e(C_n \times_L C_m) = n(m+1)$.

If we remove all edges of $C_n$ from a $C_n \times_L C_m$ graph, we have a disjoint union of $C_m$ cycles. For convenience, let us call $C_n$ the inner cycle and all copies of $C_m$ the outer cycles. We also call the edges of $C_n$ inner edges and the edges of $C_m$ outer edges.

Theorem 2.3 suggests that $v_C(0) - v_C(1)$ is maximized when $e_C(1)$ is minimized. In order to minimize $e_C(1)$, we label all the edges of $C_n$ with 1's.

We now consider the $n$ number of degree 4 vertices, each with two 1-labeled edges from $C_n$ and two currently unlabeled edges from their respective $C_m$ cycle. Ignoring any edge-friendly labeling restriction for the moment, there are three possible cases to consider in labeling the last two edges of one of the outer cycles:

(1) One edge is labeled 0 and the other 1;
(2) Both edges are labeled with 0's;
(3) Both edges are labeled with 1's.

In order to maximize $v_C(0) - v_C(1)$ in Theorem 2.3, we maximize $v_C(0)$ and minimize $v_C(1)$ at the same time. Consider any single outer cycle by itself, unattached to the inner cycle $C_n$, with each of our three possible labeling choices above. Under the first choice, the unattached outer cycle's vertex would be unlabeled and attaching it to $C_n$ will change the vertex to a 1, a net gain of one 1-vertex. With the second choice, the unattached outer cycle's vertex would be labeled 0 and attaching it to $C_n$ would result in an unlabeled vertex, a net loss of one 0-vertex. The third choice yields a vertex labeling of 1 on the unattached outer cycle and still a 1-vertex when attached to $C_n$.

We see that only the third choice keeps the edge-balance index unchanged. The other two choices reduce it by 1. Therefore, it follows that, in order to maximize $v_C(0)-v_C(1)$, we want to label the remaining edges adjacent to the degree 4 vertices with 1's.

By Theorem 2.3, the remaining unlabeled edges of our graph can be labeled in any assortment and, as long as the labeling is friendly, it yields the highest edge-balance index. However, for the purposes of the later proofs, we require that in each outer cycle at least one of the edges adjacent to the 4 degree vertex is also adjacent to a 0-edge.

We note here that in order to achieve this labeling, we must have $3n$ 1-edges in order to completely label all $n$ edges of $C_n$ and all $2n$ adjacent edges to the 4

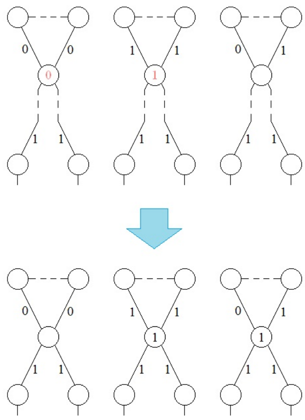

FIGURE 4. The change of the labels of the order 4 vertices.

degree vertices in the outer cycles. In order to guarantee there are enough 1-edges, $m$ must be 5 or greater.

**Theorem 2.4.** *The highest edge-balance index of $C_n \times_L C_m$ for $m \geq 5$ is*

$$\begin{cases} n & \text{if } m \text{ is odd;} \\ n & \text{if } n \text{ is even;} \\ n+1 & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

**Proof**. If $n$ is even, then we assume $n = 2s$, where $s \in \mathbb{N}$. The number of edges is then $e(C_n \times_L C_m) = 2s(m+1)$. Therefore, $e(0) = e(1) = s(m+1)$. After using our suggested labeling, we have

$$e_C(1) = e(1) - n = s(m+1) - 2s = sm - s.$$

Since $C_n$ provides no new vertices to what would be the disjoint union of outer cycles and with our labeling no vertices are changed whether the cycles are connected or disconnected, by Theorem 2.3, we have the highest edge-balance index

$$v(0) - v(1) = nm - 2e_C(1) = 2sm - 2(sm - s) = 2s = n.$$

Similarly, when $m$ is odd, we assume $m = 2t + 1$, where $t \in \mathbb{N}$. Since it is already shown that the highest edge-balance index is $n$ when $n$ is even, we only have to prove the case where both $m$ and $n$ are odd. It is then assumed that $n = 2s+1$, where $s \in \mathbb{N}$. The number of edges is then $e(C_n \times_L C_m) = (2s+1)(2t+2)$. Therefore, we have $e(0) = e(1) = (2s+1)(t+1)$. After using our suggested labeling, we have

$$e_C(1) = e(1) - n = (2s+1)(t+1) - (2s+1) = 2st + t.$$

Thus, the highest edge-balance index when $m$ is odd is

$$v(0) - v(1) = nm - 2e_C(1) = (2s+1)(2t+1) - 2(2st+t) = 2s+1 = n.$$

Finally, for $n$ is odd and $m$ is even, we assume that $n = 2s+1$ and $m = 2t$, where $s, t \in \mathbb{N}$. The number of edges is then $e(C_n \times_L C_m) = (2s+1)(2t+1) = 4st+2t+2s+1 = 2(2st+t+s)+1$. Since $e(C_n \times_L C_m)$ is odd, due to the symmetric roles of 0 and 1 in the edge-balance labeling, without loss of generality, we may assume that $e(1) \leq e(0)$. Therefore, we can see that $e(0) = 2st+t+s+1$ and $e(1) = e(1) - n = 2st+t+s$. After using our suggested labeling, we have

$$e_C(1) = 2st+t+s-(2s+1) = 2st+t-s-1.$$

Thus, the highest edge-balance index when $n$ is odd and $m$ is even is

$$v(0) - v(1) = nm - 2e_C(1) = (2s+1)(2t) - 2(2st+t-s-1) = 2s+2 = n+1.$$

This completes the proof. $\square$

## 3. The Edge-Balance Index Set of $C_n \times_L C_m$ for $m \geq 5$

From Theorem 2.4, we can conclude that

**Corollary 3.1.** *The edge-balance index set of $C_n \times_L C_m$ for $m \geq 5$ is a subset of*

$$\begin{cases} \{0,1,2,\dots,n\} & \text{if } m \text{ is odd;} \\ \{0,1,2,\dots,n\} & \text{if } n \text{ is even;} \\ \{0,1,2,\dots,n+1\} & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

In order to show that the edge-balance index set of $C_n \times_L C_m$ contains all the numbers less then the highest edge-balance index, we observe from our suggested edge labeling that we can switch a 0-edge we specified to be adjacent to one of the 1-edges which are adjacent to the 4 degree vertex in the outer cycle. Both cases are demonstrated in the Figures 5 and 6.

We can see that, no matter the edge adjacent to the 0-edge is labeled 0 or 1, this reduces the current edge-balance index by exactly 1, each time it is done to an outer cycle. This strategy enables us to prove that

**Theorem 3.2.** *The edge-balance index set of $C_n \times_L C_m$ for $m \geq 5$ is*

$$EBI(C_n \times_L C_m) = \begin{cases} \{0,1,2,\dots,n\} & \text{if } m \text{ is odd;} \\ \{0,1,2,\dots,n\} & \text{if } n \text{ is even;} \\ \{0,1,2,\dots,n+1\} & \text{if } n \text{ is odd and } m \text{ is even.} \end{cases}$$

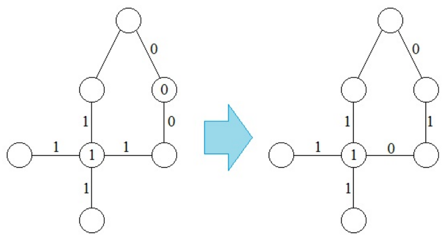


FIGURE 5. Switching edges when the adjacent edge is labeled 0.

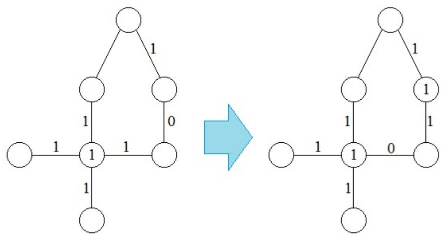


FIGURE 6. Switching edges when the adjacent edge is labeled 1.

**Proof**. When $m$ is odd or $n$ is even, since there are $n$ outer cycles, the above strategy provides the edge-balance indexes from $n-1$ all the way to 0. Therefore, the edge-balance index set is $\{0, 1, 2, \ldots, n\}$.

When $n$ is odd and $m$ is even, a similar process of $n$ outer cycles can only provides the edge-balance index to be any integer between $n$ and 1. Thus, the edge-balance index set contains $\{1, 2, 3, \ldots, n+1\}$.

To show $\mathrm{EBI}(C_n \times_L C_m)$ when $n$ is odd and $m$ is even includes 0 as well, a entirely new labeling is necessary. Since $n$ is odd, we may assume that $e(0) > e(1)$ due to symmetry. First, we label $0, 1, 0, 1, 0, 1, \ldots, 0, 1, 0$ to the edges of inner cycle. Since $n$ is odd, the very last label in the sequence must be 0. Thus, there is only one vertex, namely $v$, in the inner cycle $C_n$ to have two 0-edges as of now. Because the sequence has one more 0 then 1, we have the same number of 0- and 1-edges for outer cycles. For any outer cycle not adjacent to $v$, we label its edges by the sequence $0, 1, 0, 1$. It is easy to see that every vertex is unlabeled. For the only outer cycle adjacent to $v$, we label two edges adjacent to the order 4 vertex 0 and the outer two edges 1. Thus, for this cycle, the order 4 vertex is labeled 0, two of the adjacent order 2 vertices are unlabeled, and the not-adjacent order 2 vertex is labeled 1. Therefore, for the whole edge-balance labeling, we have only one 0-vertex and one 1-vertex. All other vertices are unlabeled. It provides an edge-balance index 0. □

## 4. The Edge-balance Index Set of $C_n \times_L C_m$ when $m$ is 3 or 4.

When $m$ is 3 or 4, as discussed before, we do not have enough number of 1-edges to be placed in outer cycles to avoid reducing the highest edge-balance index while putting back 1-edges for the inner cycle. But, we can still pair 1-edges to keep the label of the order 4 vertices unchanged as many as possible to produce the highest edge-balance index.

**Theorem 4.1.** *The highest edge-balance index of* $C_n \times_L C_4$ *is*

$$\begin{cases} \lfloor \frac{3k}{2} \rfloor & \text{if } n = 2k \text{ is even;} \\ \lfloor \frac{3k+1}{2} \rfloor + 1 & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

**Proof**. When $n = 2k$ is even, the total number of edges of $C_n \times_L C_4$ is $5n = 10k$. Thus, in an edge-balance labeling, there are $5k$ 1-edges. As in section 2, with Theorem 2.3, we maximize the edge-balance index by labeling all inner cycle edges 1. This requires $n = 2k$ 1-edges. Therefore, $e_C(1) = 3k$. So, we have $\lfloor \frac{3k}{2} \rfloor$ 1-edges pairs to be placed in the outer cycle edges of the order 4 vertices. According to the Figure 4, the other two cases cause a reduction by 1 for the edge-balance index. It affects $n - \lfloor \frac{3k}{2} \rfloor$ labels of the order 4 vertices. The highest edge-balance index becomes

$$\begin{aligned} e(0) - e(1) &= \sum_i n_i - 2e_C(1) - (\text{the number of the reduced labels}) \\ &= 4n - 2e_C(1) - \left(n - \lfloor \frac{3k}{2} \rfloor\right) \\ &= 4(2k) - 2\,(3k) - \left(2k - \lfloor \frac{3k}{2} \rfloor\right) \\ &= \lfloor \frac{3k}{2} \rfloor. \end{aligned}$$

When $n = 2k+1$ is odd, the total number of edges of $C_n \times_L C_4$ is $5n = 10k+5$. Due to the symmetry, we may assume that $e(0) > e(1)$. Thus, in an edge-balance labeling, there are $5k+2$ 1-edges. By using $n = 2k+1$ 1-edges to label inner edges, there are $e_C(1) = 3k+1$ 1-edges left for outer cycles. By the same argument, the highest edge-balance index is reduced by $n - \lfloor \frac{3k+1}{2} \rfloor$. Thus, it becomes

$$
\begin{aligned}
e(0) - e(1) &= \sum_i n_i - 2e_C(1) - (\text{the number of the reduced labels}) \\
&= 4n - 2e_C(1) - \left( n - \lfloor \frac{3k+1}{2} \rfloor \right) \\
&= 4(2k+1) - 2\,(3k+1) - \left( 2k + 1 - \lfloor \frac{3k+1}{2} \rfloor \right) \\
&= \lfloor \frac{3k+1}{2} \rfloor + 1.
\end{aligned}
$$

□

**Theorem 4.2.** *The edge-balance index of $C_n \times_L C_4$ is*

$$
EBI(C_n \times_L C_4) = \begin{cases} \left\{0, 1, 2, \dots, \lfloor \frac{3k}{2} \rfloor \right\}, & \text{if } n = 2k \text{ is even,} \\ \left\{0, 1, 2, \dots, \lfloor \frac{3k+1}{2} \rfloor + 1 \right\}, & \text{if } n = 2k+1 \text{ is odd.} \end{cases}
$$

**Proof**. When $n$ is even, by Theorem 4.1, we see that the highest edge-balance index is $\lfloor \frac{3k}{2} \rfloor$. This number also represents the number of order 4 vertices with all four edges labeled 1. Therefore, the same switching strategy from Figure 5 and 6 can produce the edge-balance index for any number between $\lfloor \frac{3k}{2} \rfloor - 1$ and 0.

Similarly, when $n$ is odd, the number of order 4 vertices with all four edges labeled by 1 for switching is enough to produce the edge-balance indexes for any number between $\lfloor \frac{3k+1}{2} \rfloor$ and 1.

A similar edge-labeling as the one in the last paragraph of the proof of Theorem 3.2 shows that EBI($C_n \times_L C_4$) when $n$ is odd includes 0 as well. This completes the proof. □

**Theorem 4.3.** *The edge-balance index of $C_n \times_L C_3$ is*

$$
EBI(C_n \times_L C_3) = \left\{0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor \right\}.
$$

**Proof**. The total number of edges of $C_n \times_L C_3$ is $4n$, which is always even. Thus, in a edge-balance labeling, there are $2n$ 1-edges. Similarly, we use $n$ 1-edges to label all inner edges in order to maximize the edge-balance index. Therefore, $e_C(1) = n$. So, we have $\lfloor \frac{n}{2} \rfloor$ 1-edges-pairs to be placed in the outer cycle edges of the order 4 vertices. According to the Figure 4, the other two cases causes a reduction by 1 for the edge-balance index. It affects $n - \lfloor \frac{n}{2} \rfloor$ labels of the order 4 vertices. The highest edge-balance index becomes

$$\begin{aligned} e(0)-e(1) &= \sum_i n_i - 2e_C(1) - (\text{the number of the reduced labels}) \\ &= 3n - 2n - \Big(n - \lfloor \frac{n}{2} \rfloor\Big)) \\ &= \lfloor \frac{n}{2} \rfloor. \end{aligned}$$

The number of order 4 vertices with all four edges labeled by 1 is $\lfloor \frac{n}{2} \rfloor$. Therefore, the same switching strategy from Figure 5 and 6 can produce the edge-balance index for any number between $\lfloor \frac{n}{2} \rfloor - 1$ and 0.

This completes the proof. □

**Acknowledgment** We would like to thank the Stonehill Undergraduate Research Experience (SURE) program for supporting and funding during summer 2010 to make this paper possible.

We would also like to thank the referee for the thorough, constructive and helpful comments and suggestions on the manuscript. Thank you very much for sharing your opinion and advice.

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