J. Indones. Math. Soc.
Vol. 30, No. 03 (2024), pp. 398–420.

LIGHTLIKE HYPERSURFACES OF AN INDEFINITE (α, β)-TYPE ALMOST CONTACT METRIC STATISTICAL MANIFOLD WITH AN (l, m)-TYPE CONNECTION

Shagun Bhatti[1] and Jasleen Kaur[2]

[1,2]**Department of Mathematics, Punjabi University, Patiala, Punjab, India**
[1] **shagun.bhatti82@gmail.com**, [2]**jass2381@gmail.com**

Abstract. This paper aims to present the theory of hypersurfaces for a novel class of manifolds called an indefinite (α, β)-type almost contact metric statistical manifold or trans-Sasakian statistical manifold with an (l, m)-type connection. The study delves into the analysis of screen semi-invariant lightlike hypersurfaces within this framework. Specific conditions on the recurrent and Lie recurrent structure tensor field have been established. Additionally, illustrative examples are provided to enhance the comprehension of the introduced concept.

Key words and phrases: lightlike hypersurface, indefinite trans-Sasakian statistical manifold, (l, m)-type connection, recurrent and Lie recurrent structure tensor field

1. INTRODUCTION

Lightlike hypersurfaces are an intriguing branch of geometry. Duggal and Bejancu [12] developed the theory of a lightlike hypersurface of a proper semi-Riemannian manifold in differential geometry. Further, this lightlike theory was studied extensively by various geometers such as Yasar et al.[5], Duggal and Sahin [13, 14]. They obtained some results on the lightlike hypersurfaces of an indefinite Sasakian manifold. In Oubi*ñ*a [11], a new class of almost contact metric structure, known as the trans-Sasakian structure, was introduced. Eventually, Chinea and Gonzalez [2] presented two subclasses of trans-Sasakian structures, the $\mathbb{C}_5$ and $\mathbb{C}_6$-structures, which contain the Kenmotsu and Sasakian structures, respectively. The lightlike hypersurfaces of various almost contact metric manifolds like indefinite trans-Sasakian, Kenmotsu, and symplectic manifolds were studied by Massamba [7], Jin [3], Massamba [6], Kang and Kim [21].

2020 Mathematics Subject Classification: 53D10, 53D15, 58A05.
Received: 04-09-2023, Accepted: 26-10-2024.

The study of geometric structures on a set of certain probability distributions resulted in the formation of an interesting branch of manifolds known as statistical manifolds, which Amari investigated [18] and thoroughly explored by Amari [19], Lauritzen [20] Later, Furuhata [8] made significant contributions to the initiation of the geometry of hypersurfaces in statistical manifolds. Afterwards, the concept of Sasakian statistical manifold and Kenmotsu statistical manifold were introduced by Furuhata et al.[10] and Furuhata et al.[9], respectively. They constructed certain results related to the real hypersurfaces and warped products of statistical manifolds. In this context, Kazan [1] demonstrated the idea of a trans-Sasakian statistical manifold, wherein various characterizations about ξ-conformal-projective flatness of a trans-Sasakian statistical manifold have been discussed. Bahadir and Tripathi investigated the theory of lightlike hypersurfaces of statistical manifolds [17] and was further studied by Bahadir et al.[16]. The geometry of lightlike hypersurfaces for the Sasakian statistical manifold was also initiated in Bahadir [15]. Thus inspired, Rani and Kaur [22] worked on the geometry of hypersurfaces of an indefinite statistical manifold. They further introduced the concept of lightlike hypersurfaces in the context of an indefinite Kaehler statistical manifold in Rani and Kaur [23] and developed significant results for Lie-recurrent structure tensor fields.

Jin [4], established a new connection on semi-Riemannian manifolds, which is defined as a linear connection $\tilde{D}$ on a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called a non-symmetric non-metric connection of type (l, m) or usually known as (l, m)-type connection, if there exist two smooth functions l and m such that $\tilde{D}$ itself and its torsion tensor $\tilde{T}$ satisfies

$$(\tilde{D}_X\tilde{g})(Y,Z) = -l\{\eta(Y)\tilde{g}(X,Z)+\eta(Z)\tilde{g}(X,Y)\}-m\{\eta(Y)\tilde{g}(\phi X,Z)+\eta(Z)\tilde{g}(\phi X,Y)\} \tag{1}$$

$$\tilde{T}(X,Y) = l\{\eta(Y)X - \eta(X)Y\} + m\{\eta(Y)\phi X - \eta(X)\phi Y\}$$

where, ϕ is a $(1,1)$-type tensor field, ν is a smooth unit vector filed and η is a 1-form defined by $\eta(X) = \tilde{g}(X,\nu)$. Here X, Y and Z are the smooth vector fields on $\tilde{M}$ where, we set $(l, m) \neq (0, 0)$.

Keeping the theory mentioned earlier in view, this paper aims to introduce the notion of an indefinite trans-Sasakian statistical manifold and presents various results on its geometry. Further, the geometry of lightlike hypersurfaces has also been initiated for the indefinite trans-Sasakian statistical manifold endowed with an (l, m)-type connection wherein various results related to its structure, geodesicity, and parallelism of vector fields have been presented. The integrability of distributions in screen semi-invariant lightlike hypersurfaces of these manifolds has been worked upon. Also, the Lie-recurrent structure tensor field of lightlike hypersurfaces of an indefinite trans-Sasakian statistical manifold with an (l, m)-type connection has also been characterized.

2. PRELIMINARIES

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold of dimension $(2n+1)$. If $\tilde{g}$ is a semi-Riemannian metric, ϕ is a $(1,1)$ tensor field, ν is a characteristic vector field and η is a 1-form, such that

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X,Y) - \eta(X)\eta(Y), \qquad \tilde{g}(\nu,\nu) = 1, \tag{2}$$

$$\phi^2(X) = -X + \eta(X)\nu, \quad \tilde{g}(X,\nu) = \eta(X), \quad \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = 0 \tag{3}$$

which follows that $\phi\nu = 0$ and $\eta o \phi = 0$ for all $X, Y \in \Gamma(TM)$, then $(\phi, \nu, \tilde{g})$ is called an almost contact metric structure on $\tilde{M}$.

Definition 2.1. [7] *An almost contact metric structure on $\tilde{M}$ is called an indefinite trans-Sasakian structure if such that*

$$(\hat{\hat{\nabla}}_X \phi)Y = \alpha\{\tilde{g}(X,Y)\nu - \eta(Y)X\} + \beta\{\tilde{g}(\phi X, Y)\nu - \eta(Y)\phi X\}, \tag{4}$$

$$\hat{\hat{\nabla}}_X \nu = -\alpha(\phi X) + \beta\{X - \eta(X)\nu\}$$

holds for any $X, Y \in \Gamma(TM)$, where $\hat{\hat{\nabla}}$ is a Levi-Civita Connection, α and β are two smooth functions. Therefore, $(\phi, \nu, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure of type (α, β).

As per Chinea and Gonzalez [2], trans-Sasakian manifold is a natural generalization of both Sasakian and Kenmotsu manifolds.

We note that trans-Sasakian structures of type $(0,0)$ are cosymplectic, of type $(0,\beta)$ are β-Kenmotsu and of type $(\alpha, 0)$ are α-Sasakian.

Let (M, g) be a hypersurface of $(\tilde{M}, \tilde{g})$ with $g = \tilde{g} \mid M$. If the induced metric g on M is degenerate, then M is called a lightlike or degenerate hypersurface of $\tilde{M}$. There exists a vector field $\xi \neq 0$ on M such that $g(\xi, X) = 0 \quad \forall\, X \in \Gamma(TM)$. The null space or radical space of $T_x(M)$ at each point $x \in M$ is a subspace $Rad(T_xM)$ defined as

$$Rad(T_xM) = \{\xi \in T_x(M) : g_x(\xi, X) = 0 \quad \forall\, X \in \Gamma(TM)\}$$

whose dimension is called the nullity degree of g.

Since g is degenerate and any null vector is perpendicular to itself, therefore $T_xM^{\perp}$ is also null and

$$Rad(T_xM) = T_xM \cap T_xM^{\perp}.$$

For a hypersurface M, dimension of $T_xM^{\perp}$ equals 1 which implies that the dimension of $Rad(T_xM)$ is also 1 and $Rad(T_xM) = T_xM^{\perp}$. Here $Rad(TM)$ is called a radical distribution of M.

Consider $S(TM)$, screen distribution, as a complementary vector bundle of $Rad(TM)$ in TM , such that

$$TM = Rad(TM) \perp S(TM) \tag{5}$$

It follows that $S(TM)$ is a non-degenerate distribution. Thus,

$$TM_{|M} = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$, known as screen transversal vector bundle, is the orthogonal complement to $S(TM)$ in $TM_{|M}$.

Theorem 2.2. [12] *Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g})$. Then there exists a unique vector bundle $tr(TM)$ known as lightlike transversal vector bundle of rank 1 over M, such that for any non-zero local normal section ξ of $Rad(TM)$, there exist a unique section N of $tr(TM)$ satisfying*

$$\tilde{g}(N, \xi) = 1 \tag{6}$$

$$\tilde{g}(N, N) = 0, \quad \tilde{g}(N, V) = 0 \ \ \forall\, V \in \Gamma(S(TM)).$$

Then the tangent bundle $T\tilde{M}$ of $\tilde{M}$ is decomposed as follows:

$$T\tilde{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM).$$

3. LIGHTLIKE HYPERSURFACES OF AN INDEFINITE (α, β)-TYPE ALMOST CONTACT METRIC STATISTICAL MANIFOLD

This section presents some basic concepts available for lightlike hypersurfaces of an indefinite statistical manifold.

Definition 3.1. [8] *A pair $(\bar{\nabla}, \tilde{g})$ is called an indefinite statistical structure on $\tilde{M}$ where $\tilde{g}$ is a semi-Riemannian metric of constant index $q \geq 1$ on $\tilde{M}$, if $\bar{\nabla}$ is torsion free and*

$$(\bar{\nabla}_X \tilde{g})(Y, Z) = (\bar{\nabla}_Y \tilde{g})(X, Z) \tag{7}$$

holds for any $X, Y, Z \in \Gamma(T\tilde{M})$.

Moreover, there exists $\bar{\nabla}^*$ which is a dual connection of $\bar{\nabla}$ with respect to $\tilde{g}$, satisfying

$$X\tilde{g}(Y, Z) = \tilde{g}(\bar{\nabla}_X Y, Z) + \tilde{g}(Y, \bar{\nabla}^*_X Z) \qquad X, Y, Z \in \Gamma(T\tilde{M})$$

If $(\tilde{M}, \tilde{g}, \bar{\nabla})$ is an indefinite statistical manifold, then so is $(\tilde{M}, \tilde{g}, \bar{\nabla}^*)$. Hence the indefinite statistical manifold is denoted by $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g})$. Then the Gauss and Weingarten formulae for dual connections as given by Furuhata [8], Bahadir [17] are as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y)$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \bar{\nabla}^*_X N = -A^*_N X + \nabla^{\perp *}_X N,$$

for $X,\ Y \in \Gamma(TM)$, $N \in \Gamma(trTM)$, where $\nabla_X Y, \nabla^*_X Y, A_N X, A^*_N X \in \Gamma(TM)$ and $h(X,Y), h^*(X,Y), \nabla^{\perp}_X N, \nabla^{\perp *}_X N \in \Gamma(tr(TM))$.

Here ∇, ∇^* are called induced connections on M and A_N, A^*_N are called shape operators with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ respectively. Also, we denote by B and B^*, the second fundamental forms with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$. Then

$$B(X,Y) = \tilde{g}(h(X,Y),\xi), \quad B^*(X,Y) = \tilde{g}(h^*(X,Y),\xi),$$
$$\tau(X) = \tilde{g}(\nabla^{\perp}_X N,\xi), \quad \tau^*(X) = \tilde{g}(\nabla^{\perp *}_X N,\xi).$$

It follows that

$$h(X,Y) = B(X,Y)N, \quad h^*(X,Y) = B^*(X,Y)N$$
$$\nabla^{\perp}_X N = \tau(X)N, \quad \nabla^{\perp *}_X N = \tau^*(X)N$$

Hence,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + B^*(X,Y)N \tag{8}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \bar{\nabla}^*_X N = -A^*_N X + \tau^*(X)N \tag{9}$$

as per Bahadir and Tripathi [17], the relation between dual connections using the Gauss formula is described as

$$Xg(Y,Z) = g(\bar{\nabla}_X Y,Z) + g(Y,\bar{\nabla}^*_X Z)$$
$$= g(\nabla_X Y,Z) + g(Y,\nabla^*_X Z) + B(X,Y)\theta(Z) + B^*(X,Z)\theta(Y)$$

where θ is a 1-form such that $\theta(X) = \tilde{g}(X,N)$.

From the above equation, it is concluded that the induced connections ∇ and ∇^* are not dual connections and a lightlike hypersurface of a statistical manifold need not be a statistical manifold. Also, the induced connections ∇ and ∇^* and the second fundamental forms B and B^* are symmetric.

Further, using Gauss and Weingarten formulae, the following holds:

$$(\nabla_X g)(Y,Z)+(\nabla^*_X g)(Y,Z) = B(X,Y)\theta(Z)+B^*(X,Z)\theta(Y)+B^*(X,Y)\theta(Z)+B(X,Z)\theta(Y)$$

Let P denote the projection morphism of TM on $S(TM)$ with respect to the decomposition (5). Then

$$\nabla_X PY = \nabla'_X PY + h'(X,PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X,PY)$$
$$\nabla_X \xi = -A'_{\xi} X + \nabla'^{\perp}_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_{\xi} X + \nabla^{*\prime\perp}_X \xi$$

holds for all $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\nabla'_X PY$, $\nabla^{*\prime}_X PY$, $A'_{\xi} X$ and $A^{*\prime}_{\xi} X \in \Gamma(S(TM))$, $\nabla', \nabla^{*\prime}$ and $\nabla'^{\perp}, \nabla^{*\prime\perp}$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. Here $h', h^{*\prime}$ and $A', A^{*\prime}$ are respectively called screen second fundamental forms and screen shape operators of $S(TM)$.

The local second fundamental forms of $S(TM)$ are defined as

$$C(X,PY) = \tilde{g}(h'(X,PY),N), \quad C^*(X,PY) = \tilde{g}(h^{*\prime}(X,PY),N),$$

$$\epsilon(X) = g(\nabla'^{\perp}_X \xi, N), \quad \epsilon^*(X) = g(\nabla^{*\prime\perp}_X \xi, N) \quad \forall\, X, Y \in \Gamma(TM).$$

Therefore,

$$h'(X, PY) = C(X, PY)\xi, \quad h^{*\prime}(X, PY) = C^*(X, PY)\xi,$$

$$\nabla'^{\perp}_X \xi = -\tau(X)\xi, \quad \nabla^{*\prime\perp}_X \xi = -\tau^*(X)\xi,$$

$$\nabla_X PY = \nabla'_X PY + C(X, PY)\xi, \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + C^*(X, PY)\xi,$$

$$\nabla_X \xi = -A'_\xi X - \tau(X)\xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X - \tau^*(X)\xi \quad \forall\, X, Y \in \Gamma(TM) \tag{10}$$

where $\epsilon(X) = -\tau(X)$.

Using the above equation, the induced objects are related as:

$$\begin{aligned} B(X, \xi) + B^*(X, \xi) = 0, \quad g(A_N X + A^*_N X, N) = 0, \\ C(X, PY) = g(A^*_N X, PY), \quad C^*(X, PY) = g(A_N X, PY). \end{aligned} \tag{11}$$

From the equations (6), (7),(8),(9) and (10), the following propositions hold:

Proposition 3.2. [23] *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then the second fundamental forms B and B^* are related to the shape operators $A'_\xi X$ and $A^{*\prime}_\xi X$ of $S(TM)$ as follows:*

$$g(A'_\xi X, PY) = B^*(X, PY), \quad g(A^{*\prime}_\xi X, PY) = B(X, PY).$$

Therefore, equation (11) gives,

$$B(A^{*\prime}_\xi X, Y) = B(X, A^{*\prime}_\xi Y), \quad B^*(A'_\xi X, Y) = B^*(X, A'_\xi Y).$$

Proposition 3.3. [17] *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then, the shape operator of any screen distribution of a lightlike hypersurface is symmetric concerning the second fundamental form of M.*

Proposition 3.4. [17] *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$. Then, the second fundamental forms B and B^* are not degenerate.*

Also, for the dual connections, the following holds:

$$B(X, Y) = g(A^{*\prime}_\xi X, Y) - B^*(X, \xi)\theta(Y)$$

$$B^*(X, Y) = g(A'_\xi X, Y) - B(X, \xi)\theta(Y),$$

using above equations, $A^{*\prime}_\xi \xi + A'_\xi \xi = 0$.

3.1. **Indefinite (α,β)-almost contact metric statistical manifold.**

Following Furuhata et al. [10], we consider a Levi-Civita connection $\hat{\hat{\bar{\nabla}}}$ with respect to $\tilde{g}$ such that $\hat{\hat{\bar{\nabla}}} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$. For a statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1,2)$ tensor K of a torsion-free affine connection $\bar{\nabla}$ and Levi-Civita connection $\hat{\hat{\bar{\nabla}}}$ is defined as

$$K(X,Y) = K_X Y = \bar{\nabla}_X Y - \hat{\hat{\bar{\nabla}}}_X Y. \tag{12}$$

Since $\bar{\nabla}$ and $\hat{\hat{\bar{\nabla}}}$ are torsion free, then

$$K_X Y = K_Y X, \qquad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z) \tag{13}$$

holds for any $X, Y, Z \in \Gamma(T\tilde{M})$.

Moreover, $K(X,Y) = \hat{\hat{\bar{\nabla}}}_X Y - \bar{\nabla}^*_X Y$ then, $K(X,Y) = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y)$.

Definition 3.5. *Let $(\tilde{g}, \phi, \nu)$ be an indefinite trans-Sasakian structure on $\tilde{M}$. Then, a quadruplet $(\bar{\nabla} = \hat{\hat{\bar{\nabla}}} + K, \tilde{g}, \phi, \nu)$ is known as an indefinite trans-Sasakian statistical structure on $\tilde{M}$ if $(\bar{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}$ and the condition*

$$K_X \phi Y = -\phi K_X Y \tag{14}$$

holds for any $X, Y \in \Gamma(T\tilde{M})$.

Therefore, $(\tilde{M}, \bar{\nabla}, \tilde{g}, \phi, \nu)$ is called an indefinite trans-Sasakian statistical manifold or (α,β)-type almost contact metric statistical manifold. If $(\tilde{M}, \bar{\nabla}, \tilde{g}, \phi, \nu)$ is an indefinite trans-Sasakian statistical manifold, then so is $(\tilde{M}, \bar{\nabla}^*, \tilde{g}, \phi, \nu)$.

Theorem 3.6. *Let $(\tilde{M}, \bar{\nabla}, \bar{\nabla}^*, \tilde{g})$ be an indefinite statistical manifold with an almost contact metric structure $(\tilde{g}, \phi, \nu)$. Then $(\tilde{M}, \bar{\nabla}, \bar{\nabla}^*, \tilde{g}, \phi, \nu)$ is said to be an indefinite (α,β)-type almost contact metric statistical manifold $\tilde{M}$ if and only if*

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y = \alpha\{g(X,Y)\nu - \eta(Y)X\} + \beta\{\tilde{g}(\phi X, Y)\nu - \eta(Y)\phi X\} \quad \textit{and} \tag{15}$$

$$\bar{\nabla}_X \nu = -\alpha(\phi X) + \beta\{X - \eta(X)\nu\} + \eta(\bar{\nabla}_X \nu)\nu \tag{16}$$

hold $\forall\, X, Y \in \Gamma(T\tilde{M})$ on $\tilde{M}$.

Proof. Let $\tilde{M}$ be an indefinite (α,β)-type almost contact metric statistical manifold. Then, (4) and (12) implies

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y = K_X \phi Y + \hat{\hat{\bar{\nabla}}}_X \phi Y - \phi \hat{\hat{\bar{\nabla}}}_X Y + \phi K_X Y$$

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y = \alpha\{g(X,Y)\nu - \eta(Y)X\} + \beta\{\tilde{g}(\phi X, Y)\nu - \eta(Y)\phi X\}.$$

Now by replacing $\bar{\nabla}$ to $\bar{\nabla}^*$ and Y to ν in above, $\phi \bar{\nabla}_X \nu = \alpha(X - \nu) + \beta(\phi X)$.

Hence (16) follows from (3).

Conversely, on replacing Y by ϕY in (15),

$$\phi\left(\bar{\nabla}_X\phi^2Y-\phi\bar{\nabla}^*_X\phi Y\right)=0,\ \text{where } \eta(\phi Y)=0 \text{ and } \phi\nu=0.$$

Since $\tilde{M}$ is an indefinite statistical manifold, therefore (3) and (16) imply

$$\begin{aligned}0&=-\phi\bar{\nabla}_XY+\eta(Y)\phi\bar{\nabla}_X\nu+\bar{\nabla}^*_X\phi Y-\eta(\bar{\nabla}^*_X\phi Y)\nu\\&=-\phi\bar{\nabla}_XY+\alpha\{\eta(Y)X-\tilde{g}(X,Y)\nu\}+\beta\{\eta(Y)\phi X+\tilde{g}(\phi Y,X)\nu\}+\bar{\nabla}^*_X\phi Y.\end{aligned}$$

Hence

$$\bar{\nabla}^*_X\phi Y-\phi\bar{\nabla}_XY=\alpha\{g(X,Y)\nu-\eta(Y)X\}+\beta\{\tilde{g}(\phi X,Y)\nu-\eta(Y)\phi X\} \tag{17}$$

Further, by (15) and (17) respectively,

$$(\hat{\bar{\nabla}}_X\phi)Y-\alpha\{\tilde{g}(X,Y)\nu-\eta(Y)X\}-\beta\{\tilde{g}(\phi X,Y)\nu-\eta(Y)\phi X\}=K_X\phi Y+\phi K_XY$$

and

$$(\hat{\bar{\nabla}}_X\phi)Y-\alpha\{\tilde{g}(X,Y)\nu-\eta(Y)X\}-\beta\{\tilde{g}(\phi X,Y)\nu-\eta(Y)\phi X\}=-K_X\phi Y-\phi K_XY.$$

Remark 3.7. *Let $(\tilde{g},\phi,\nu)$ be an indefinite trans-Sasakian structure on $\tilde{M}$. So, by setting*

$$K_XY=\eta(X)\eta(Y)\nu$$

for any $X,Y\in\Gamma(T\tilde{M})$ such that K satisfies (13) and (14), an indefinite trans-Sasakian statistical structure $(\bar{\nabla}^\lambda=\hat{\bar{\nabla}}+\lambda K,\tilde{g},\phi,\nu)$ is obtained on $\tilde{M}$ for $\lambda\in\mathbb{C}^\infty(\tilde{M})$.

Inspired from Massamba [7], the basic structure of indefinite trans-Sasakian manifold has been consolidated with statistical structure and elaborated with an example.

Example 3.8. *Let $\tilde{M}$ be a 7-dimensional manifold defined by $\tilde{M}=\{y\in\mathbb{R}^7: y_7\neq 0\}$, where $y=(y_1,y_2,...,y_7)$ are the standard coordinates in $\mathbb{R}^7$. Considering the vector fields $\{e_1,e_2,...,e_7\}$, linearly independent at each point of $\tilde{M}$, as a combination of frames $\left\{\dfrac{\partial}{\partial x_i}\right\}$.*

Let $\tilde{g}$ be the semi-Riemannian metric defined as $\tilde{g}(e_i,e_j)=0,\ \forall\, i\neq j, i,j=1,2,...,7$ and $\tilde{g}(e_k,e_k)=1,\forall\ k=1,2,3,4,7;\ \tilde{g}(e_m,e_m)=-1,\forall\ m=5,6$. Also, let η be the 1-form defined by $\eta(Z)=\tilde{g}(Z,e_7)$, for any $Z\in\mathbb{X}(\tilde{M})$, where $\mathbb{X}(\tilde{M})$ is the set of all differentiable vector fields on $\tilde{M}$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1=-e_2,\ \phi e_2=e_1,\ \phi e_3=-e_4,\ \phi e_4=e_3,\ \phi e_5=-e_6,\ \phi e_6=e_5,\ \phi e_7=0.$$

Then, using the linearity of ϕ and $\tilde{g}$, we have $\eta(e_7) = 1$, $\phi^2 X = -X + \eta(X)e_7$, $\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)$, for any $X, Y \in \mathbb{X}(\tilde{M})$. Thus, for $e_7 = \nu$, $(\tilde{M}, \tilde{g}, \phi, \nu, \eta)$ is an almost contact metric manifold.

Let $\hat{\hat{\nabla}}$ be the Levi-Civita connection with respect to the metric $\tilde{g}$ and let us choose the vector fields $e_1, e_2, e_3, ..., e_7$ to be

$$e_i = e^{y_7} \sum_{j=1}^{7} f_{ij}(y_1, ..., y_6) \frac{\partial}{\partial y_j}, \qquad det(f_{ij}) \neq 0$$

where functions f_{ij} are defined such that the action of $\hat{\hat{\nabla}}$, on the basis $\{e_1, e_2, ...e_7\}$, given by

$$\hat{\hat{\nabla}}_{e_1} e_1 = \nu, \quad \hat{\hat{\nabla}}_{e_1} e_2 = -\frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_2} e_1 = -y_2 e^{y_7} e_2 + \frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_2} e_2 = y_2 e^{y_7} e_1 + \nu,$$

$$\hat{\hat{\nabla}}_{e_3} e_4 = y_3 e^{y_7} e_3 - \frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_3} e_3 = -y_3 e^{y_7} e_4 + \nu, \quad \hat{\hat{\nabla}}_{e_4} e_3 = \frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_4} e_4 = \nu,$$

$$\hat{\hat{\nabla}}_{e_5} e_5 = -\nu, \quad \hat{\hat{\nabla}}_{e_5} e_6 = \frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_6} e_5 = y_6 e^{y_7} e_6 - \frac{1}{2} e^{2y_7} \nu, \quad \hat{\hat{\nabla}}_{e_6} e_6 = -y_6 e^{y_7} e_5 - \nu,$$

and $\hat{\hat{\nabla}}_{e_i} e_j = 0$, $\forall\, i \neq j, i, j = 1, 2, 3, ..., 6$ such that $\tilde{g}(\phi e_i, e_j) = 0$.

Now, for $i = 1, 2, 3, ..., 6$

$[e_1, e_2] = y_2 e^{y_7} e_2 - e^{2y_7} \nu, \quad [e_3, e_4] = y_3 e^{y_7} e_3 - e^{2y_7} \nu, \quad [e_5, e_6] = -y_6 e^{y_7} e_6 + e^{2y_7} \nu$

and $[e_i, e_7] = -e_i$.

For $i, j = 1, 2, ..., 6$, the m^{th}-component of the Lie brackets $[e_i, e_j]$ is given by

$$[e_i, e_j]_m = e^{2y_7} \sum_{k=1}^{6} (f_{ik} \frac{\partial}{\partial y_k}(f_{jm}) - f_{jk} \frac{\partial}{\partial y_k}(f_{im})) + e^{2y_7}(f_{i7} f_{jm} - f_{j7} f_{im}).$$

By Koszul's formula i.e.

$$2\tilde{g}(\hat{\hat{\nabla}}_X Y, Z) = X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) - \tilde{g}(X, [Y, Z]) - \tilde{g}(Y, [X, Z]) + \tilde{g}(Z, [X, Y])$$

following expressions are derived

$$\hat{\hat{\nabla}}_{e_1} e_7 = -e_1 + \frac{1}{2} e^{2y_7} e_2, \quad \hat{\hat{\nabla}}_{e_2} e_7 = -\frac{1}{2} e^{2y_7} e_1 - e_2, \quad \hat{\hat{\nabla}}_{e_3} e_7 = -e_3 + \frac{1}{2} e^{2y_7} e_4,$$

$$\hat{\hat{\nabla}}_{e_4} e_7 = -\frac{1}{2} e^{2y_7} e_3 - e_4, \quad \hat{\hat{\nabla}}_{e_5} e_7 = -e_5 + \frac{1}{2} e^{2y_7} e_6, \quad \hat{\hat{\nabla}}_{e_6} e_7 = -\frac{1}{2} e^{2y_7} e_5 - e_6,$$

Therefore, using above, it is concluded that $(\tilde{M}, \tilde{g}, \phi, \nu, \eta)$ is an indefinite trans-Sasakian manifold of type $(\frac{1}{2} e^{2y_7}, -1)$.

Using Remark (3.7) and taking $\lambda = 1$,

$$\bar{\nabla}_{e_1} e_1 = \nu, \quad \bar{\nabla}_{e_1} e_2 = -\frac{1}{2} e^{2y_7} \nu, \quad \bar{\nabla}_{e_2} e_1 = -y_2 e^{y_7} e_2 + \frac{1}{2} e^{2y_7} \nu, \quad \bar{\nabla}_{e_2} e_2 = y_2 e^{y_7} e_1 + \nu,$$

$$\bar{\nabla}_{e_3}e_4 = y_3e^{y_7}e_3 - \frac{1}{2}e^{2y_7}\nu, \quad \bar{\nabla}_{e_3}e_3 = -y_3e^{y_7}e_4 + \nu, \quad \bar{\nabla}_{e_4}e_3 = \frac{1}{2}e^{2y_7}\nu, \quad \bar{\nabla}_{e_4}e_4 = \nu,$$

$$\bar{\nabla}_{e_5}e_5 = -\nu, \quad \bar{\nabla}_{e_5}e_6 = \frac{1}{2}e^{2y_7}\nu, \quad \bar{\nabla}_{e_6}e_5 = y_6e^{y_7}e_6 - \frac{1}{2}e^{2y_7}\nu, \quad \bar{\nabla}_{e_6}e_6 = -y_6e^{y_7}e_5 - \nu,$$

$$\bar{\nabla}_{e_1}e_7 = -e_1 + \frac{1}{2}e^{2y_7}e_2, \quad \bar{\nabla}_{e_2}e_7 = -\frac{1}{2}e^{2y_7}e_1 - e_2, \quad \bar{\nabla}_{e_3}e_7 = -e_3 + \frac{1}{2}e^{2y_7}e_4,$$

$$\bar{\nabla}_{e_4}e_7 = -\frac{1}{2}e^{2y_7}e_3 - e_4, \quad \bar{\nabla}_{e_5}e_7 = -e_5 + \frac{1}{2}e^{2y_7}e_6, \quad \bar{\nabla}_{e_6}e_7 = -\frac{1}{2}e^{2y_7}e_5 - e_6.$$

Here, $(\bar{\nabla}, \tilde{g})$ *is a statistical structure and since* $K_{e_i}\phi e_j + \phi K_{e_i}e_j = 0$ *holds* $\forall\, i, j = 1, 2, ..., 7$ *then, from remark (3.7),* $(\bar{\nabla}, \tilde{g}, \phi, \eta, \nu)$ *is an indefinite trans-Sasakian statistical structure on* $\tilde{M}$*. Similarly, the above equations for dual connection* $\bar{\nabla}^*$ *can be obtained using* $\bar{\nabla}^*_X Y = \hat{\bar{\nabla}}_X Y - \eta(X)\eta(Y)\nu$.

Thus, $(\bar{\nabla} = \hat{\bar{\nabla}} + K, \tilde{g}, \phi, \nu)$ *defines an indefinite trans-Sasakian statistical structure on* $\tilde{M}$*.*

3.2. **Results on lightlike hypersurfaces.**

Let (M, g) be a lightlike hypersurface of an indefinite trans-Sasakian statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*, \phi, \nu)$, where g is the degenerate metric induced on M. Therefore, for any $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$, using (2) and (3), following holds:

$$\tilde{g}(\xi, \nu) = 0, \quad \tilde{g}(N, \nu) = 0 \qquad (18)$$
$$\phi^2\xi = -\xi, \quad \phi^2 N = -N.$$

Proposition 3.9. *Let* M *be a lightlike hypersurface of an indefinite trans-Sasakian satistical manifold* $\tilde{M}$ *such that the characteristic vector field* ν *is tangent to* M*. Then*

$$g(\phi\xi, N) = \frac{1}{\alpha}\{-g(A^*_N\xi, \nu) + \beta\}, \quad g(\phi\xi, \phi N) = 1 \qquad (19)$$

where ξ, N *are local sections of* $Rad(TM)$ *and* $ltr(TM)$*, respectively.*

Proof. For an indefinite trans-Sasakian statistical manifold $\tilde{M}$

$$g(\phi\xi, N) = \frac{1}{\alpha}g\{\bar{\nabla}_\xi\nu + \beta(\xi - \eta(\xi)\nu) + \eta(\bar{\nabla}_\xi\nu)\nu, N\} = \frac{1}{\alpha}\{-g(A^*_N\xi, \nu) + \beta\}$$

and $g(\phi\xi, \phi N) = 1$ have been obtained using (2), (9) and (16).

The above proposition leads to the following decomposition:

$$S(TM) = \{\phi Rad(TM) \oplus \phi ltr(TM)\} \perp L_o \perp <\nu> \qquad (20)$$

where L_o is non-degenerate and ϕ-invariant distribution.

If distributions on M are denoted by

$$L = Rad(TM) \perp \phi Rad(TM) \perp L_o, \quad L' = \phi ltr(TM), \tag{21}$$

then L is invariant and L' is anti-invariant distributions under ϕ.

Also,

$$TM = L \oplus L' \perp \nu. \tag{22}$$

Consider two null vector fields U and W such that

$$U = -\phi N, \quad W = -\phi \xi \tag{23}$$

and their corresponding 1-forms

$$u(X) = \bar{g}(X, W), \quad w(X) = \bar{g}(X, U).$$

Denote by S, the projection morphism of $T\tilde{M}$ on the distribution L. Then,

$$X = SX + u(X)U \tag{24}$$

for any $X \in \Gamma(T\tilde{M})$. Applying ϕ to (24), we have

$$\phi X = \bar{\phi} X + u(X)N \tag{25}$$

where $\bar{\phi}$ is a tensor field of type $(1,1)$ defined on M by $\bar{\phi} X = \phi SX$.

Using (3),

$$\bar{\phi}^2 X = -X + \eta(X)\nu + u(X)U. \tag{26}$$

Since $\bar{\phi} U = 0$, we obtain $\bar{\phi}^3 + \bar{\phi} = 0$ from (26), which shows that $\bar{\phi}$ is an f-structure on M.

Using (18) and (23),

$$\bar{g}(U, W) = 1$$

which imples that $< U > \oplus < W >$ is non-degenerate vector bundle of $S(TM)$ with rank 2. From (20) and (21), the following decompositions hold:

$$S(TM) = \{U \oplus W\} \perp L_o \perp < \nu >, \quad L = Rad(TM) \perp < W > L_o, \quad L' = < U > . \tag{27}$$

Let P and Q be two projections of TM into L and L', respectively. Then $X = PX + QX + \eta(X)\nu$, for any $X \in \Gamma(TM)$. Therefore (2), (3), (25) and (27) give

$$\bar{\phi}^2 X = -X + \eta(X)\nu + u(X)U \tag{28}$$

where $QX = u(X)U$ and $\phi PX = \bar{\phi} X$. Also, using (25), following identities hold:

$$g(\bar{\phi} X, \bar{\phi} Y) = g(X, Y) - \eta(X)\eta(Y) - u(X)w(Y) - u(Y)w(X) \tag{29}$$

$$g(\bar{\phi} X, Y) = -g(X, \bar{\phi} Y) - u(Y)\theta(X) - u(X)\theta(Y) \tag{30}$$

$$\bar{\phi}\nu = 0, \quad g(\bar{\phi} X, \nu) = 0 \quad \forall\, X, Y \in \Gamma(TM). \tag{31}$$

Thus, the following proposition:

Proposition 3.10. *For a lightlike hypersurface M of an indefinite trans-Sasakian statistical manifold $\tilde{M}$, $\bar{\phi}$ need not be an almost contact metric structure.*

Proof. Proof follows from (28)-(31).

Example 3.11. *Following example (3.8), let M be a hypersurface defined by*

$$M = \{y \in \tilde{M} : y_5 = y_4,\ f_{4i} = f_{5j} = 0,\ f_{44} \neq 0,\ f_{55} \neq 0\}$$

of an indefinite trans-Sasakian statistical manifold $(\tilde{M}, \tilde{g}, \phi, \eta, \nu)$.

The tangent space TM is spanned by $\{Z_i\}$, where $Z_1 = e_1,\ Z_2 = e_2,\ Z_3 = e_3,\ Z_4 = e_4 - e_5,\ Z_5 = e_6,\ Z_6 = \nu$. Further, $E = e_4 - e_5$ spans the distribution $TM^\perp$ of rank 1. Therefore, $TM^\perp \subset TM$ and M is a 6-dimensional lightlike hypersurface of $\tilde{M}$. The transversal bundle $ltr(TM)$ is spanned by $N = \frac{1}{2}(e_4 + e_5)$. From decomposition (20) and the almost contact structure of $\tilde{M}$, L_o is spanned by $\{H, \phi H\}$, where $H = Z_1$, $\phi H = -Z_2$ and the distributions ν, $\phi Rad(TM)$ and $\phi ltr(TM)$ are spanned, respectively, by ν, $\phi D = Z_3 + Z_5$ and $\phi N = \frac{1}{2}(Z_3 - Z_5)$.

Hence, M is a lightlike hypersurface of an indefinite trans-Sasakian statistical manifold $\tilde{M}$.

4. LIGHTLIKE HYPERSURFACES OF AN INDEFINTE (α, β)-TYPE ALMOST CONTACT METRIC STATISTICAL MANIFOLD WITH AN (l, m)-TYPE CONNECTION

4.1. (l, m)-type connection.

For a Levi-Civita connection $\hat{\bar{\nabla}}$ on an indefinite (α, β)-type almost contact metric statistical manifold $(\tilde{M}, \phi, \tilde{g}, \nu)$ where $\hat{\bar{\nabla}} = \frac{1}{2}\{\bar{\nabla} + \bar{\nabla}^*\}$, set

$$\tilde{D}_X Y = \bar{\nabla}_X Y - K_X Y + \eta(Y)\{lX + m\phi X\} \tag{32}$$

and

$$\tilde{D}_X Y = \bar{\nabla}^*_X Y + K_X Y + \eta(Y)\{lX + m\phi X\}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

Since $\bar{\nabla}$ and $\bar{\nabla}^*$ are torsion free, the (l, m)-type connection $\tilde{D}$ as given in (1) is obtained.

As $\tilde{M}$ admits a tensor field ϕ of type $(1,1)$, then for any $X, Y \in \Gamma(T\tilde{M})$,

$$(\tilde{D}_X\phi)Y = \alpha\{\tilde{g}(X,Y)\nu - \eta(Y)X\} + \beta\{\tilde{g}(\phi X, Y)\nu - \eta(Y)\phi X\} - \eta(Y)\{l\phi X - mX + m\eta(X)\nu\}.$$

On replacing Y by ν in the above equation, the following has been derived:

$$\tilde{D}_X\nu = (m-\alpha)\phi X + (l+\beta)X - \beta\eta(X)\nu. \tag{33}$$

Let M be a lightlike hypersurface of an indefinite (α,β)-type almost contact metric statistical manifold $(\tilde{M},\tilde{g})$ with (l,m)-type connection $\tilde{D}$. Let D be the induced linear connection on M from $\tilde{D}$. Then, the corresponding Gauss formula is given by

$$\tilde{D}_X Y = D_X Y + \tilde{B}(X,Y)N \tag{34}$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y \in \Gamma(TM)$ and $\tilde{B}$ is the local second fundamental form on M.

Therefore, (8) and (32) gives

$$D_X Y = \nabla_X Y - K_X Y + \eta(Y)\{lX + m\bar{\phi}X\}, \quad \tilde{B}(X,Y) = B(X,Y) + m\eta(Y)u(X),$$
$$D_X Y = \nabla^*_X Y + K_X Y + \eta(Y)\{lX + m\bar{\phi}X\}, \quad \tilde{B}(X,Y) = B^*(X,Y) + m\eta(Y)u(X). \tag{35}$$

Similarly, the Weingarten formula is given by

$$\tilde{D}_X N = -\tilde{A}_N X + \tilde{\tau}(X)N, \tag{36}$$

where

$$\tilde{A}_N X = A_N X + K_X N, \quad \tilde{\tau}(X) = \tau(X),$$
$$\tilde{A}_N X = A^*_N X - K_X N, \quad \tilde{\tau}(X) = \tau^*(X).$$

Now, (25), (1) and (34) gives

$$(D_X g)(Y,Z) = \tilde{B}(X,Z)\theta(Y) + \tilde{B}(X,Y)\theta(Z) - l\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\} \tag{37}$$
$$-m\{\eta(Y)g(\bar{\phi}X,Z) + \eta(Y)u(X)\theta(Z) + \eta(Z)g(\bar{\phi}X,Y) + \eta(Z)u(X)\theta(Y)\}$$
$$T(X,Y) = l\{\eta(Y)X - \eta(X)Y\} + m\{\eta(Y)\bar{\phi}X - \eta(X)\bar{\phi}Y\}$$
$$\tilde{B}(X,Y) - \tilde{B}(Y,X) = m\{\eta(Y)u(X) - \eta(X)u(Y)\} \tag{38}$$

Following are some results about the geometry of lightlike hypersurfaces of indefinite (α,β)-type almost contact metric statistical manifold with (l,m)-type connection.

Lemma 4.1. *Let M be a lightlike hypersurface of an indefinite (α,β)-type almost contact metric statistical manifold $\tilde{M}$ with an (l,m)-type connection $\tilde{D}$. Then the induced connection D on M is not an (l,m)-type connection.*

PROOF. Proof follows from (37).

Theorem 4.2. *Let $\tilde{M}$ be an indefinite (α,β)-type almost contact metric statistical manifold with an (l,m)-type connection $\tilde{D}$. Then, for a lightlike hypersurface M of $\tilde{M}$ such that ν is tangent to M, $\tilde{B}$ is symmetric if and only if $m = 0$.*

Proof. Since $\tilde{B}(X,Y)-\tilde{B}(Y,X)=m\{\eta(Y)u(X)-\eta(X)u(Y)\}$, therefore $\tilde{B}$ is symmetric, if $m=0$.
Conversely, if $\tilde{B}(X,Y)=\tilde{B}(Y,X)$, then by replacing X, Y with ν, U respectively in (38), $m=0$.

Now, let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. Therefore, for any $X,Y\in\Gamma(TM)$ and $\xi\in\Gamma(Rad(TM))$

$$D_X PY = D'PY + C'(X,PY)\xi, \quad D_X\xi = -\tilde{A}'_\xi X - \tau'(X)\xi$$

holds, where D and D' are the induced linear connections on M and $S(TM)$, respectively. Also, C' is the local second fundamental form on $S(TM)$, $\tilde{A}'$ is the shape operator and τ' is the 1-form on M.

Here,

$$D'_X PY = \nabla'_X PY - K_X PY + \eta(PY)\{lX + m\bar{\phi}X\}, \quad C'(X,PY)\xi = C(X,PY)\xi,$$

$$D'_X PY = \nabla^{*\prime}_X PY + K_X PY + \eta(PY)\{lX + m\bar{\phi}X\}, \quad C'(X,PY)\xi = C^*(X,PY)\xi.$$

Also,

$$\tilde{A}'_\xi X = A'_\xi X + K_X\xi, \quad \tau'(X) = \tau(X),$$

$$\tilde{A}'_\xi X = A^{*\prime}_\xi X - K_X\xi, \quad \tau'(X) = \tau^*(X).$$

Now, (11) and (35) lead to

$$\tilde{B}(X,\xi) = 0, \quad \tilde{B}(\xi,X) = 0.$$

Further, the relations between local second fundamental forms with their shape operators are as follows:

$$\tilde{B}(X,Y) = \frac{1}{2}\{g(A^{*\prime}_\xi X,Y) + g(A'_\xi X,Y)\} + m\eta(Y)u(X) \tag{39}$$

$$C'(X,PY) = C(X,PY) = g(A^*_N X,PY) = g(\tilde{A}_N X + K_X N, PY). \tag{40}$$

$$\tilde{g}(\tilde{A}'_\xi X,N) = 0, \quad \tilde{g}(\tilde{A}_N X,N) = 0.$$

Putting $X=\xi$ in (39),

$$A^{*\prime}_\xi\xi + A'_\xi\xi = 0, \quad \tilde{D}_X\xi = -\tilde{A}'_\xi X - \tau'(X)\xi. \tag{41}$$

On applying $\tilde{D}_X$ to $\tilde{g}(\xi,\nu)$ and using (39) and (41),

$$g(\tilde{A}'_\xi X,\nu) = -\alpha u(X), \quad \tilde{B}(X,\nu) = (m-\alpha)u(X). \tag{42}$$

Similarly, by applying $\tilde{D}_X$ to $\tilde{g}(\nu,N)$ and from (36) and (40),

$$g(\tilde{A}_N X,\nu) = -\alpha w(X) + \beta\theta(X), \quad C'(X,\nu) = -\alpha w(X) + \beta\theta(X) + \eta(K_X N). \tag{43}$$

Using (25) and (34) in (33), following has been derived:

$$D_X\nu = (m-\alpha)\bar{\phi}X + (l+\beta)X - \beta\eta(X)\nu.$$

Lemma 4.3. *Let M be a lightlike hypersurface of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$ with (l, m)-type connection $\tilde{D}$. Then,*

$$D_X U = \bar{\phi}(\tilde{A}_N X) + \tilde{\tau}(X)U - (\alpha\theta(X) + \beta w(X))\nu,$$

$$\tilde{B}(X, U) = C'(X, W) - g(K_X N, W),$$

holds for any $X, Y \in \Gamma(TM)$.

Proof. Since $\tilde{M}$ is an indefinite trans-Sasakian statistical manifold, then

$$\bar{\nabla}^*_X U + \alpha(\theta(X)\nu) + \beta(w(X)\nu) = \phi A_N X + \tau(X)U \quad \text{follows by using (9).}$$

By applying (l, m)-connection and using (34) in above equation,

$$D_X U + \tilde{B}(X, U)N = -\alpha\theta(X)\nu - \beta w(X)\nu + \bar{\phi}(\tilde{A}_N X) + u(\tilde{A}_N X)N + \tilde{\tau}(X)U$$

which implies

$$D_X U = \bar{\phi}(\tilde{A}_N X) + \tilde{\tau}(X)U - (\alpha\theta(X) + \beta w(X))\nu,$$

and

$$\tilde{B}(X, U) = u(\tilde{A}_N X) = g(\tilde{A}_N X, W) = C'(X, W) - g(K_X N, W).$$

Lemma 4.4. *Let M be a lightlike hypersurface of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$ with (l, m)-type connection $\tilde{D}$. Then,*

$$D_X W = \bar{\phi}(\tilde{A}'_\xi X) - \tilde{\tau}(X)W - \beta u(X)\nu.$$

$$\tilde{B}(X, W) = u(\tilde{A}'_\xi X) = g(\tilde{A}'_\xi X, W).$$

holds for any $X, Y \in \Gamma(TM)$.

Proof. Replacing Y by ξ in Gauss equation and using (10),

$$\bar{\nabla}_X \xi = -A'_\xi X - \tau(X)\xi + B(X, \xi)N.$$

On applying ϕ to above equation $\phi\bar{\nabla}_X \xi = -\phi A'_\xi X + \tau(X)W - B(X, \xi)U$.

Since $\tilde{M}$ is an indefinite trans-Sasakian statistical manifold with (l, m)-type connection, therefore

$$-\bar{\nabla}^*_X W - \beta u(X)\nu = -\phi A'_\xi X + \tau(X)W,$$

which implies

$$-\tilde{D}_X W + K_X W - \beta u(X)\nu = -\phi(\tilde{A}'_\xi X - K_X \xi) + \tau(X)W.$$

Now from (25) and (34), it follows that

$$-D_X W - \tilde{B}(X, W)N - \beta u(X)\nu = -\tilde{\phi}(\tilde{A}'_\xi X) - u(\tilde{A}'_\xi X)N + \tilde{\tau}(X)W$$

Therefore, the desired result holds when comparing tangential and normal parts.

Example 4.5. *Consider the indefinite trans-Sasakian statistical manifold* $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*, \phi, \nu)$ *mentioned in example (3.8). Applying* (l, m)*-connection* $\tilde{D}$ *on* $\tilde{M}$ *and using (32), the following derivations have been made:*

$$\tilde{D}_{e_1}e_1 = \nu, \quad \tilde{D}_{e_1}e_2 = -\frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_2}e_1 = -y_2e^{y_7}e_2+\frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_2}e_2 = y_2e^{y_7}e_1+\nu,$$

$$\tilde{D}_{e_3}e_4 = y_3e^{y_7}e_3-\frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_3}e_3 = -y_3e^{y_7}e_4+\nu, \quad \tilde{D}_{e_4}e_3 = \frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_4}e_4 = \nu,$$

$$\tilde{D}_{e_5}e_5 = -\nu, \quad \tilde{D}_{e_5}e_6 = \frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_6}e_5 = y_6e^{y_7}e_6-\frac{1}{2}e^{2y_7}\nu, \quad \tilde{D}_{e_6}e_6 = -y_6e^{y_7}e_5-\nu,$$

$$\tilde{D}_{e_1}e_7 = (l-1)e_1 + (\frac{1}{2}e^{2y_7} - m)e_2, \quad \tilde{D}_{e_2}e_7 = (l - \frac{1}{2}e^{2y_7})e_1 + (m-1)e_2,$$

$$\tilde{D}_{e_3}e_7 = (l-1)e_3 + (\frac{1}{2}e^{2y_7} - m)e_4, \quad \tilde{D}_{e_4}e_7 = (m - \frac{1}{2}e^{2y_7})e_3 + (l-1)e_4,$$

$$\tilde{D}_{e_5}e_7 = (l-1)e_5 + (\frac{1}{2}e^{2y_7} - m)e_6, \quad \tilde{D}_{e_6}e_7 = (m - \frac{1}{2}e^{2y_7})e_5 + (l-1)e_6.$$

Also, for $e_7 = \nu$*, above equations satisfy (33) i.e.*

$$\tilde{D}_X\nu = (m-\alpha)\phi X + (l+\beta)X - \beta\eta(X)\nu$$

with $\alpha = \frac{1}{2}e^{2y_7}$ *and* $\beta = -1$*. Therefore* $(\tilde{M}, \tilde{g}, \phi, \eta, \nu)$ *is an indefinite* $(\frac{1}{2}e^{2y_7}, -1)$*-type almost contact metric statistical manifold with an* (l, m)*-type connection* $\tilde{D}$*.*

Theorem 4.6. *For a lightlike hypersurface* M *of an indefinite* (α, β)*-type almost contact metric statistical manifold* $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*, \phi, \nu)$ *with an* (l, m)*-type connection* $\tilde{D}$*, following identities hold:*

$$(D_X\bar{\phi})Y = u(Y)\tilde{A}_NX - \tilde{B}(X,Y)U - (l+\beta)\eta(Y)\bar{\phi}X + X(m-\alpha)\eta(Y) \tag{44}$$
$$+\{\alpha g(X,Y) + \beta g(\bar{\phi}X, Y) + \beta u(X)\theta(Y) - m\eta(X)\eta(Y)\}\nu \text{ and}$$
$$(D_Xu)(Y) = -u(Y)\tilde{\tau}(X) - \tilde{B}(X, \bar{\phi}Y) - (l+\beta)\eta(Y)u(X) \tag{45}$$

for any $X, Y \in \Gamma(TM)$*.*

Proof. From the notion of indefinite (α, β)-type almost contact metric statistical manifold,

$$\bar{\nabla}_X\phi Y - \phi\bar{\nabla}^*_XY = \alpha\{g(X,Y)\nu - \eta(Y)X\} + \beta\{\bar{g}(\phi X, Y)\nu - \eta(Y)\phi X\}. \tag{46}$$

Then application of (l, m)-type connection to (46) gives

$$\tilde{D}_X\phi Y - \phi\{\tilde{D}_XY - \eta(Y)(lX + m\phi X)\} = \alpha(g(X,Y)\nu - \eta(Y)X) + \beta(-\eta(Y)\phi X + g(\phi X, Y)\nu).$$

Now, using (3), (23), (25), (34) and (36), following holds:

$$D_X\bar{\phi}Y + \tilde{B}(X, \bar{\phi}Y)N - u(Y)\tilde{A}_NX + u(Y)\tilde{\tau}(X)N + Xu(Y)N - \bar{\phi}D_XY - u(D_XY)N$$
$$+\tilde{B}(X,Y)U + \eta(Y)\{l\bar{\phi}X + lu(X)N - mX + m\eta(X)\nu\}$$
$$= \alpha\{g(X,Y)\nu - \eta(Y)X\} + \beta\{-\eta(Y)\bar{\phi}X - \eta(Y)u(X)N + g(\bar{\phi}X, Y)\nu + u(X)\theta(Y)\nu\}.$$

The required result follows on comparing tangential and normal parts.

Theorem 4.7. *Let (M, g, ∇, ∇^*) be a lightlike hypersurface of an indefinite trans-Sasakian statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*, \phi, \nu)$ with an (l, m)-type connection $\tilde{D}$. Then, for the null vector fields U and W, following hold:*

$$D_X W = \bar{\phi}\tilde{A}'_{\xi} X - \tau'(X)W - \beta u(X)\nu, \tag{47}$$

$$\tilde{\tau}(X) = u(D_X U), \quad \forall\, X \in \Gamma(TM). \tag{48}$$

Proof. By taking $Y = U$ in (45) and using (23), (25),

$$Xu(U)N - u(D_X U)N = -u(U)\tilde{\tau}(X)N - \tilde{B}(X, \bar{\phi}U)N - (l + \beta)\eta(U)u(X)N$$

which implies $\tilde{\tau}(X) = u(D_X U)$ or $\tau(X) = u(\nabla_X U) - u(K_X U)$.

Now, on replacing Y by ξ in (44) and using (6), (30),

$$-D_X W - \bar{\phi}(-\tilde{A}'_{\xi} X - \tau'(X)\xi) = -\tilde{B}(X, \xi)U + \{\alpha g(X, \xi) + \beta u(X)\}\nu.$$

Therefore, $D_X W = \bar{\phi}\tilde{A}'_{\xi} X - \tau'(X)W - \beta u(X)\nu \quad \forall\, X \in \Gamma(TM)$.

We now establish the following assertion:

Remark 4.8. *Let M be a hypersurface of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$ with an (l, m)-type connection $\tilde{D}$. Then, M is said to be totally geodesic with respect to $\tilde{D}$ if $\tilde{B}(X, Y) = 0 \;\forall\; X, Y \in \Gamma(TM)$, as $2\tilde{B}(X, Y) = B(X, Y) + B^*(X, Y) + 2m\eta(Y)u(X)N$.*

Theorem 4.9. *A lightlike hypersurface M of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$ with an (l, m)-type connection $\tilde{D}$, is totally geodesic with respect to $\tilde{D}$ if and only if*

$$(D_X \bar{\phi})Y = \{\alpha g(X, Y) + \beta g(\bar{\phi}X, Y)\}\nu,$$

and

$$\tilde{A}_N X = -\bar{\phi} D_X U + \{\beta\theta(X) - \alpha w(X)\}\nu \tag{49}$$

hold for all $X \in \Gamma(TM)$ and $Y \in \Gamma(L)$.

Proof. Let $Y \in \Gamma(L)$ then, $u(Y) = 0$. So, it follows from (44) that

$$(D_X \bar{\phi})Y = -\tilde{B}(X, Y)U + \alpha g(X, Y)\nu + \beta g(\bar{\phi}X, Y)\nu.$$

Also, by taking $Y = U$ in (44) leads to,

$$-\bar{\phi} D_X U = \tilde{A}_N X - \tilde{B}(X, U)U + \alpha w(X)\nu - \beta\theta(X)\nu.$$

Using remark (4.8), the required conditions hold.

Theorem 4.10. *Let M be a lightlike hypersurface of $\tilde{M}$, where $\tilde{M}$ is an indefinite (α, β)-type almost contact metric statistical manifold with an (l, m)-type connection $\tilde{D}$. If the vector field W is parallel with respect to $\tilde{D}$, then following holds for any $X, Y \in \Gamma(TM)$:*

$$\tilde{A}'_{\xi}X = \eta(\tilde{A}'_{\xi}X)\nu + u(\tilde{A}'_{\xi}X)U$$

and

$$\tau'(X) = 0 \quad \forall\, X \in \Gamma(TM).$$

Proof. Applying $\bar{\phi}$ on (47),

$$\bar{\phi}D_X W = -\tilde{A}'_{\xi}X + \eta(\tilde{A}'_{\xi}X)\nu + u(\tilde{A}'_{\xi}X)U - \tau'(X)\xi.$$

The vector field W being parallel with respect to $\tilde{D}$ implies

$$\tilde{A}'_{\xi}X = \eta(\tilde{A}'_{\xi}X)\nu + u(\tilde{A}'_{\xi}X)U$$

and

$$\tau'(X) = 0.$$

Theorem 4.11. *Let M be a lightlike hypersurface of $\tilde{M}$, where $\tilde{M}$ is an indefinite (α, β)-type almost contact metric statistical manifold with an (l, m)-type connection $\tilde{D}$. If the vector field U is parallel with respect to $\tilde{D}$, then*

$$\tilde{A}_N X = u(\tilde{A}_N X)U + \eta(\tilde{A}_N X)\nu$$

and

$$\tilde{\tau}(X) = 0 \quad \forall\, X \in \Gamma(TM)$$

holds for any $X, Y \in \Gamma(TM)$.

Proof. On applying $\bar{\phi}$ to (49) and using (28),

$$\bar{\phi}\tilde{A}_N X = -\bar{\phi}^2 D_X U + \{\beta\theta(X) - \alpha w(X)\}\bar{\phi}\nu.$$

Since $\bar{\phi}\nu = 0$, therefore

$$\bar{\phi}\tilde{A}_N X = D_X U - \eta(D_X U)\nu - u(D_X U)U.$$

From (48), it follows that

$$\bar{\phi}\tilde{A}_N X = D_X U - \eta(D_X U)\nu, \quad \tilde{\tau}(X) = 0. \tag{50}$$

Now the application of $\bar{\phi}$ to (50) and parallelism of U implies

$$\tilde{A}_N X = u(\tilde{A}_N X)U + \eta(\tilde{A}_N X)\nu.$$

4.2. **Screen semi-invariant lightlike hypersurfaces.**

Inspired by [15], we propose the following definition:

Definition 4.12. *Let (M, g, ∇, ∇^*) be a lightlike hypersurface of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$. Then, M is said to be a screen semi-invariant lightlike hypersurface of $(\tilde{M}, \bar{\nabla}, \bar{\nabla}^*, \tilde{g}, \phi, \nu)$ if*

$$\phi(ltr(TM)) \subset S(TM) \quad \text{and} \quad \phi(Rad(TM)) \subset S(TM).$$

Theorem 4.13. *Let $\tilde{M}$ be an indefinite (α, β)-type almost contact metric statistical manifold with (l, m)-type connection $\tilde{D}$. Let M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then the distribution $L \perp< \nu >$ is integrable if and only if*

$$\tilde{B}(X, \bar{\phi}Y) = \tilde{B}(Y, \bar{\phi}X) \qquad \forall\, X, Y \in \Gamma(L \perp< \nu >).$$

Proof. Let $X, Y \in \Gamma(L \perp< \nu >)$. Then, (22), (44) and (45) implies

$$(D_X\bar{\phi})Y = -\tilde{B}(X, Y)U + \{\alpha g(X, Y) + \beta g(\bar{\phi}X, Y)\}\nu$$

and $\tilde{B}(X, \bar{\phi}Y) = g(D_XY, W)$ since $Y \in \Gamma(L \perp< \nu >)$ if and only if $u(Y) = 0$.

Similarly, $\tilde{B}(Y, \bar{\phi}X) = g(D_YX, W)$ which gives

$$\tilde{B}(X, \bar{\phi}Y) - \tilde{B}(Y, \bar{\phi}X) = g(D_XY - D_YX, W).$$

If $X, Y \in \Gamma(L \perp< \nu >)$, so does $[X, Y]$. Consequently, $L \perp< \nu >$ is integrable with respect to $\bar{\nabla}$ if and only if $g([X, Y], W) = 0$. Therefore $g(\nabla_XY - \nabla_YX, W) = 0$.

The application of (l, m)-type connection to the above equation alongwith (35) results in

$$g(D_XY - D_YX, W) = 0$$

which proves the assertion.

Theorem 4.14. *For a screen semi-invariant lightlike hypersurface M of an indefinite (α, β)-type almost contact metric statistical manifold $\tilde{M}$ with (l, m)-type connection $\tilde{D}$, the distribution $L' \perp< \nu >$ is integrable if and only if*

$$u(X)\tilde{A}_NY - u(Y)\tilde{A}_NX = \beta\{u(X)\theta(Y) - u(Y)\theta(X)\}\nu \qquad \forall\, X, Y \in \Gamma(L' \perp< \nu >)$$

Proof. $X \in \Gamma(L' \perp< \nu >)$ if and only if $\bar{\phi}X = 0$. Using (22) in (44),

$$\bar{\phi}D_XY = -u(Y)\tilde{A}_NX + \tilde{B}(X, Y)U - \{\alpha g(X, Y) + \beta u(X)\theta(Y)\}\nu \tag{51}$$

for $X, Y \in \Gamma(L' \perp< \nu >)$.

Now, if $X, Y \in \Gamma(L' \perp < \nu >)$, so is $[X, Y] \in \Gamma(L' \perp < \nu >)$. Further, $L' \perp < \nu >$ is integrable with respect to $\bar{\nabla}$ if and only if

$$\bar{\phi}[X, Y] = 0.$$

Using the definition of (l, m)-type connection, above equation gives

$$\bar{\phi}(D_X Y - D_Y X) = 0. \tag{52}$$

Using (22) and (35), we conclude that $\tilde{B}$ is symmetric. Therefore, from the direct calculations and using (51) and (52) following hold:

$$u(X)\tilde{A}_N Y - u(Y)\tilde{A}_N X = \beta\{u(X)\theta(Y) - u(Y)\theta(X)\}\nu.$$

5. RECURRENT AND LIE RECURRENT STRUCTURE TENSOR FIELD

Definition 5.1. *The structure tensor field $\bar{\phi}$ of a lightlike hypersurface M with respect to (l, m)-type connection $\tilde{D}$ is said to be recurrent if there exists a 1-form μ on M such that*

$$(D_X \bar{\phi})Y = \mu(X)\bar{\phi}Y. \tag{53}$$

Theorem 5.2. *For a recurrent structure tensor field $\bar{\phi}$, there exists a lightlike hypersurface M of an indefinite trans-Sasakian statistical manifold $\tilde{M}$ with (l, m)-type connection $\tilde{D}$ if $mW = l\xi$.*

Proof. Since M is recurrent lightlike hypersurface, (44) implies

$$\mu(X)\bar{\phi}Y = u(Y)\tilde{A}_N X - \tilde{B}(X, Y)U - (l + \beta)\eta(Y)\bar{\phi}X + X(m - \alpha)\eta(Y) \tag{54}$$
$$+\{\alpha g(X, Y) + \beta g(\bar{\phi}X, Y) + \beta u(X)\theta(Y) - m\eta(X)\eta(Y)\}\nu.$$

Taking $Y = \xi$ in (54) and using (26), we obtain,

$$\mu(X)W = -\beta u(X)\nu. \tag{55}$$

Since $g(U, W) = 0$, therefore by taking scalar product of U with above equation $\mu(X) = 0$.

Similarly, taking scalar product with ν to (55), we obtain $\beta = 0$. As $\mu = 0$, $\bar{\phi}$ is parallel with respect to D. Using (42) and by putting $Y = \nu$ in (54), we have

$$(m - \alpha)\{X - u(X)U - \eta(X)\nu\} = l\bar{\phi}X. \tag{56}$$

Again, taking scalar product of ν with (54) gives

$$0 = u(Y)g(\tilde{A}_N X, \nu) - \alpha g(X, Y) - \alpha\eta(X)\eta(Y).$$

Considering the skew-symmetric part of the above equation and then adding both equations, we get

$$u(X)g(\tilde{A}_N Y,\nu) - u(Y)g(\tilde{A}_N X,\nu) = 0. \tag{57}$$

Now from (43), we have $g(\tilde{A}_N X,\nu) = -\alpha w(X)$. It follows from (57) that $\alpha = 0$.

Hence on replacing X by W, (56) leads to $mW = l\xi$.

Theorem 5.3. *For a lightlike hypersurface M of an indefinite trans-Sasakian statistical manifold $\tilde{M}$ with (l,m)-type connecton $\tilde{D}$, if $\bar{\phi}$ is recurrent, then*

$$(D_X\bar{\phi})Y = 0,$$

$$\tilde{A}_N X = \tilde{B}(X,U)U - \{\alpha w(X) - \beta\theta(X)\}\nu, \tag{58}$$

$$\tilde{B}(X,U)u(Y) = \frac{1}{2}\{g({A^*}'_\xi X + A'_\xi X, Y)\} + \{\alpha - (l+\beta)\theta(W)\}\eta(Y)u(X).$$

Proof. Considering $\mu(X) = 0$, the assertion follows from (53).

Further, on replacing Y by U, (54) becomes

$$\tilde{A}_N X = \tilde{B}(X,U)U - \{\alpha w(X) - \beta\theta(X)\}\nu.$$

Therefore, by taking scalar product of W with (54) and using (30), (39), (58),

$$\tilde{B}(X,U)u(Y) = \frac{1}{2}\{g({A^*}'_\xi X + A'_\xi X, Y)\} + \{\alpha - (l+\beta)\theta(W)\}\eta(Y)u(X)$$

follows from the hypothesis.

Definition 5.4. *The structure tensor field $\bar{\phi}$ of M is said to be Lie recurrent if there exists a 1-form φ on M such that*

$$(\mathbb{L}_X\bar{\phi})Y = \varphi(X)\bar{\phi}Y \tag{59}$$

where, $\mathbb{L}_X$ denotes the Lie-derivative on M with respect to X, i.e.,

$$(\mathbb{L}_X\bar{\phi})Y = [X,\bar{\phi}Y] - \bar{\phi}[X,Y]. \tag{60}$$

Note: *The structure tensor field $\bar{\phi}$ is called Lie parallel if $\mathbb{L}_X\bar{\phi} = 0$.*

Theorem 5.5. *Let M be a lightlike hypersurface of an indefinite (α,β)-type almost contact metric statistical manifold $\tilde{M}$ with (l,m)-type connection $\tilde{D}$. If the structure tensor field $\bar{\phi}$ is Lie recurrent, it is not Lie parallel.*

Proof. Considering (59) and (60), then

$$\varphi(X)\bar{\phi}Y = \bar{\nabla}_X\bar{\phi}Y - \bar{\nabla}_{\bar{\phi}Y}X - \bar{\phi}[\bar{\nabla}_X Y - \bar{\nabla}_Y X].$$

Applying (l,m)-connection in the above equation and using (44),

$$\varphi(X)\bar{\phi}Y = (D_X\bar{\phi})Y - D_{\bar{\phi}Y}X + \bar{\phi}D_Y X + \eta(Y)\{l\bar{\phi}X - mX + m\eta(X)\nu + mu(X)U\}$$

$$\varphi(X)\bar{\phi}Y = u(Y)\tilde{A}_N X - \tilde{B}(X,Y)U - \eta(Y)\{\beta\bar{\phi}X + \alpha X\} + mu(X)\eta(Y)U \quad (61)$$
$$+\{\alpha g(X,Y) + \beta g(\bar{\phi}X,Y) + \beta u(X)\theta(Y)\}\nu - D_{\bar{\phi}Y}X + \bar{\phi}D_Y X.$$

Replacing Y by ξ,

$$-\varphi(X)W = \beta u(X)\nu + D_W X + \bar{\phi}D_\xi X \quad (62)$$

Taking the scalar product of W and ν with (62),

$$u(D_W X) = \frac{1}{\alpha}u(D_\xi X)\{g(A^*_N\xi,\nu) - \beta\} \quad \text{and} \quad \eta(D_W X) = -\beta u(X)$$

follows from (19).

Again, replacing Y by W, (61) implies

$$\varphi(X)\xi = -\tilde{B}(X,W)U + \alpha u(X)\nu - D_\xi X + \bar{\phi}D_W X \quad (63)$$

On application of $\bar{\phi}$ on (63) and from (36),

$$\varphi(X)W = \beta u(X)\nu + D_W X + \bar{\phi}D_\xi X + u(D_W X)U. \quad (64)$$

Therefore, from (64) and (62), $\varphi(X)W = \frac{1}{2}u(D_W X)U \neq 0$, Hence $\bar{\phi}$ is not Lie parallel.

REFERENCES

[1] Kazan, A., Conformally-projectively flat trans-Sasakian statistical manifolds, *Physica A Stat. Mech. Appl.*, **535** (2019), 122441.

[2] Chinea, D. and Gonzalez, C., A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.*, **156** (1990), 15-36.

[3] Jin, D.H., Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection, *Bull. Korean Math. Soc.*, **53**(6) (2016), 1771-1783.

[4] Jin, D.H., Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (l,m)-type connection, *J. Korean Math. Soc.*, **55**(5) (2018), 1075–1089.

[5] Yasar, E., Çoeken, A.C. and Yuecesan, A., Lightlike hypersurfaces of semi-Riemannian manifold with semi-symmetric metric connection, *Kuwait J. Sci.*, **34**(2A) (2007), 11-24.

[6] Massamba, F., On semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds, *J. Geom.*, **95** (2009), 73–89.

[7] Massamba, F., Lightlike hypersurfaces in indefinite trans-Sasakian manifolds, *Results Math.*, **63**(1-2) (2013), 251–287.

[8] Furuhata, H., Hypersurfaces in statistical manifolds, *Differential Geom. Appl.*, **27**(3) (2009), 420-429.

[9] Furuhata, H., Hasegawa, I. Okuyama, Y. and Sato, K., Kenmotsu statistical manifolds and warped product, *J. Geom.*, **108**(3) (2017), 1175-1191.

[10] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K. and Shahid, M.H., Sasakian statistical manifolds, *J. Geom. Phys.*, **117** (2017), 179-186.

[11] Oubiña, J.A., New class of almost contact metric structure, *Publ. Math. Debrecen*, **32** (1985), 187-193.

[12] Duggal, K.L. and Bejancu, A., *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Kluwer Academic, Dordrecht, **364**, 1996.

[13] Duggal, K.L. and Sahin, B., Lightlike submanifolds of indefinite Sasakian manifolds, *Int. J. Math. Math. Sci.*, **2007** (2007), 21 Pages.

[14] Duggal, K.L. and Sahin, B., *Differential Geometry of Lightlike Submanifolds*, Springer Science & Business Media, 2011.

[15] Bahadir, O., On lightlike geometry of indefinite Sasakian statistical manifolds, *AIMS math.*, **6**(11) (2021), 12845-12862.

[16] Bahadir, O., Siddiqui, A.N., Glbahar, M. and Alkhaldi, A.H., Main curvatures identities on lightlike hypersurfaces of statistical manifolds and their characterizations, *Mathematics*, **10**(13) (2022), 2290.

[17] Bahadir, O. and Tripathi, M.M., Geometry of lightlike hypersurfaces of a statistical manifold, *WSEAS Trans Math.*, **22** (2023), 466-474.

[18] Amari, S., *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics, Springer, New York, **28**, 1985.

[19] Amari, S., Barndorff-Nielsen, O.E., Kass, R.E., Lauritzen, S.L. and Rao, C.R., *Differential Geometry in Statistical Inference*, Lecture Notes-Monograph Series, **10**, 1987.

[20] Lauritzen, S.L., *Statistical manifolds*, Differential geometry in statistical inference, **10** (1987), 163–216.

[21] Kang, T.H. and Kim, S.K., Lightlike hypersurfaces of indefinite cosymplectic manifolds, *Int. Math. Forum*, **2**(67) (2007), 3303-3316.

[22] Rani, V. and Kaur, J., Lightlike hypersurfaces of an indefinite statistical manifold, *Adv. Appl. Math. Sci.*, **20**(9) (2021), 1995-2013.

[23] Rani, V. and Kaur, J., On the structure of lightlike hypersurfaces of an indefinite Kaehler statistical manifold, *Differ. Geom.-Dyn. Syst.*, **23** (2021), 221-234.