J. Indones. Math. Soc.
Vol. 30, No. 03 (2024), pp. 447–467.

UNVEILING THE RELATIONSHIP BETWEEN M-POLYNOMIAL BASED TOPOLOGICAL INDICES AND INVERSE GRAPHS OF FINITE CYCLIC GROUPS: A COMPREHENSIVE STUDY

K. Mageshwaran[1], S. Gopinath[2*], R. Siluvaidasan[3]

[1,3]Department of Mathematics, Rajalakshmi Engineering College, Tamil Nadu, India
[1]maheamadha@gmail.com, [3]siluvaidasan.r@rajalakshmi.edu.in
[2*]Department of Mathematics, Koneru Lakshmaiah Education Foundation, Andhra Pradesh, India
[2*]gopinathmathematics@gmail.com

Abstract. In the discipline of graph theory, topological indices are extremely important. The M-polynomial is a powerful tool for determining a graph's topological indices. The use of M-polynomials to describe macro-molecules and biochemical networking is a novel concept. Also, the M-polynomial of various micro-structural allows us to calculate a variety of topological indices. The chemical substances and biochemical networks are correlated with their chemical characteristics and bioactive compounds using these findings. In this research, we use the M-polynomial to create special essential topological indices of inverse graphs on finite cyclic groups, such as Randic, Zagreb, Augmented Zagreb, Harmonic, Inverse sum, and Symmetric division degree indices.

Key words and phrases: degree, M-polynomial, topological indices, inverse graph

1. Introduction

Graph theory is a major sub-field in mathematics that analyses and discusses many configurations. Numerical methods are used to answer difficulties that arise mostly during the study of features and combinations of different patterns in this discipline. Graph theory is a new and easily grasped mathematical concept with several implications in fields as diverse as biochemistry, medicine, computer programming, and operations research. Graph theory explains the various characteristics of networks [1].

The topological indices in graphs display the appropriate technical or organizational as well as a variety of other features. They depend heavily on vertex

2020 Mathematics Subject Classification:05C05, 05C10, 05C25, 05C75
Received: 07-11-2023, accepted: 11-11-2024.

lengths, vertex degrees, or the structure described by the matrix. Several network topological indices are crucial in computer-aided modeling, molecular chemistry, and medicinal research. Polynomials appear in a variety of shapes and sizes, including Hosoya polynomials, Pi-polynomials, and M-polynomials. The M-polynomial is still the most comprehensive polynomial, including significantly more knowledge on the graph in consideration [2]. The topological indices are often computed using the parameters, but they can alternatively be found using the derivatives and integrals on M-polynomials of the specified network. The primary aspect of this polynomial is that this could provide accurate representations for small molecules with more than ten degrees of freedom. One can refer to transferring chemical descriptors to algebraic structures in [38, 39, 40, 41].

Let G be a simple finite connected graph, $V(\mathsf{G})$ is its vertex set and $E(\mathsf{G})$ is its edge set with $|V(\mathsf{G})| = p$ and $|E(\mathsf{G})| = q$. The number of edges that are incident to a graph's vertex is called its degree. A vertex ϑ_i's degree is typified by $d_{\mathsf{G}}(\vartheta_i)$.

2. Fundamentals

To calculate topological indices (TIs) for the graphs under consideration, the following criteria and theorems were utilised. Wiener index will be the first topological index, which was created in 1947 [3]. Randic index (RI) was first developed in [6], just after the effectiveness of the Wiener index [4, 5]. The RI formula is defined as:

$$RI(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \frac{1}{\sqrt{d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j})}}.$$

B. Bollobas [7] with D. Amic [8] created and described the broad Randic index separately. Certain mathematicians and chemists have frequently used these due to their favorable and essential conclusions throughout the field of cognitive chemistry. See references [9, 10, 11, 12] for a survey of these findings. The generalized and inverse Randic indexes are calculated as follows:

$$RI_\alpha(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} (d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j}))^\alpha,$$
$$RIR_\alpha(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \frac{1}{(d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j}))^\alpha}.$$

This must be the most comprehensive index yet, and it has been thoroughly researched [6, 13]. The initial and subsequent Zagreb indices (ZIs) are mentioned as follows in [14]:

$$ZI_f(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} d_{\mathsf{G}}(\vartheta_i) + d_{\mathsf{G}}(\vartheta_{i+j}),$$

$$ZI_s(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j}).$$

We refer to [15, 16, 17, 18, 19] for more information on the implementations of these indexes. The formula for the redefined ZI is as follows:

$$^{m}ZI_s(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \frac{1}{d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j})}.$$

The given graph's symmetric division degree index has recently been implemented [20]. It's a major parameter [21] that is applied to find the maximum contact domain in chemistry,

$$SSD(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \left(\frac{min(d_{\mathsf{G}}(\vartheta_i), d_{\mathsf{G}}(\vartheta_{i+j}))}{max(d_{\mathsf{G}}(\vartheta_i), d_{\mathsf{G}}(\vartheta_{i+j}))} + \frac{max(d_{\mathsf{G}}(\vartheta_i), d_{\mathsf{G}}(\vartheta_{i+j}))}{min(d_{\mathsf{G}}(\vartheta_i), d_{\mathsf{G}}(\vartheta_{i+j}))}\right).$$

Other notable TIs include harmonic index (HI) [22], which seems to be a version of RI and has the following mathematical model:

$$HI(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \frac{2}{d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j})}.$$

The way of describing the inverse sum index is as follows [23]:

$$ISI(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \frac{d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j})}{d_{\mathsf{G}}(\vartheta_i) + d_{\mathsf{G}}(\vartheta_{i+j})}.$$

The heat of production of alkanes is best approximated by the augmented Zagreb index [24, 25]. It's written as [26]:

$$AZI(\mathsf{G}) = \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \left(\frac{d_{\mathsf{G}}(\vartheta_i)d_{\mathsf{G}}(\vartheta_{i+j})}{d_{\mathsf{G}}(\vartheta_i) + d_{\mathsf{G}}(\vartheta_{i+j}) - 2}\right)^3.$$

The topological indexes mentioned above are significant for chemical researchers [27, 28, 29, 30, 31], and computing these indices requires a lot of computer work [32]. M-polynomial was created to save computing work [33], and this basic polynomial may be used to obtain practically every degree-dependent index [34]. This polynomial's mathematical model is

$$\begin{aligned} M[\mathsf{G};\tau,\omega] &= \sum_{\vartheta_i\vartheta_{i+j}\in E(\mathsf{G})} \left(\tau^{d_{\mathsf{G}}(\vartheta_i)}\omega^{d_{\mathsf{G}}(\vartheta_{i+j})}\right) \\ &= \sum_{\delta\le s\le t\le\triangle} m_{st}\left(\tau^s\omega^t\right). \end{aligned}$$

Definition 2.1. [35, 36, 37] Let $(\Upsilon, *)$ be a finite group and $S = \{\mu \in \Upsilon | \mu \neq \mu^{-1}\}$. The inverse graph $G_S(\Upsilon)$ associated with Υ is the graph with set of vertices corresponds Υ so that two vertices are different μ and ν are adjacent iff $\mu\nu \in S$ (or) $\nu\mu \in S$.

Since Υ is taken as the cyclic group $\mathbb{Z}_\wp$, S will be the non self-invertible set elements of $\mathbb{Z}_\wp$. Then the inverse graph obtained from $\mathbb{Z}_\wp$ will be typified by $\mathsf{G} = \mathsf{G}_S(\mathbb{Z}_\wp)$. The inverse graph $\mathsf{G} = \mathsf{G}_S(\mathbb{Z}_{11})$ is illustrated in Figure 1.

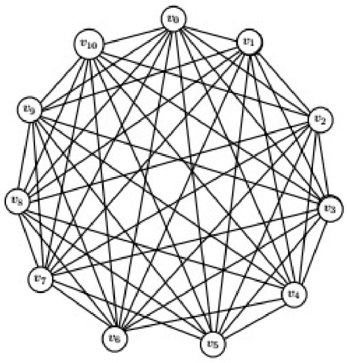

Figure 1. $\mathsf{G}_S(\mathbb{Z}_{11})$

3. Main results

The goal of this study is to compute the M-polynomial for $\mathsf{G}_S(\mathbb{Z}_\wp)$. The important mathematical models on several topological indices for $\mathsf{G}_S(\mathbb{Z}_\wp)$ via M-polynomial are presented in this section.

Theorem 3.1. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer , then* $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = (\wp-1)\left(\tau^{\wp-2}\omega^{\wp-1}\right) + \left(\frac{\wp^2-4\wp+3}{2}\right)\left(\tau^{\wp-2}\omega^{\wp-2}\right)$.

Proof. **1)** Let $\wp \geq 3$ be an odd integer, then the inverse graph $\mathsf{G}_S(\mathbb{Z}_\wp)$ associated with a finite cyclic group $\mathbb{Z}_\wp$ has the only self-invertible element 0. Let $V(\mathsf{G}) = \{\vartheta_0, \vartheta_1, \vartheta_2 \ldots, \vartheta_{\wp-1}\}$ and $E(\mathsf{G}) = \{\vartheta_i\vartheta_{i+j} : 0 \leq i \leq \wp-1, 1 \leq j \leq \wp-1 \textit{ and } j \neq \wp-2i\}$. Note that $\vartheta_i\vartheta_{\wp-i} \notin E(\mathsf{G})$ for each i so that $0 \leq i \leq \wp-1$. Since $\wp$ is odd, vertex's degree in $\mathsf{G}_S(\mathbb{Z}_\wp)$ is $d_\mathsf{G}(\vartheta_0) = \wp-1$ and $d_\mathsf{G}(\vartheta_i) = \wp-2$, here $1 \leq i \leq \wp-1$; $i \neq 0$. Now $|V(\mathsf{G}_S(\mathbb{Z}_\wp))| = \wp$ and by the fundamental theorem of graph theory, $\sum_{i=0}^{\wp-1} d_\mathsf{G}(\vartheta_i) = 2q$. We have,

$$\begin{aligned}\sum_{i=0}^{\wp-1} d_\mathsf{G}(\vartheta_i) &= d_\mathsf{G}(\vartheta_0) + \sum_{i=1}^{\wp-1} d_\mathsf{G}(\vartheta_i)\\ &= (\wp-1) + (\wp-1)(\wp-2)\\ &= (\wp-1)^2.\end{aligned}$$

Hence, we recognise that $2q = (\wp-1)^2$ and we obtain that $|E(\mathsf{G}_S(\mathbb{Z}_\wp))| = \dfrac{(\wp-1)^2}{2}$. We now consider these two cases separately:

Case (i) If ϑ_0 is adjacent to ϑ_j with $0 \leq 1 \leq j \leq \wp - 1$, then $d_{\mathsf{G}}(\vartheta_0) = \wp - 1$ and $d_{\mathsf{G}}(\vartheta_j) = \wp - 2$ for $j \neq 0$. In the inverse graph, ϑ_0 is adjacent to $\wp - 1$ vertices, i.e. $|E_1(\mathsf{G}_S(\mathbb{Z}_\wp))| = \wp - 1$.

Case (ii) If ϑ_i is adjacent to ϑ_{i+j} with $1 \leq i \leq \wp - 1$, $1 \leq j \leq \wp - 1$, then $d_{\mathsf{G}}(\vartheta_i) = \wp - 2$ and $d_{\mathsf{G}}(\vartheta_{i+j}) = \wp - 2$ for $i + j \neq 0$. In the inverse graph, ϑ_i, for $i \neq 0$, is adjacent to $\wp - 2$ vertices and

$$\begin{aligned} |E_2(\mathsf{G}_S(\mathbb{Z}_\wp))| &= |E(\mathsf{G}_S(\mathbb{Z}_\wp))| - |E_1(\mathsf{G}_S(\mathbb{Z}_\wp))| \\ &= \frac{(\wp - 1)^2}{2} - (\wp - 1) \\ &= \frac{\wp^2 - 4\wp + 3}{2}. \end{aligned}$$

From the above two cases, we obtain the edges based on the end vertices degree:

$$\begin{aligned} E_1(\mathsf{G}_S(\mathbb{Z}_\wp) &= \{\vartheta_0\vartheta_j \in E_1(\mathsf{G}_S(\mathbb{Z}_\wp)) : d_{\mathsf{G}}(\vartheta_0) = \wp - 1, d_{\mathsf{G}}(\vartheta_j) = \wp - 2\}, \\ E_2(\mathsf{G}_S(\mathbb{Z}_\wp) &= \{\vartheta_i\vartheta_{i+j} \in E_2(\mathsf{G}_S(\mathbb{Z}_\wp)) : d_{\mathsf{G}}(\vartheta_i) = \wp - 2, d_{\mathsf{G}}(\vartheta_{i+j}) = \wp - 2\} \end{aligned}$$

such that,

$$\begin{aligned} |E_1(\mathsf{G}_S(\mathbb{Z}_\wp))| &= \wp - 1, \\ |E_2(\mathsf{G}_S(\mathbb{Z}_\wp))| &= \frac{\wp^2 - 4\wp + 3}{2}. \end{aligned}$$

Now, the concept of M-polynomial, we get the following value, and Figure 2 illustrates the M-polynomial with $\wp = 5$.

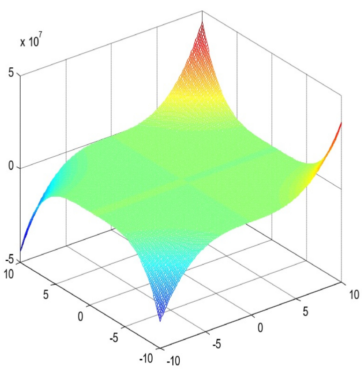

FIGURE 2. Graphical representation of M-polinomial with $\wp = 5$

$$M[\mathsf{G}_S(\mathbb{Z}_\wp); \tau, \omega] = \sum_{\delta \leq s \leq t \leq \triangle} m_{st}\big(\tau^s \omega^t\big)$$

$$
\begin{aligned}
&= \sum_{\wp-2\leq\wp-1} m_{(\wp-2)(\wp-1)}\big(\tau^{\wp-2}\omega^{\wp-1}\big)\\
&+ \sum_{\wp-2\leq\wp-2} m_{(\wp-2)(\wp-2)}\big(\tau^{\wp-2}\omega^{\wp-2}\big)\\
&= |E_1(\mathsf{G}_S(\mathbb{Z}_\wp))|\big(\tau^{\wp-2}\omega^{\wp-1}\big) + |E_2(\mathsf{G}_S(\mathbb{Z}_\wp))|\big(\tau^{\wp-2}\omega^{\wp-2}\big)\\
&= (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)\big(\frac{\wp^2-4\wp+3}{2}\big)\big(\tau^{\wp-2}\omega^{\wp-2}\big).
\end{aligned}
$$

□

Corollary 3.2. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer, then*

$$ZI_f(\mathsf{G}_S(\mathbb{Z}_\wp)) = \wp^3 - 4\wp^2 + 6\wp - 3.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\left(\frac{\wp^2-4\wp+3}{2}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$
\begin{aligned}
D_\tau(f(\tau,\omega)) &= (\wp-1)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + (\frac{\wp^2-4\wp+3}{2})(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + (\frac{\wp^2-4\wp+3}{2})(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
ZI_f(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D_\tau f + D_\omega f \mid_{\tau=\omega=1}\\
&= (\wp-1)(\wp-2) + (\wp-1)(\wp-1) + (\wp^2-4\wp+3)(\wp-2)\\
&= \wp^3 - 4\wp^2 + 6\wp - 3.
\end{aligned}
$$

□

Corollary 3.3. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer, then*

$$ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^4 - 6\wp^3 + 15\wp^2 - 18\wp + 8}{2}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\left(\frac{\wp^2-4\wp+3}{2}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$
\begin{aligned}
D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
D_\tau D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-1}\big)\\
&\quad + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).
\end{aligned}
$$

Hence, we have

$$\begin{aligned} ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D_\tau D_\omega f \mid_{\tau=\omega=1} \\ &= (\wp-1)(\wp-1)(\wp-2) + (\wp-1)(\wp-1) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)(\wp-2) \\ &= (\wp^3-4\wp^2+5\wp-2) + \frac{\wp^4-8\wp^3+23\wp^2-4\wp+12}{2} \\ &= \frac{\wp^4-6\wp^3+15\wp^2-18\wp+8}{2}. \end{aligned}$$

□

Corollary 3.4. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *be an odd integer, then*

$$^{m}ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^2-2\wp-1}{2\wp^2-8\wp+8}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + (\frac{\wp^2-4\wp+3}{2})\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$\begin{aligned} S_\omega(f(\tau,\omega)) &= \frac{\wp-1}{\wp-1}\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2(\wp-2)}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big). \\ S_\tau S_\omega(f(\tau,\omega)) &= \frac{\wp-1}{(\wp-1)(\wp-2)}\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2(\wp-2)(\wp-2)}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big). \end{aligned}$$

Hence, we have

$$\begin{aligned} ^{m}ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau S_\omega f \mid_{\tau=\omega=1} \\ &= \frac{\wp-1}{(\wp-1)(\wp-2)} + \frac{\wp^2-4\wp+3}{2(\wp-2)(\wp-2)} \\ &= \frac{1}{\wp-2} + \frac{\wp^2-4\wp+3}{2(\wp-2)(\wp-2)} \\ &= \frac{\wp^2-2\wp-1}{2\wp^2-8\wp+8}. \end{aligned}$$

□

Corollary 3.5. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *be an odd integer then*

$$RI_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) = (\wp-1)^{\alpha+1}(\wp-2)^\alpha + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^{2\alpha}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$D_\omega^\alpha(f(\tau,\omega)) = (\wp-1)(\wp-1)^\alpha\big(\tau^{\wp-2}\omega^{\wp-1}\big)$$

$$+\left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^{\alpha}\left(\tau^{\wp-2}\omega^{\wp-2}\right).$$

$$D_\tau^\alpha D_\omega^\alpha(f(\tau,\omega)) = (\wp-1)(\wp-1)^{\alpha}(\wp-2)^{\alpha}\left(\tau^{\wp-2}\omega^{\wp-1}\right) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^{2\alpha}\left(\tau^{\wp-2}\omega^{\wp-2}\right).$$

Hence, we have

$$\begin{aligned} RI_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D_\tau^\alpha D_\omega^\alpha f \mid_{\tau=\omega=1} \\ &= (\wp-1)(\wp-1)^{\alpha}(\wp-2)^{\alpha} + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^{2\alpha} \\ &= (\wp-1)^{\alpha+1}(\wp-2)^{\alpha} + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^{2\alpha}. \end{aligned}$$

□

Corollary 3.6. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer, then*

$$RIR_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{1}{(\wp-1)^{\alpha-1}(\wp-2)^{\alpha}} + \frac{\wp^2-4\wp+3}{2(\wp-2)^{2\alpha}}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\left(\tau^{\wp-2}\omega^{\wp-1}\right) + \left(\frac{\wp^2-4\wp+3}{2}\right)\left(\tau^{\wp-2}\omega^{\wp-2}\right)$ then

$$S_\omega^\alpha(f(\tau,\omega)) = \frac{\wp-1}{(\wp-1)^{\alpha}}\left(\tau^{\wp-2}\omega^{\wp-1}\right) + \frac{\wp^2-4\wp+3}{2(\wp-2)^{\alpha}}\left(\tau^{\wp-2}\omega^{\wp-2}\right).$$

$$S_\tau^\alpha S_\omega^\alpha(f(\tau,\omega)) = \frac{\wp-1}{(\wp-1)^{\alpha}(\wp-2)^{\alpha}}\left(\tau^{\wp-2}\omega^{\wp-1}\right) + \frac{\wp^2-4\wp+3}{2(\wp-2)^{2\alpha}}\left(\tau^{\wp-2}\omega^{\wp-2}\right).$$

Hence, we have

$$\begin{aligned} RIR_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau S_\omega f \mid_{\tau=\omega=1} \\ &= \frac{\wp-1}{(\wp-1)^{\alpha}(\wp-2)^{\alpha}} + \frac{\wp^2-4\wp+3}{2(\wp-2)^{2\alpha}} \\ &= \frac{1}{(\wp-1)^{\alpha-1}(\wp-2)^{\alpha}} + \frac{\wp^2-4\wp+3}{2(\wp-2)^{2\alpha}}. \end{aligned}$$

□

Corollary 3.7. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer, then*

$$SSD(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^3-4\wp^2+5\wp-1}{\wp-2}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\left(\dfrac{\wp^2-4\wp+3}{2}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$D_\omega(f(\tau,\omega)) = (\wp-1)(\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).$$

$$S_\tau D_\omega(f(\tau,\omega)) = \frac{(\wp-1)(\wp-1)}{\wp-2}\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \frac{\wp^2-4\wp+3}{2}\big(\tau^{\wp-2}\omega^{\wp-2}\big).$$

$$S_\omega(f(\tau,\omega)) = \frac{\wp-1}{\wp-1}\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2(\wp-2)}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big).$$

$$D_\tau S_\omega(f(\tau,\omega)) = (\wp-2)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \frac{\wp^2-4\wp+3}{2}\big(\tau^{\wp-2}\omega^{\wp-2}\big).$$

Hence, we have

$$\begin{aligned} SSD(\mathsf{G}_S(\mathbb{Z}_\wp)) &= (S_\tau D_\omega f + D_\tau S_\omega f)\mid_{\tau=\omega=1} \\ &= \frac{(\wp-1)(\wp-1)}{\wp-2} + \frac{\wp^2-4\wp+3}{2} + (\wp-2) + \frac{\wp^2-4\wp+3}{2} \\ &= \frac{\wp^2-2\wp+1}{\wp-2} + \wp^2 - 3\wp + 1 \\ &= \frac{\wp^3-4\wp^2+5\wp-1}{\wp-2}. \end{aligned}$$

□

Corollary 3.8. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) be an odd integer then*

$$HI(\mathsf{G}_S(\mathbb{Z}_\wp)) = 2\left(\frac{\wp-1}{2\wp-3} + \frac{\wp^2-4\wp+3}{2(2\wp-4)}\right).$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\left(\frac{\wp^2-4\wp+3}{2}\right)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$J(f(\tau,\omega)) = (\wp-1)\big(\tau^{2\wp-3}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)\big(\tau^{2\wp-4}\big).$$

$$2S_\tau J(f(\tau,\omega)) = 2\left(\frac{\wp-1}{2\wp-3}\tau^{2\wp-3} + \frac{\wp^2-4\wp+3}{2(2\wp-4)}\tau^{2\wp-4}\right).$$

Hence, we have

$$HI(\mathsf{G}_S(\mathbb{Z}_\wp)) = 2S_\tau J f\mid_{\tau=1} = 2\left(\frac{\wp-1}{2\wp-3} + \frac{\wp^2-4\wp+3}{2(2\wp-4)}\right).$$

□

Corollary 3.9. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *be an odd integer then*

$$ISI(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{(\wp-1)^2(\wp-2)}{2\wp-3} + \frac{(\wp^2-4\wp+3)(\wp-2)^2}{2(2\wp-4)}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\Big(\frac{\wp^2-4\wp+3}{2}\Big)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$\begin{aligned}
D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
D_\tau D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-1}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)\\
&\quad(\wp-2)\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
JD_\tau D_\omega(f(\tau,\omega)) &= (\wp-1)(\wp-1)(\wp-2)\big(\tau^{2\wp-3}\big) + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^2\big(\tau^{2\wp-4}\big).\\
S_\tau JD_\tau D_\omega(f(\tau,\omega)) &= \frac{(\wp-1)^2(\wp-2)}{2\wp-3}\big(\tau^{2\wp-3}\big) + \frac{\wp^2-4\wp+3}{2(2\wp-4)}(\wp-2)^2\big(\tau^{2\wp-4}\big).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
ISI(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau JD_\tau D_\omega f\mid_{\tau=1}\\
&= \frac{(\wp-1)^2(\wp-2)}{2\wp-3} + \frac{(\wp^2-4\wp+3)(\wp-2)^2}{2(2\wp-4)}.
\end{aligned}$$

□

Corollary 3.10. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *be an odd integer, then*

$$AZI(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{(\wp-1)^4(\wp-2)^3}{(2\wp-5)^3} + \frac{(\wp^2-4\wp+3)(\wp-2)^6}{2(2\wp-6)^3}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = (\wp-1)\big(\tau^{\wp-2}\omega^{\wp-1}\big)+\Big(\frac{\wp^2-4\wp+3}{2}\Big)\big(\tau^{\wp-2}\omega^{\wp-2}\big)$ then

$$\begin{aligned}
D_\omega^3(f(\tau,\omega)) &= (\wp-1)(\wp-1)^3\big(\tau^{\wp-2}\omega^{\wp-1}\big)\\
&\quad + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^3\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
D_\tau^3 D_\omega^3(f(\tau,\omega)) &= (\wp-1)(\wp-1)^3(\wp-2)^3\big(\tau^{\wp-2}\omega^{\wp-1}\big)\\
&\quad + \left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^6\big(\tau^{\wp-2}\omega^{\wp-2}\big).\\
JD_\tau^3 D_\omega^3(f(\tau,\omega)) &= (\wp-1)^4(\wp-2)^3\big(\tau^{2\wp-3}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^6\left(\tau^{2\wp-4}\right).\\
Q_{-2}JD_\tau^3D_\omega^3(f(\tau,\omega)) &= (\wp-1)^4(\wp-2)^3\left(\tau^{2\wp-5}\right)\\
&+\left(\frac{\wp^2-4\wp+3}{2}\right)(\wp-2)^6\left(\tau^{2\wp-6}\right).\\
S_\tau^3Q_{-2}JD_\tau^3D_\omega^3(f(\tau,\omega)) &= \frac{(\wp-1)^4(\wp-2)^3}{(2\wp-5)^3}\left(\tau^{2\wp-5}\right)\\
&+\frac{\wp^2-4\wp+3}{2(2\wp-6)^3}(\wp-2)^6\left(\tau^{2\wp-6}\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
AZI(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau^3Q_{-2}JD_\tau^3D_\omega^3 f\mid_{\tau=1}\\
&= \frac{(\wp-1)^4(\wp-2)^3}{(2\wp-5)^3}+\frac{(\wp^2-4\wp+3)(\wp-2)^6}{2(2\wp-6)^3}.
\end{aligned}
$$

□

The following Figure 3 shows that, the graphical representation the M-polynomial based topological indices for $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ is odd.

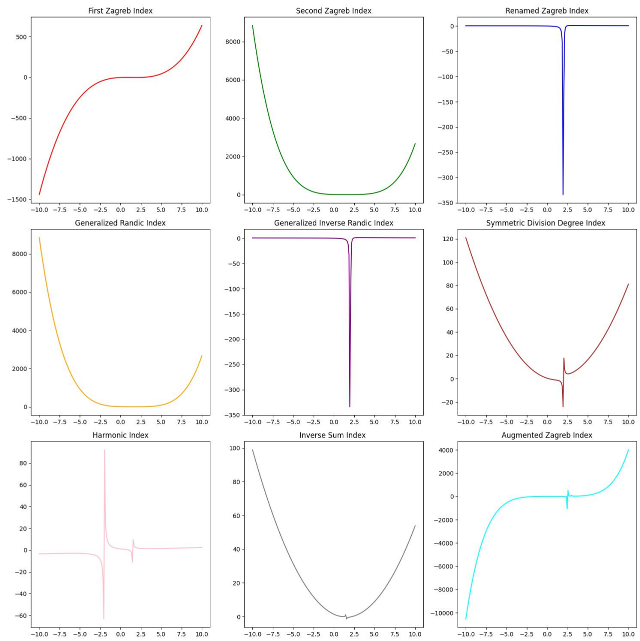

FIGURE 3. $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ is odd

Theorem 3.11. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *be an even integer, then*

$$M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

Proof. Let $\wp \geq 3$ is even, then $\mathsf{G}_S(\mathbb{Z}_\wp)$ associated with the finite cyclic group $\mathbb{Z}_\wp$ has the self-invertible elements are 0 and $\frac{n}{2}$. Let $V(\mathsf{G}) = \{\vartheta_0, \vartheta_1, \vartheta_2, \ldots, \vartheta_{\wp-1}\}$ and $E(\mathsf{G}) = \{\vartheta_i\vartheta_{i+j} : 0 \leq i \leq \wp-1,\ 1 \leq j \leq \wp-1 \ \textit{and}\ j \neq \wp-2i\}$. Note that $\vartheta_i\vartheta_{\wp-i} \notin E(\mathsf{G})$ and $\vartheta_i\vartheta_{\frac{\wp}{2}} \notin E(\mathsf{G})$ for each i so that $0 \leq i \leq \wp-1$. As $\wp$ is even, vertex's degree of $\mathsf{G}_S(\mathbb{Z}_\wp)$ is given by

$$d_{\mathsf{G}}(\vartheta_i) = \begin{cases} \wp-2 & \text{for } i \in \{0,\ \frac{\wp}{2}\} \\ \wp-3 & \text{for } i \notin \{0,\ \frac{\wp}{2}\} \text{ and } 1 \leq i \leq \wp-1. \end{cases}$$

From the above function, exactly two vertices of $\mathsf{G}_S(\mathbb{Z}_\wp)$ has the degree $\wp-2$ and all of the rest $\wp-2$ vertices of $\mathsf{G}_S(\mathbb{Z}_\wp)$ has degree $\wp-3$. Now, $|V(\mathsf{G}_S(\mathbb{Z}_\wp))| = \wp$ and $\sum_{i=0}^{\wp-1} d_{\mathsf{G}}(\vartheta_i) = 2q$. We have

$$\begin{aligned}\sum_{i=0}^{\wp-1} d_{\mathsf{G}}(\vartheta_i) &= d_G(\vartheta_0) + \sum_{i=1}^{\frac{\wp}{2}-1} d_{\mathsf{G}}(\vartheta_i) + d_{\mathsf{G}}(\vartheta_{\frac{\wp}{2}}) + \sum_{i=\frac{\wp}{2}+1}^{\wp-1} d_{\mathsf{G}}(\vartheta_i)\\ &= \wp-2 + \frac{\wp-2}{2}(\wp-3) + \wp-2 + \frac{\wp-2}{2}(\wp-3)\\ &= \wp^2-3\wp+2.\end{aligned}$$

Hence, we know that $2q = \wp^2-3\wp+2$ implying $|E(\mathsf{G}_S(\mathbb{Z}_\wp))| = \dfrac{\wp^2-3\wp+2}{2}$.

Case (i) If ϑ_0 is adjacent to ϑ_j and ϑ_0 is non-adjacent to all $\vartheta_{\frac{\wp}{2}}$ with $0 \leq i \leq \wp-1$, $0 \leq j \leq \wp-1$ and $j \neq 0, \frac{\wp}{2}$ then $d_{\mathsf{G}}(\vartheta_0) = \wp-2$ and $d_{\mathsf{G}}(\vartheta_j) = \wp-3$ for $j \neq 0$ and $\vartheta_{\frac{\wp}{2}}$. In the inverse graph, ϑ_0 is adjacent to $\wp-2$ vertices giving $|E_1(\mathsf{G}_S(\mathbb{Z}_\wp))| = \wp-2$.

Case (ii) If $\vartheta_{\frac{\wp}{2}}$ is adjacent to ϑ_j and $\vartheta_{\frac{\wp}{2}}$ is non-adjacent to ϑ_0 with $0 \leq i \leq \wp-1$, $0 \leq j \leq \wp-1$ and $j \neq 0, \frac{\wp}{2}$, then $d_{\mathsf{G}}(\vartheta_{\frac{\wp}{2}}) = \wp-2$ and $d_{\mathsf{G}}(\vartheta_j) = \wp-3$ for $j \neq 0,\ \frac{\wp}{2}$. In the inverse graph, $\vartheta_{\frac{\wp}{2}}$ is adjacent to $\wp-2$ vertices giving $|E_2(\mathsf{G}_S(\mathbb{Z}_\wp))| = \wp-2$.

Case (iii) If ϑ_i is adjacent to ϑ_{i+j} with $1 \leq i \leq \wp-1, 0 \leq j \leq \wp-1$ and $j \neq 0, \frac{\wp}{2}$, then $d_G(\vartheta_i) = \wp-3$ and $d_{\mathsf{G}}(\vartheta_{i+j}) = \wp-3$ for $i+j \neq 0,\ \frac{\wp}{2}$. In the inverse graph, ϑ_i, for $i \neq 0,\ \frac{\wp}{2}$ is adjacent to $\wp-3$ vertices and hence

$$\begin{aligned}|E_3(\mathsf{G}_S(\mathbb{Z}_\wp))| &= |E(\mathsf{G}_S(\mathbb{Z}_\wp))| - |E_1(\mathsf{G}_S(\mathbb{Z}_\wp))| - |E_2(\mathsf{G}_S(\mathbb{Z}_\wp))|\\ &= \frac{\wp^2-3\wp+2}{2} - 2(\wp-2)\\ &= \frac{\wp^2-7\wp+10}{2}.\end{aligned}$$

Hence, we know that $2q = (\wp-1)^2$ and we obtain that $|E(\mathsf{G}_S(\mathbb{Z}_\wp))| = \frac{(\wp-1)^2}{2}$. Now, the concept of M-polynomial, we get the following value, and Figure 4 illustrates the M-polynomial with $\wp = 10$.

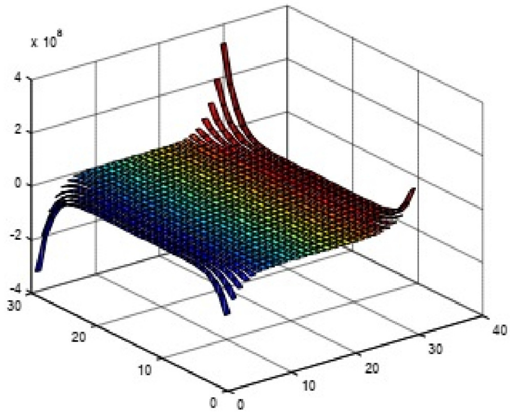

FIGURE 4. Graphical representation of M-polinomial with $\wp = 10$

$$\begin{aligned} M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] &= \sum_{\delta\le s\le t\le\triangle} m_{st}\left(\tau^s\omega^t\right) \\ &= \sum_{\wp-3\le\wp-2} m_{(\wp-3)(\wp-2)}\left(\tau^{\wp-3}\omega^{\wp-2}\right) \\ &+ \sum_{\wp-3\le\wp-2} m_{(\wp-3)(\wp-2)}\left(\tau^{\wp-3}\omega^{\wp-2}\right) \\ &+ \sum_{\wp-3\le\wp-3} m_{(\wp-3)(\wp-3)}\left(\tau^{\wp-3}\omega^{\wp-3}\right) \\ &= |E_1(\mathsf{G}_S(\mathbb{Z}_\wp))|\left(\tau^{\wp-3}\omega^{\wp-2}\right) + |E_2(\mathsf{G}_S(\mathbb{Z}_\wp))|\left(\tau^{\wp-3}\omega^{\wp-2}\right) \\ &+ |E_3(\mathsf{G}_S(\mathbb{Z}_\wp))|\left(\tau^{\wp-3}\omega^{\wp-3}\right) \\ &= 2(\wp-2)\left(\tau^{\wp-3}\omega^{\wp-2}\right) + \left(\frac{\wp^2-7\wp+10}{2}\right)\left(\tau^{\wp-3}\omega^{\wp-3}\right). \end{aligned}$$

□

Corollary 3.12. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *is even then*

$$ZI_f(\mathsf{G}_S(\mathbb{Z}_\wp)) = \wp^3 - 6\wp^2 + 13\wp - 10.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\left(\tau^{\wp-3}\omega^{\wp-2}\right) + \left(\frac{\wp^2-7\wp+10}{2}\right)\left(\tau^{\wp-3}\omega^{\wp-3}\right)$ then

$$D_\tau(f(\tau,\omega)) = 2(\wp-2)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

$$D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

Hence, we have

$$\begin{aligned} ZI_f(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D_\tau f + D_\omega f \,|_{\tau=\omega=1} \\ &= 2(\wp-2)(\wp-3) + 2(\wp-2)(\wp-2) + (\wp^2-7\wp+10)(\wp-3) \\ &= \wp^3 - 6\wp^2 + 13\wp - 10. \end{aligned}$$

□

Corollary 3.13. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) is even then*

$$ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^4 - 9\wp^3 + 33\wp^2 - 59\wp + 42}{2}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-1}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

$$\begin{aligned} D_\tau D_\omega(f(\tau,\omega)) &= 2(\wp-2)(\wp-2)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-2}\big) \\ &\quad + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big). \end{aligned}$$

Hence, we have

$$\begin{aligned} ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D_\tau D_\omega f \,|_{\tau=\omega=1} \\ &= 2(\wp-2)(\wp-2)(\wp-3) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)(\wp-3) \\ &= 2(\wp^3-7\wp^2+16\wp-12) + \left(\frac{\wp^4-13\wp^3+61\wp^2-123\wp+90}{2}\right) \\ &= \frac{\wp^4-9\wp^3+33\wp^2-59\wp+42}{2}. \end{aligned}$$

□

Corollary 3.14. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) is even then*

$${}^{m}ZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^2-3\wp-2}{2\wp^2-12\wp+18}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\Big(\frac{\wp^2-7\wp+10}{2}\Big)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$S_\omega(f(\tau,\omega)) = \frac{2(\wp-2)}{\wp-2}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2(\wp-3)}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$S_\tau S_\omega(f(\tau,\omega)) = \frac{2(\wp-2)}{(\wp-2)(\wp-3)}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2(\wp-3)(\wp-3)}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

Hence, we have

$$\begin{aligned}{}^mZI_s(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau S_\omega f\,|_{\tau=\omega=1}\\ &= \frac{2(\wp-2)}{(\wp-2)(\wp-3)} + \frac{\wp^2-7\wp+10}{2(\wp-3)(\wp-3)}\\ &= \frac{2}{\wp-3} + \frac{\wp^2-7\wp+10}{2(\wp-3)(\wp-3)}\\ &= \frac{\wp^2-3\wp-2}{2\wp^2-12\wp+18}.\end{aligned}$$

□

Corollary 3.15. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *is even then*

$$RI_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) = 2(\wp-2)^{\alpha+1}(\wp-3)^\alpha + \frac{(\wp^2-7\wp+10)(\wp-2)^{2\alpha}}{2}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$D^\alpha_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)^\alpha\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^\alpha\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$\begin{aligned}D^\alpha_\tau D^\alpha_\omega(f(\tau,\omega)) &= 2(\wp-2)(\wp-2)^\alpha(\wp-3)^\alpha\big(\tau^{\wp-3}\omega^{\wp-2}\big)\\ &\quad+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^{2\alpha}\big(\tau^{\wp-3}\omega^{\wp-3}\big).\end{aligned}$$

Hence, we have

$$\begin{aligned}RI_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) &= D^\alpha_\tau D^\alpha_\omega f\,|_{\tau=\omega=1}\\ &= 2(\wp-2)(\wp-2)^\alpha(\wp-3)^\alpha + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^{2\alpha}\\ &= 2(\wp-2)^{\alpha+1}(\wp-3)^\alpha + \frac{(\wp^2-7\wp+10)(\wp-2)^{2\alpha}}{2}.\end{aligned}$$

□

Corollary 3.16. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *is even then*

$$RIR_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{2}{(\wp-2)^{\alpha-1}(\wp-3)^\alpha} + \frac{\wp^2-7\wp+10}{2(\wp-3)^{2\alpha}}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$S_\omega^\alpha(f(\tau,\omega)) = \frac{2(\wp-2)}{(\wp-2)^\alpha}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \frac{\wp^2-7\wp+10}{2(\wp-3)^\alpha}\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$S_\tau^\alpha S_\omega^\alpha(f(\tau,\omega)) = \frac{2(\wp-2)}{(\wp-2)^\alpha(\wp-3)^\alpha}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \frac{\wp^2-7\wp+10}{2(\wp-3)^{2\alpha}}\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

Hence, we have

$$\begin{aligned} RIR_\alpha(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau S_\omega f \mid_{\tau=\omega=1} \\ &= \frac{2(\wp-2)}{(\wp-2)^\alpha(\wp-3)^\alpha} + \frac{\wp^2-7\wp+10}{2(\wp-3)^{2\alpha}} \\ &= \frac{2}{(\wp-2)^{\alpha-1}(\wp-3)^\alpha} + \frac{\wp^2-7\wp+10}{2(\wp-3)^{2\alpha}}. \end{aligned}$$

□

Corollary 3.17. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *is even then*

$$SSD(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{\wp^3-6\wp^2+11\wp-4}{\wp-3}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$S_\tau D_\omega(f(\tau,\omega)) = \frac{2(\wp-2)(\wp-2)}{\wp-3}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \frac{\wp^2-7\wp+10}{2}\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$S_\omega(f(\tau,\omega)) = \frac{2(\wp-2)}{\wp-2}\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \left(\frac{\wp^2-7\wp+10}{2(\wp-3)}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$D_\tau S_\omega(f(\tau,\omega)) = 2(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-2}\big) + \frac{\wp^2-7\wp+10}{2}\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$

Hence, we have

$$SSD(\mathsf{G}_S(\mathbb{Z}_\wp)) = (S_\tau D_\omega f + D_\tau S_\omega f) \mid_{\tau=\omega=1}$$

$$= \frac{2(\wp-2)(\wp-2)}{\wp-3} + \frac{\wp^2-7\wp+10}{2} + 2(\wp-3) + \frac{\wp^2-7\wp+10}{2}$$
$$= \frac{2\wp^2-8\wp+8}{\wp-3} + \wp^2-5\wp+4$$
$$= \frac{\wp^3-6\wp^2+11\wp-4}{\wp-3}.$$

□

Corollary 3.18. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) is even then*

$$HI(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{4(\wp-2)}{2\wp-5} + \frac{\wp^2-7\wp+10}{2\wp-6}.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$J(f(\tau,\omega)) = 2(\wp-2)\big(\tau^{2\wp-5}\big) + \left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{2\wp-6}\big).$$
$$2S_\tau J(f(\tau,\omega)) = 2\left(\frac{2(\wp-2)}{2\wp-5}\tau^{2\wp-5} + \frac{\wp^2-7\wp+10}{2(2\wp-6)}\tau^{2\wp-6}\right).$$

Hence, we have

$$HI(\mathsf{G}_S(\mathbb{Z}_\wp)) = 2S_\tau Jf\mid_{\tau=1}$$
$$= 2\left(\frac{2(\wp-2)}{2\wp-5} + \frac{\wp^2-7\wp+10}{2(2\wp-6)}\right)$$
$$= \frac{4(\wp-2)}{2\wp-5} + \frac{\wp^2-7\wp+10}{2\wp-6}.$$

□

Corollary 3.19. *If $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ (≥ 3) is even then*

$$ISI(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{2(\wp-2)^2(\wp-3)}{2\wp-5} + \frac{(\wp^2-7\wp+10)(\wp-3)^2}{2(2\wp-6)}.$$

Proof. Let$M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\frac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)$$
$$+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-3}\big).$$
$$D_\tau D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)(\wp-3)\big(\tau^{\wp-3}\omega^{\wp-2}\big)$$

$$+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)(\wp-3)\left(\tau^{\wp-3}\omega^{\wp-3}\right).$$

$$JD_\tau D_\omega(f(\tau,\omega)) = 2(\wp-2)(\wp-2)(\wp-3)\left(\tau^{2\wp-5}\right) + \left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^2\left(\tau^{2\wp-6}\right).$$

$$S_\tau JD_\tau D_\omega(f(\tau,\omega)) = \frac{2(\wp-2)^2(\wp-3)}{2\wp-5}\left(\tau^{2\wp-5}\right) + \frac{\wp^2-7\wp+10}{2(2\wp-6)}(\wp-3)^2\left(\tau^{2\wp-6}\right).$$

Hence, we have

$$\begin{aligned} ISI(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau JD_\tau D_\omega f \mid_{\tau=1} \\ &= \frac{2(\wp-2)^2(\wp-3)}{2\wp-5} + \frac{(\wp^2-7\wp+10)(\wp-3)^2}{2(2\wp-6)}. \end{aligned}$$

□

The following Figure 5 shows that, the graphical representation the M-polynomial based topological indices for $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ is even.

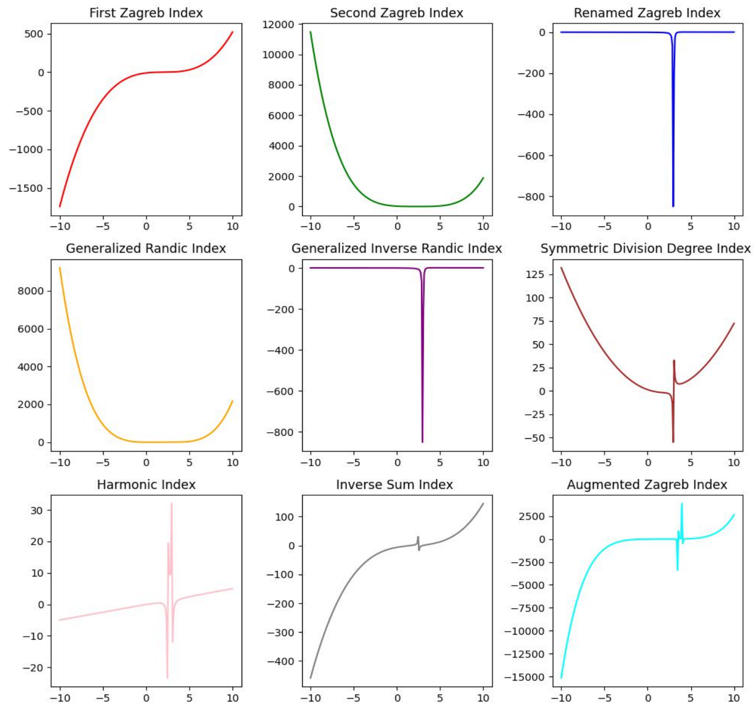

Figure 5. $\mathsf{G}_S(\mathbb{Z}_\wp)$ with $\wp$ is even

Corollary 3.20. *If* $\mathsf{G}_S(\mathbb{Z}_\wp)$ *with* $\wp$ (≥ 3) *is even then*

$$AZI(\mathsf{G}_S(\mathbb{Z}_\wp)) = \frac{2(\wp-2)^4(\wp-3)^3}{(2\wp-7)^3} + \frac{\wp^2-7\wp+10}{2(2\wp-8)^3}(\wp-3)^6.$$

Proof. Let $M[\mathsf{G}_S(\mathbb{Z}_\wp);\tau,\omega] = f(\tau,\omega) = 2(\wp-2)\big(\tau^{\wp-3}\omega^{\wp-2}\big)+\left(\dfrac{\wp^2-7\wp+10}{2}\right)\big(\tau^{\wp-3}\omega^{\wp-3}\big)$ then

$$\begin{aligned}
D_\omega^3(f(\tau,\omega)) &= 2(\wp-2)(\wp-2)^3\big(\tau^{\wp-3}\omega^{\wp-2}\big)\\
&\quad+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^3\big(\tau^{\wp-3}\omega^{\wp-3}\big).\\
D_\tau^3D_\omega^3(f(\tau,\omega)) &= 2(\wp-2)(\wp-2)^3(\wp-3)^3\big(\tau^{\wp-3}\omega^{\wp-2}\big)\\
&\quad+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^6\big(\tau^{\wp-3}\omega^{\wp-3}\big).\\
JD_\tau^3D_\omega^3(f(\tau,\omega)) &= 2(\wp-2)^4(\wp-3)^3\big(\tau^{2\wp-5}\big)\\
&\quad+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^6\big(\tau^{2\wp-6}\big).\\
Q_{-2}JD_\tau^3D_\omega^3(f(\tau,\omega)) &= 2(\wp-2)^4(\wp-3)^3\big(\tau^{2\wp-7}\big)\\
&\quad+\left(\frac{\wp^2-7\wp+10}{2}\right)(\wp-3)^6\big(\tau^{2\wp-8}\big).\\
S_\tau^3Q_{-2}JD_\tau^3D_\omega^3(f(\tau,\omega)) &= \frac{2(\wp-2)^4(\wp-3)^3}{(2\wp-7)^3}\big(\tau^{2\wp-7}\big)\\
&\quad+\frac{\wp^2-7\wp+10}{2(2\wp-8)^3}(\wp-3)^6\big(\tau^{2\wp-8}\big).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
AZI(\mathsf{G}_S(\mathbb{Z}_\wp)) &= S_\tau^3Q_{-2}JD_\tau^3D_\omega^3 f\mid_{\tau=1}\\
&= \frac{2(\wp-2)^4(\wp-3)^3}{(2\wp-7)^3} + \frac{\wp^2-7\wp+10}{2(2\wp-8)^3}(\wp-3)^6.
\end{aligned}$$

□

4. Conclusion

In this research, we focused on M-polynomial-based topological indices for a fresh category of graphs, $\mathsf{G}_S(\mathbb{Z}_\wp)$. The limitations for M-polynomial-based topological indices of the inverse graph from n ordered finite cyclic groups are obtained by making generalizations. The index's boundaries are crisp, and the issues are closely related to science features, which is a unique feature of this study. However, finding M-polynomial-based topological indices for inverse graphs from quaternion, symmetric, and dihedral groups in general remains a major challenge. Additionally, the degree-focused topological indices of inverse graphs from arbitrary finite groups, and the eccentricity and distance-focused topological indices of inverse graphs from arbitrary finite groups are being investigated.

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