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EFFECTS OF INVERSION LAYER ON THE ATMOSPHERIC POLLUTANT DISPERSION FROM A HIGH CHIMNEY

Fidelis Nofertinus Zai[a], Agus Yodi Gunawan[b,*]

Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, 40132 Bandung, Indonesia, [a]fidelisnofertinus@gmail.com, [b]ayodi@itb.ac.id

Abstract. An inversion layer is a layer in the lower atmosphere at a certain height through which there is no transport of pollutants. It plays as a significant factor in the formation of air pollutants where they are trapped. In this paper, a mathematical model describing an atmospheric pollutant dispersion from a high chimney in the presence of an inversion layer is constructed. The aim of the model is to predict the concentration of pollutants at ground level. The advection-diffusion equation governs the concentration of a pollutant released into the air. An analytical solution procedure via the integral transforms is presented for the steady-state case. Solutions are entirely determined by two parameters, i.e., the source strength emanating from the chimney and the height of the inversion layer. The pollutant concentration on the ground level with some multiple source formations will be explored, and also for various values of inversion layer height. Results show that the lower the inversion layer, the higher the pollutant concentration on the ground level is.

Key words and Phrases: inversion layer, dispersion, advection-diffusion equation, integral transforms.

1. INTRODUCTION

Air pollution is a very complex phenomenon, nowadays especially, the problem of air pollution must be accompanied by human activity, characterized by emissions from chimneys that are a source of air pollution [15]. In this moment, air quality standards that should be satisfying have always been a problem, especially in developing countries [4]. Air pollutants that come from various sources will affect humans and the environment directly or indirectly [7]. The impact of air pollution is not only on public health but also reduces economic levels and quality of life

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[12]. According to this, one of the important things that can be done is to view the pollutant concentration on the ground level [5].

An approach to measure and analyze pollutant concentrations is by modeling the dispersion of atmospheric pollutants. That model will be used to determine pollutant distribution, air quality, and regulate emission sources. The term dispersion, in this case, is used to describe the combination of diffusion and advection (due to wind) that occurs in the air. Under these circumstances, diffusion is the process by which air pollutants are transported from one part of the system to another as a result of random molecular motion [1]. The advection-diffusion equation can be used to describe air pollutant concentrations.

The model is solved analytically, so the solution from the model will be used in carrying out the simulation. This analytical solution is especially useful for studying pollutant transport, as it gives the possibility to perform parameter sensitivity analysis and estimate emission sources. The simplest exact solution is called a Gaussian plume, corresponding to a point source emitting contaminants in the direction of the wind in the infinite domain [10]. The Gaussian plume model has been widely used in various applications, such as the transportation of smoke from volcanic eruptions [14] and the spread of odors from the farm facility [8].

An earlier study of pollutant concentration modeling was previously done by Stockie [10], by modeling the dispersion of atmospheric pollutants using the advection-diffusion equation. In the atmosphere, the air temperature decreases with increasing altitude. However, there are situations where an inversion or increase in temperature occurs. This layer is an inversion layer that causes pollutants to be trapped. So in this study, the model that has been obtained will be expanded by adding an inversion layer in the atmosphere which will affect the domain. In addition, the model will be analyzed through a dimensionless model and it will be used to see how the wind direction influences the formation of a given number of chimneys.

2. MODEL FORMULATION

This study will discuss pollutant concentrations, with the pollutant concentrations at location $\vec{x} = (x, y, z) \in \mathbb{R}^3$ $[m]$ and at time $t \geq 0$ will be considered by using the basic model that has been obtained by Stockie [10]. This can be described by a function $C(\vec{x}, t)$ $[kg/m^3]$ which is constructed based on the law of conservation of mass, to be more precise, expressed in differential form in Equation (1).

$$\frac{\partial C}{\partial t} + \nabla \cdot \vec{J} = S. \tag{1}$$

where $S(\vec{x}, t)$ $[kg/m^3]$ denote the emission sources and the vector functions $\vec{J}(\vec{x}, t)$ $[kg/m^2 s]$ denote the pollutant mass flux resulting from the process of diffusion and advection. In atmospheric diffusion, Fick's law states that the diffusion flux is proportional to the concentration gradient or can be written as $\vec{J}_D = -K\nabla C$. The negative sign indicates that diffusion moves from an area of high concentration to an area of low concentration and K $[m^2/s]$ is the diffusion coefficient. In addition, in the pollutant transport process, there is also a linear advection process which is caused by wind or can be written as $\vec{J}_A = C\vec{u}$ with $\vec{u}$ $[m/s]$ is the wind speed. From these two contributing factors, we obtained the equation $\vec{J} = \vec{J}_D + \vec{J}_A = C\vec{u} - K\nabla C$. So, from Equation (1) we get the advection-diffusion equation,

$$\frac{\partial C}{\partial t} + \nabla \cdot (C\vec{u}) = \nabla \cdot (K\nabla C) + S. \tag{2}$$

Equation (2) will be used in constructing an atmospheric pollutant dispersion model, with the presence of an inversion layer as illustrated in Figure 1.

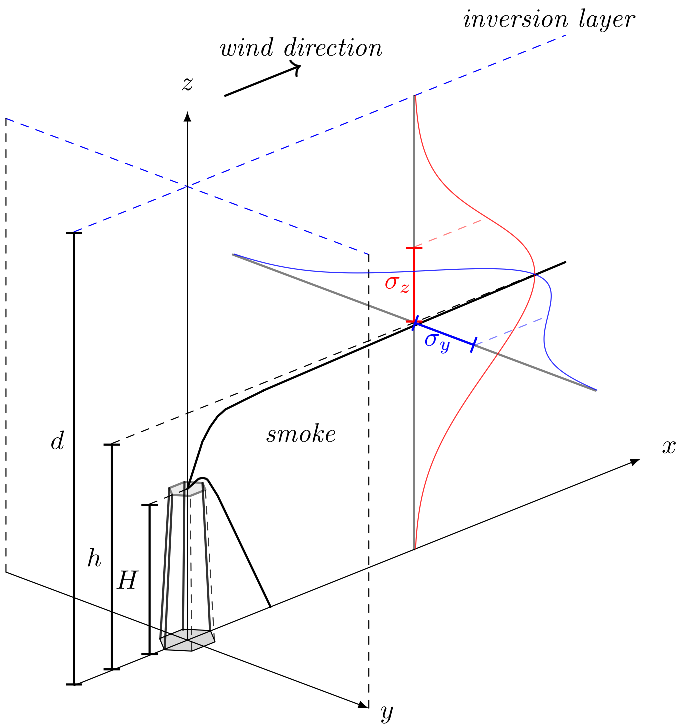


FIGURE 1. Illustration of atmospheric pollutant dispersion with inversion layer

The atmospheric pollutant dispersion model will be constructed based on the following assumptions [10]:

(1) The level of emission source contaminants is considered to be constant with value Q $[kg/s]$ from a single point source $\vec{x} = (0, 0, H)$ $[m]$ located at a height H from the ground surface as in Figure 1. So it can be written as

$$S(\vec{x}) = Q\delta(x)\delta(y)\delta(z - H),$$

where δ $[m^{-1}]$ is a Dirac delta function.

(2) Wind speed is constant and parallel to the positive x-axis, so $\vec{u} = (u, 0, 0)$ for $u \geq 0$. Furthermore, it is assumed that the wind speed is uniform with respect to time.

(3) The solution is steady (not dependent on time) and Q is constant.

(4) The *Eddy* diffusion is a function of the wind direction x only and the diffusion is isotropic (uniform in all directions) $K_x(x) = K_y(x) = K_z(x) =: K(x)$.

(5) The wind speed is large enough so that diffusion on the direction of x-axis is much smaller than advection, $K_x \partial_x^2 C$ can be ignored.

(6) Topographic variations can be neglected so that the ground surface can be taken as a plane $z = 0$ (ground level).

Based on the assumptions above, Equation (2) becomes

$$u\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial y^2} + K\frac{\partial^2 C}{\partial z^2} + Q\delta(x)\delta(y)\delta(z - H), \tag{3}$$

with $0 \leq x < \infty$, $-\infty < y < \infty$, $0 \leq z \leq d$ and boundary conditions,

$$C(0, y, z) = 0, \quad C(\infty, y, z) = 0, \quad C(x, \pm\infty, z) = 0.$$

In addition, the boundary conditions at ground level can be obtained,

$$K\frac{\partial C}{\partial z}(x, y, 0) = p,$$

where $p \neq 0$ is the rate in case of pollutant deposition occurs on the ground surface and in this case it is assumed that p can be determined empirically [13]. On the inversion layer, pollutants will be isolated. Thus, by using the Neumann boundary conditions, the boundary conditions at the inversion layer can be obtained,

$$K\frac{\partial C}{\partial z}(x, y, d) = 0.$$

Based on the Stackgold Theorem [9], the equivalent equation is obtained from Equation (3) for $x > 0$:

$$\begin{aligned} u\frac{\partial C}{\partial x} &= K\frac{\partial^2 C}{\partial y^2} + K\frac{\partial^2 C}{\partial z^2}, \\ C(0, y, z) &= \frac{Q}{u}\delta(y)\delta(z - H), \end{aligned} \tag{4}$$

with other boundary conditions, same as before.

3. SOLUTION METHODOLOGY

In the model that has been obtained, the Eddy diffusion (K) is a highly challenging function to determine [10]. In practice the standard deviation function of the pollutant distribution based on the width and height of the pollutant, σ_y and σ_z as shown in Figure 1 is applied [3]. These parameters are defined as follows,

$$\sigma^2(x) = \frac{2}{u}\int_0^x K(\xi)\mathrm{d}\xi. \tag{5}$$

The values of σ_y and σ_z can be obtained using various forms, one of which is the simple power law, where $\sigma_y = R_y x^{r_y}$ and $\sigma_z = R_z x^{r_z}$. While, R_y, r_y, R_z, and r_z are coefficients obtained based on atmospheric stability conditions [6].

To simplify obtaining an analytical solution, a variable transformation is performed by defining,

$$r = \frac{1}{u}\int_0^x K(\xi)\mathrm{d}\xi. \tag{6}$$

with r $[m^2]$ and obtain that $\sigma^2 = 2r$. So Equation (4) become,

$$\frac{\partial c}{\partial r} = \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}, \tag{7}$$

with $c(r, y, z) := C(x, y, z)$ and the boundary conditions for c which is the same as Equation (4) by replacing the variable x with r. Suppose a solution of Equation (7) is $c(r, y, z) = \frac{Q}{u} a(r, y) \cdot b(r, z)$. Thus, two diffusion equations are obtained:

$$\begin{aligned} &\frac{\partial a}{\partial r} = \frac{\partial^2 a}{\partial y^2}, \text{ with } 0 \le r < \infty,\ -\infty < y < \infty, \\ &a(0, y) = \delta(y),\ a(\infty, y) = 0,\ a(r, \pm\infty) = 0. \end{aligned} \tag{8}$$

and

$$\begin{gathered} \frac{\partial b}{\partial r} = \frac{\partial^2 b}{\partial z^2}, \text{ with } 0 \le r < \infty,\ 0 \le z \le d, \\ b(0, z) = \delta(z - H),\ b(\infty, z) = 0,\ \frac{\partial b}{\partial z}(r, 0) = p,\ \frac{\partial b}{\partial z}(r, d) = 0. \end{gathered} \tag{9}$$

The equation (8) solved by Laplace transform for variables r because $r \in [0, \infty)$ and the Fourier transform for variables y because $y \in (-\infty, \infty)$. While Equation (9) is an eigenvalue problem [11], so it has a solution in the form of a Fourier series. So a solution is obtained from Equation (7):

$$\begin{aligned} c(r, y, z) =& \frac{Qp}{u\sqrt{4\pi r}} \exp\left(-\frac{y^2}{4r}\right) \times \\ & \left(-\frac{z^2 + 2r}{2d} + z - \frac{d}{3} + \frac{2d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi z}{d}\right)\exp\left(-r\left(\frac{n\pi}{d}\right)^2\right)\right) + \\ & \frac{Q}{ud\sqrt{\pi r}}\exp\left(-\frac{y^2}{4r}\right)\left(\frac{1}{2} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi z}{d}\right)\cos\left(\frac{n\pi H}{d}\right)\exp\left(-r\left(\frac{n\pi}{d}\right)^2\right)\right). \end{aligned} \tag{10}$$

Next, the solution is expressed in terms of the standard deviation of the pollutant distribution. By substituting $\sigma_y = 2r$ into the solution of Equation (8) and substituting $\sigma_z = 2r$ into the solution of Equation (9), we obtain the solution in terms of the standard deviation of the pollutant distribution (σ) as follows,

$$\begin{aligned} C(x,y,z) =& \frac{Qp}{u\sigma_y\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) \times \\ & \left(-\frac{z^2+\sigma_z^2}{2d} + z - \frac{d}{3} + \frac{2d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi z}{d}\right)\exp\left(-\left(\frac{n\pi\sigma_z}{d\sqrt{2}}\right)^2\right)\right) \\ & + \frac{Q\sqrt{2}}{ud\sigma_y\sqrt{\pi}}\exp\left(-\frac{y^2}{2\sigma_y^2}\right) \times \\ & \left(\frac{1}{2} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi z}{d}\right)\cos\left(\frac{n\pi H}{d}\right)\exp\left(-\left(\frac{n\pi\sigma_z}{d\sqrt{2}}\right)^2\right)\right). \end{aligned} \tag{11}$$

4. SIMULATION

The model simulation will be carried out in two parts, namely the model with the Eddy diffusion coefficient (K) and the model with the standard deviation of the pollutant distribution (σ). The simulation of the model with the Eddy diffusion coefficient is carried out based on the parameter values in Table 1, whereas the Eddy diffusion coefficient is assumed to be constant. Know that information in Table 1 is taken from [10] without mentioning the units.

TABLE 1. Parameter Model with Eddy Diffusion Coefficient (K)

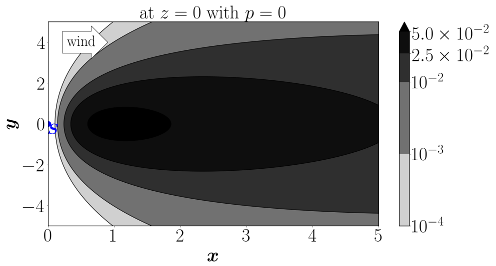
Parameter	Value	Reference
Source contaminant level (Q)	1	[10]
Wind velocity (u)	1	[10]
The height of the pollutant source (H)	2	[10]
Eddy diffusion coefficient (K)	1	[10]
The height of the inversion layer (d)	10	Assumed

Next, a model with a standard deviation of pollutant distribution (σ) is used in simulations with multiple sources. The simulation of the model with the standard deviation of the pollutant distribution is carried out based on the parameter values in Table 2.

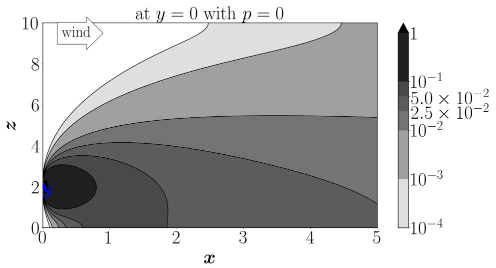
TABLE 2. Parameter model with σ

Parameter	Value	Units	Reference
Source contaminant level (Q)	$[1.1,\ 2.5,\ 0.16,\ 0.16] \times 10^{-3}$	kg/s	[10]
Wind velocity (u)	5	m/s	[10]
The height of the pollutant source (H)	$[15, 35, 15, 15]$	m	[10]
$(R_y,\ r_y,\ R_z,\ r_z)$	$(0.32,\ 0.78,\ 0.22,\ 0.78)$	-	[6]
The height of the inversion layer (d)	100	m	Assumed

In the dispersion of atmospheric pollutants with an inversion layer, a simulation will be carried out based on the analytical solutions that have been obtained. By using Equation (10), simulation of atmospheric pollutant dispersion are obtain based on the parameter values in Table 1, as follows:



C level	Area
$C \geq 0.05$	1.6374
$0.025 \leq C < 0.05$	16.0356
$0.01 \leq C < 0.025$	16.0429
$0.001 \leq C < 0.01$	10.5887
$0.0001 \leq C < 0.001$	1.9419



C level	Area
$C \geq 1$	0.0412
$0.1 \leq C < 1$	1.3388
$0.05 \leq C < 0.1$	3.9404
$0.025 \leq C < 0.05$	10.9498
$0.01 \leq C < 0.025$	9.1992
$0.001 \leq C < 0.01$	12.5079
$0.0001 \leq C < 0.001$	5.2992

FIGURE 2. Pollutants Concentration with $p = 0$.

In Figure 2, the results are obtained in the $x - y$ contour with $z = 0$ (ground level) and the $x - z$ contour with $y = 0$. Here, it is chosen that $p = 0$. This indicates that there is no deposition of pollutants on the ground surface. Meanwhile, if pollutant deposition occurs on the ground surface the pollutant concentration on the ground surface will decrease as shown in Figure 3 (see concentration levels).

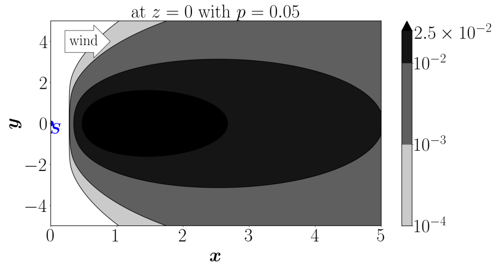
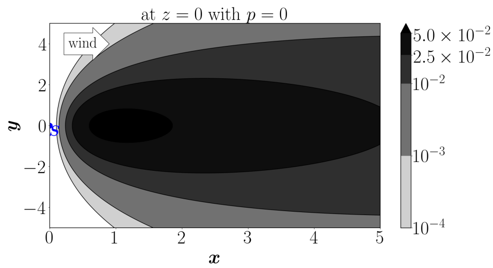
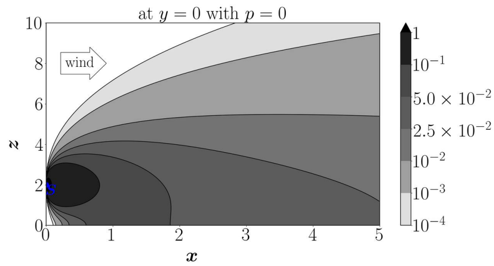


Figure 3. Concentration of Pollutants at ground level with $p = 0.05$.

Based on Equation (10), a simulation is also carried out when $d \to \infty$. This is done to see the results when the inversion layer is far above the ground. The simulation obtained will be compared with the results obtained by Stockie [10].



C level	Area
$C \geq 0.05$	1.6374
$0.025 \leq C < 0.05$	16.0356
$0.01 \leq C < 0.025$	16.0429
$0.001 \leq C < 0.01$	10.5887
$0.0001 \leq C < 0.001$	1.9419

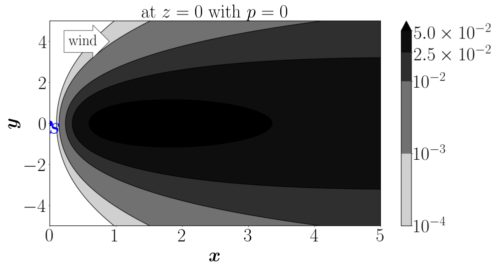


C level	Area
$C \geq 1$	0.0412
$0.1 \leq C < 1$	1.3388
$0.05 \leq C < 0.1$	3.9404
$0.025 \leq C < 0.05$	10.9498
$0.01 \leq C < 0.025$	9.1986
$0.001 \leq C < 0.01$	11.9199
$0.0001 \leq C < 0.001$	5.7673

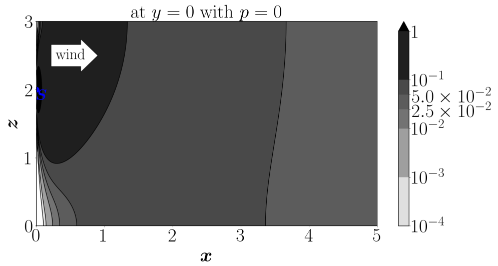
Figure 4. Pollutants Concentration with $p = 0$ and $d = 100000$.

Based on Figure 4, the total area of the contour $x - z$ for $d = 100000$ is small than $d = 10$. This is due to the effect of the inversion layer which traps pollutants. So the lower the height of the inversion layer the higher the pollutant concentration. The simulation image obtained is consistent with result that was obtaine by Stockie [10].

Simulations were also conducted for very low inversion layer positions to see how the height of the inversion layer affects pollutant concentrations, as shown in Figure 5.



C level	Area
$C \geq 0.05$	5.0870
$0.025 \leq C < 0.05$	19.4568
$0.01 \leq C < 0.025$	12.6087
$0.001 \leq C < 0.01$	7.2175
$0.0001 \leq C < 0.001$	1.8833

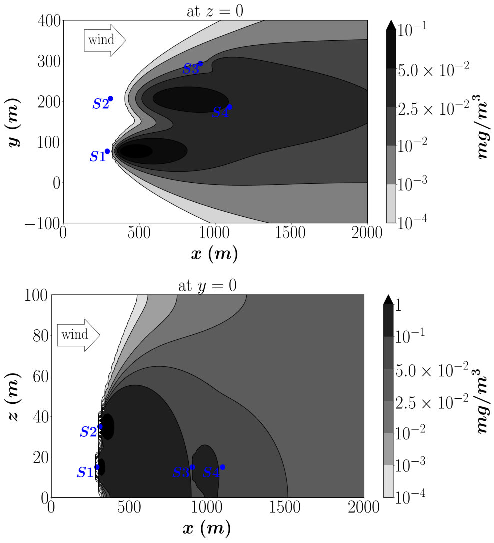


C level	Area
$C \geq 1$	0.0408
$0.1 \leq C < 1$	2.0773
$0.05 \leq C < 0.1$	8.0333
$0.025 \leq C < 0.05$	4.5873
$0.01 \leq C < 0.025$	0.0715
$0.001 \leq C < 0.01$	0.0674
$0.0001 \leq C < 0.001$	0.0274

FIGURE 5. Pollutant concentration with $d = 3$

Figure 5 shows the pollutant concentration results with a very low inversion layer position. Based on the simulation, it is known that the concentration of pollutants at the ground surface is very high and the area affected by pollutants is larger. This is caused by the position of the inversion layer very close to the ground surface, so more pollutants are isolated.

Furthermore, by using Equation (11), a simulation of the dispersion of atmospheric pollutants with multiple sources is obtained based on the parameter values in Table 2, as shown in Figure 6. The simulation is obtained by the location of the source on the x, y, z-axis, i.e., S_1 (288, 77, 15), S_2 (308, 207, 35), S_3 (900, 293, 15), and S_4 (1093, 186, 15).

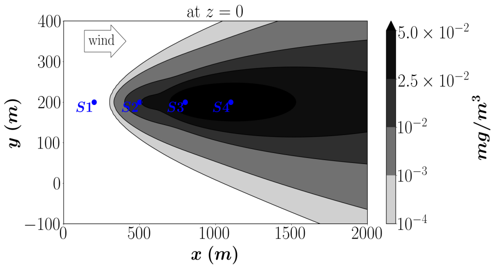


C level	Area
$C \geq 0.1$	3333.0066
$0.05 \leq C < 0.1$	17155.1809
$0.025 \leq C < 0.05$	212430.1539
$0.01 \leq C < 0.025$	314380.9430
$0.001 \leq C < 0.01$	173218.3119
$0.0001 \leq C < 0.001$	28918.7335

C level	Area
$C \geq 1$	1097.9316
$0.1 \leq C < 1$	19468.6795
$0.05 \leq C < 0.1$	22625.2328
$0.025 \leq C < 0.05$	41544.9466
$0.01 \leq C < 0.025$	68679.5412
$0.001 \leq C < 0.01$	6371.9243
$0.0001 \leq C < 0.001$	2058.6217

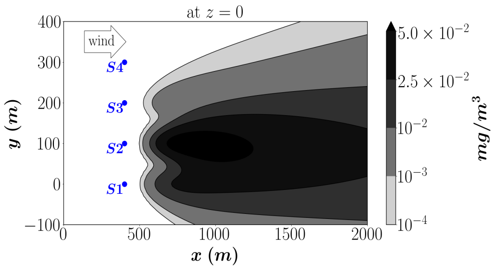
FIGURE 6. Pollutant concentration with multiple sources (chimney).

Based on Figure 6, pollutant concentration simulations with various sources (chimneys) are obtained. The simulation shows that the concentration of pollutants around the chimney is very high. In addition, pollutants also move from a point source and spread in the direction of the wind. This gives the same results as the simulation when using K. Furthermore, a simulation is also carried out to see the concentration of pollutants on the surface by varying the location of the sources, where the emission sources are arranged to form a straight line that is parallel and perpendicular to the wind direction (x-axis). The simulation was carried out based on the parameter values in Table 2, with the concentration results at the ground surface as shown in Figure 7.



C level	Area
$C \geq 0.05$	60778.3551
$0.025 \leq C < 0.05$	140280.3647
$0.01 \leq C < 0.025$	124399.5687
$0.001 \leq C < 0.01$	194588.7658
$0.0001 \leq C < 0.001$	106852.2694

(A) Source location parallel to the $x-$axis



C level	Area
$C \geq 0.05$	31271.4440
$0.025 \leq C < 0.05$	189981.3744
$0.01 \leq C < 0.025$	162925.2034
$0.001 \leq C < 0.01$	177727.6738
$0.0001 \leq C < 0.001$	88226.6444

(B) Source location perpendicular to the $x-$axis

FIGURE 7. Pollutant concentration by varying location of emission sources

Based on Figure 7, the simulation is obtained by the location of the source on the x, y, z-axis. For the source location parallel to the x-axis, given the source location, i.e., S_1 (200, 200, 35), S_2 (500, 200, 35), S_3 (800, 200, 35), and S_4 (1100, 200, 35). For the source location perpendicular to the x-axis, given the source location, i.e., S_1 (400, 0, 35), S_2 (400, 100, 35), S_3 (400, 200, 35), and S_4 (400, 300, 35). Furthermore, by observing the pollutant concentration on the ground level ($x - y$ contour), it is obtained that if the chimneys are located in a straight line and perpendicular to the wind direction, the high pollutant area on the ground is wider than that of the parallel one.

5. CONCLUSION

Based on the results of research that has been done, it can be obtained that the concentration of pollutants on the ground surface can be viewed using the advection-diffusion equation. Furthermore, the existence of an inversion layer causes pollutants to be trapped, resulting in a wider area affected by pollutants, compared to the absence of the inversion layer. Meanwhile, it is found that, if the inversion layer is very close to the emission source (chimney), it results in very high pollutant concentrations on the ground. For the multiple sources, it is obtained that if the chimneys are located in a straight line and perpendicular to the wind direction, the high pollutant area on the ground is wider than that of the parallel one. For future research, modification of the model will be carried out by adding vegetation factors that can reduce pollutant concentrations.

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