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A NEW NOTION OF INNER PRODUCT IN A SUBSPACE OF n-NORMED SPACE

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Abstract. Given an n-normed space X for $n \geq 2$, we investigate the completeness of Y (as a subspace of X) with respect to a new norm that correspond to this new inner product on Y. Next, we introduce the angle on a subspace Y of n-normed space X.

Key words and Phrases: n-norm, completness, inner product, angles.

1. INTRODUCTION

An inner product is an important functional in mathematical analysis. On this functional in vector space, we can introduce the orthogonality, the norm, the angle between two subspaces, the n-norm, and the n-inner product (see [8, 10, 16, 19]). Now let X be a real vector space. We recall that an inner product on X is a mapping $\langle \cdot, \cdot \rangle : X^2 \to \mathbb{R}$ such that satisfying

(1) $\langle x, x \rangle \geq 0$ for all $x \in X$; $x = 0$ if and only if $\langle x, x \rangle = 0$;
(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$;
(3) $\langle \gamma x, y \rangle = \gamma \langle x, y \rangle$ for all $x \in X$ and for any scalars $\gamma \in \mathbb{R}$;
(4) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$, for all $x, x', y \in X$.

A pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space. For instance on $\mathbb{R}^n$, we introduce

$$\langle x, y \rangle_{\mathbb{R}^n} := \sum_{j=1}^{n} x_j y_j \tag{1}$$

for every $x, y \in \mathbb{R}^n$. Of course, the functional $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ satisfies 1-4. We can measure "the length" of $x \in \mathbb{R}^n$ using $\|x\|_{\mathbb{R}^n} = \sqrt{\langle x, x \rangle_{\mathbb{R}^n}}$. It is called a norm of x. In

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general, we now recall a norm on X and explain its properties. It is a function $\|\cdot\| : X \to \mathbb{R}$ which satifies

(1) $\|x\| \geq 0$, for all $x \in X$; $\|x\| = 0$ if and only if $x = 0 \in X$;
(2) $\|\gamma x\| = |\gamma|\,\|x\|$, for all $x \in X$ and for all scalar $\gamma \in \mathbb{R}$;
(3) $\|x + x'\| \leq \|x\| + \|x'\|$ for all $x, x' \in X$.

We call that a pair of $(X, \|\cdot\|)$ is a normed space.

A normed space is the inner product space if it satisfies the parallelogram law (see [16]). There have been many researchers' efforts in formulating "the inner product" in normed space. See [15, 20] for a new inner products with a weighted on ℓ^p. Related results inner product and semi-inner product may also be found in [1, 7, 21, 22].
Here we shall formulate a new inner product using n-norm on a real vector space X $(dim(X) \geq n)$. Recall that an *n-norm* on X is a function $\|\cdot, \ldots, \cdot\| : X \times \cdots \times X \longrightarrow \mathbb{R}$ which satisfies the following four properties:

(1) $a_1, \ldots, a_n$ are linearly dependent if and only if $\|a_1, \ldots, a_n\| = 0$;
(2) $\|a_1, \ldots, a_n\|$ invariant under permutation;
(3) $\|\gamma a_1, \ldots,_n\| = |\gamma|\|a_1, \ldots, a_n\|$ for any any $a_1, \ldots, a_n \in X$ and for every $\gamma \in \mathbb{R}$;
(4) $\|a_1, \ldots, a_{n-1}, b + c\| \leq \|a_1, \ldots, a_{n-1}, b\| + \|a_1, \ldots, a_{n-1}, c\|$ for every b, c $a_1,\ldots,a_{n-1} \in X$.

Now we call that the pair $(X, \|\cdot, \ldots, \cdot\|)$ is an *n-normed space*. Usually, the interpretation of $\|x_1, \cdots, x_n\|$ is the volume of the n-dimensional parallelepiped spanned $a_1, ..., a_n \in X$. The development of the theory of 2-normed spaces was started since the late 1960's. Gähler had an idea to generalize an area in a real vector space. The theory of n-normed spaces for $n \geq 2$ was developed in the late 1960's [4, 5, 6]. Recent results can be found, for instance, in [2, 9, 13, 11, 17].

In this article, we will define an inner product on a subspace of $(X, \|\cdot, \cdots, \cdot\|)$. We also discuss the completeness of the subspace that equipped with the inner product. Motivated by this fact, we want to have *a simple* and *good* definition of angle on a subspace of n-normed space.

2. MAIN RESULTS

In this part, we can define an inner product on a subspace of $(X, \|\cdot, \cdots, \cdot\|)$. Next, we also discuss the completeness of the subspaces that equipped with this inner product.

2.1. **A New Inner Product.** Suppose that $(X, \|\cdot, \cdots, \cdot\|)$ is a n-normed space. Now, take a fixed set of linearly independent vectors

$$A = \{f_1, \cdots, f_n\} \subset X. \tag{2}$$

Then we have that

$$Y := \operatorname{span}(A) \tag{3}$$

is a subspace of X. For $g \in Y$, there is $a_g = (a_{1g}, \cdots, a_{ng}) \in \mathbb{R}^n$ such that $g = \sum_{i=1}^{n} a_{ig} f_i$.

On a normed space, there are many notions of orthogonality [3]. Two of them are Pythagorean orthogonality and isosceles orthogonality, as introduced by R.C. James in [14]. Note that if the normed space is the inner product space, then the two types of orthogonality coincide with the definition of orthogonality on the inner product space. Moreover, we think that all the properties about orthogonality in a normed space can be applied to an n-normed space. Now, we give a quadratic formula on $\mathbb{R}$.

Lemma 2.1. *For every $c_1, \cdots, c_n, d_1, \cdots, d_n \in \mathbb{R}$, we have*

$$\sum_{j=1}^{n} (c_j + d_j)^2 - \left(\sum_{j=1}^{n} c_i^2 + \sum_{j=1}^{n} d_i^2 \right) = 2 \sum_{j=1}^{n} c_i d_i.$$

PROOF. We give $c_i, d_i \in \mathbb{R}$ for $j = 1, \cdots, n$. First, we observe that

$$(c_j + d_j)^2 - c_j^2 - d_j^2 = c_j^2 + d_j^2 + 2c_j d_j - c_j^2 - d_j^2 = 2c_j d_j,$$

where $j = 1, \cdots, n$. Hence $\sum_{j=1}^{n} (c_j + d_j)^2 - \left(\sum_{j=1}^{n} c_j^2 + \sum_{j=1}^{n} d_j^2 \right) = 2 \sum_{j=1}^{n} c_j d_j$. $\square$

In Lemma 2.1, if the real numbers above are viewed as vectors $x = (c_1, \ldots, c_n)$, $y = (d_1, \ldots, d_n)$ in $\mathbb{R}^n$ and $\sum_{j=1}^{n} c_j d_j = 0$, then x and y are said to be orthogonal. As a result, the two vectors form a right triangle with hypotenuse $x + y$. One may say that x and y are orthogonal (Pythagorean type). The above lemma will be used to prove the following result.

Proposition 2.2. *Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space, (2) and (3). Then we obtain*

$$\begin{aligned} 2 \left(\sum_{j=1}^{n} a_{jg} a_{jh} \right) \|f_1, \cdots, f_n\|^2 = & \sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 \\ & - \sum_{\{j_2, \ldots, j_n\} \subseteq \{1, \cdots, n\}} \|g, f_{j_2}, \cdots, f_{j_n}\|^2 \\ & - \sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \|h, f_{j_2}, \cdots, f_{j_n}\|^2 \end{aligned}$$

for every $g, h \in Y$.

PROOF. Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space, (2) and (3). We observe that for $\alpha, \beta \in \mathbb{R}$, we have $\|\alpha f_1 + \beta f_j, f_2, \cdots, f_n\| = |\alpha| \|f_1, \cdots, f_n\|$, for every $j = 2, \cdots, n$. Because $Y := \text{span}\{f_1, \cdots, f_n\}$ then for $g \in Y$, there is $a_g = (a_{1g}, \cdots, a_{ng}) \in \mathbb{R}^n$ such that $g = \sum\limits_{i=1}^{n} a_{jg} f_i$. Moreover,

$$\begin{aligned} \|g, f_2, \cdots, f_n\| &= \left\| \sum_{i=1}^{n} a_{ig} f_i, f_2, \cdots, f_n \right\| \\ &= \|a_{1g} f_1, f_2, \cdots, f_n\| \\ &= |a_{1g}| \, \|f_1, f_2, \cdots, f_n\| . \end{aligned}$$

Consequently, for every $g, h \in Y$ we have

$$\sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \|g, f_{j_2}, \cdots, f_{j_n}\|^2 = \sum_{j=1}^{n} a_{jg}^2 \, \|f_1, \cdots, f_n\|^2 , \tag{4}$$

$$\sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \|h, f_{j_2}, \cdots, f_{j_n}\|^2 = \sum_{j=1}^{n} a_{jh}^2 \, \|f_1, \cdots, f_n\|^2 . \tag{5}$$

and

$$\sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 = \sum_{j=1}^{n} (a_{jg} + a_{jh})^2 \, \|f_1, \cdots, f_n\|^2 . \tag{6}$$

Using (4), (5), (6) and by Lemma 2.1,

$$\begin{aligned} &\sum_{\{j_2, \cdots, j_n\} \subseteq \{1, \cdots, n\}} \left(\|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 - \|g, f_{j_2}, \cdots, f_{j_n}\|^2 - \|h, f_{j_2}, \cdots, f_{j_n}\|^2 \right) \\ &\qquad = 2 \left(\sum_{j=1}^{n} a_{jg} a_{jh} \right) \|f_1, \cdots, f_n\|^2 \end{aligned}$$

holds. The proof is complete. $\square$

Another quadratic formula for real numbers is also presented in the following lemma.

Lemma 2.3. *For every $c_1, \cdots, c_n, d_1, \cdots, d_n \in \mathbb{R}$, we obtain*

$$\sum_{j=1}^{n} \left((c_j + d_j)^2 - (c_j - d_j)^2 \right) = 4 \sum_{j=1}^{n} c_j d_j .$$

PROOF. Suppose that $c_j, d_j \in \mathbb{R}$ for $j = 1, \cdots, n$. We know that

$$(c_j + d_j)^2 - (c_j - d_j)^2 = (c_j^2 + d_j^2 + 2c_j d_j) - (c_j^2 + d_j^2 - 2c_j d_j) = 2c_j d_j,$$

$j = 1, \cdots, n$. Consequently, $\sum\limits_{j=1}^{n} \left((c_j + d_j)^2 - (c_j - d_j)^2 \right) = 4 \sum\limits_{j=1}^{n} c_j d_j$. $\square$

Furthermore, like Lemma 2.1, the real numbers in the Lemma 2.3 can also be seen as two vectors $x = (c_1, \ldots, c_n)$ and $y = (d_1, \ldots, d_n)$ in $\mathbb{R}^n$. For $\sum_{j=1}^{n} c_j d_j = 0$, we have

$$\sum_{j=1}^{n}(c_j + d_j)^2 = \sum_{j=1}^{n}(c_j - d_j)^2$$

or we write that x and y are orthogonal isoceles. Next, in an n-normed space, we use Lemma 2.3 to explain the following proposition.

Proposition 2.4. *If $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space, (2) and (3), then*

$$\begin{aligned} 4\left(\sum_{j=1}^{n} a_{jg}a_{ih}\right)\|f_1, \cdots, f_n\|^2 = & \sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 \\ & - \sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \|g - h, f_{j_2}, \cdots, f_{j_n}\|^2 \end{aligned}$$

for any $g, h \in Y$.

PROOF. By the above assumptions and the property

$$\|\alpha f_1 + \beta f_j, f_2, \cdots, f_n\| = |\alpha|\|f_1, \cdots, f_n\|,$$

with $\alpha, \beta \in \mathbb{R}$ and for any $j = 1, \cdots, n$, we obtain that

$$\|g, f_2, \cdots, f_n\| = |a_{1g}|\,\|f_1, f_2, \cdots, f_n\|$$

holds for every $g \in Y$. Now check that

$$\sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 = \sum_{j=1}^{n}(a_{jg} + a_{jh})^2 \|f_1, \cdots, f_n\|^2, \quad (7)$$

for every $g, h \in Y$, and

$$\sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \|g - h, f_{j_2}, \cdots, f_{j_n}\|^2 = \sum_{j=1}^{n}(a_{jg} - a_{jh})^2 \|f_1, \cdots, f_n\|^2. \quad (8)$$

for every $g, h \in Y$. Next, we use (7), (8), and Lemma 2.3 to obtain

$$\begin{aligned} & \sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \left(\|g + h, f_{j_2}, \cdots, f_{j_n}\|^2 - \|g - h, f_{j_2}, \cdots, f_{j_n}\|^2\right) \\ & \qquad = 4\left(\sum_{j=1}^{n} a_{jg}a_{jh}\right)\|f_1, \cdots, f_n\|^2 \end{aligned}$$

for any $g, h \in Y$. □

Now, recall (2) and (3) to define

$$\begin{aligned}\langle g, h\rangle_A &:= \frac{1}{4} \sum_{\{j_2,\cdots,j_n\}\subseteq\{1,\cdots,n\}} \left(\|g+h, f_{j_2}, \cdots, f_{j_n}\|^2 - \|g-h, f_{j_2}, \ldots, f_{j_n}\|^2 \right) \\ &= \left(\sum_{j=1}^{n} a_{jg} a_{jh} \right) \|f_1, \cdots, f_n\|^2 . \end{aligned} \tag{9}$$

for every $g, h \in Y$. The above functional is defined using Proposition 2.4. It is also similar to **polarization law**, while we want to check a norm (derived by an inner product or not). Another way, one may use Proposition 2.2 to obtain (9). Now, note that the coefficients of the linear combination of the vectors in Y can be viewed as an element of $\mathbb{R}^n$. For example, the coefficients of $g \in Y$ can be seen as $a_g = (a_{1g}, \cdots, a_{ng}) \in \mathbb{R}^n$. Thus, the relation between (1) and (9) is expressed as follows

$$\langle g, h\rangle_A = \langle a_g, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 , \tag{10}$$

for every $g, h \in Y$. Furthermore using (1), we shall prove that (10) is an inner product as follows.

Theorem 2.5. *If $(X, \|\cdot, \cdots, \cdot\|)$ is a n-normed space, (2) and (3), then the mapping $\langle \cdot, \cdot\rangle_A$ is an inner product on Y.*

Proof. Using all of the assumptions of this proposition, $\langle \cdot, \cdot\rangle_A$ will be checked that it satisfies the properties of inner product.

(1) (**non negative**) Take $g \in Y$. We have $\langle a_g, a_g\rangle_{\mathbb{R}^n} \geq 0$ and $\|f_1, \cdots, f_n\| > 0$, so we obtain $\langle g, g\rangle_A = \langle a_g, a_g\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 \geq 0$.

($\Rightarrow$) We give $\langle g, g\rangle_A = \langle a_g, a_g\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 = 0$. Since $\|f_1, \cdots, f_n\| > 0$, then

$$\langle a_g, a_g\rangle_{\mathbb{R}^n} = \left(\sum_{i=1}^{n} a_{ig}^2 \right) = 0.$$

Consequently, we obtain $a_{jg} = 0$ for every $j = 1, \cdots, n$. So, $g = \sum_{j=1}^{n} a_{jg} f_j = 0$ holds.

($\Leftarrow$) Suppose that $g = 0$. Since $a_g = 0 \in \mathbb{R}^n$, then $\langle a_g, a_g\rangle_{\mathbb{R}^n} = 0$ holds. We obtain $\langle g, g\rangle_A = 0$.

(2) (**commutative**) Take $g, h \in Y$. Check that

$$\langle g, h\rangle_A = \langle a_g, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 = \langle a_h, a_g\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 = \langle h, g\rangle_A .$$

(3) (**homogen**) Take $g, h \in Y$ and $\alpha \in \mathbb{R}$. We have

$$\begin{aligned}\langle \alpha g, h\rangle_A &= \langle \alpha a_g, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 \\ &= \alpha \langle a_g, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 \\ &= \alpha \langle g, h\rangle_A .\end{aligned}$$

(4) (**distributive**) Take $g_1, g_2, h \in Y$. We observe that

$$\begin{aligned}\langle g_1 + g_2, h\rangle_A &= \langle a_{g_1} + a_{g_2}, a_h\rangle_{\mathbb{R}^n} \|f_1, \dots, f_n\|^2 \\ &= \langle a_{g_1}, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 + \langle a_{g_2}, a_h\rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2 \\ &= \langle g_1, h\rangle_A + \langle g_2, h\rangle_A .\end{aligned}$$

Hence, $\langle \cdot, \cdot\rangle_A$ is an inner product on Y. □

We also give a norm that is inducted by the inner product on Y.

Corollary 2.6. *Lets $g \in Y$. The following*

$$\|g\|_A := \sqrt{\langle a_g, a_g\rangle_{\mathbb{R}^n}} \|f_1, \cdots, f_n\| = \|a_g\|_{\mathbb{R}^n} \|f_1, \cdots, f_n\| \tag{11}$$

defines the norm on Y.

Because $Y = \text{span}\ (A)$ with $A = \{f_1, \cdots, f_n\}$ then we have the following corollary.

Corollary 2.7. *In $(Y, \langle \cdot, \cdot\rangle_A)$, set $A = \{f_1, \cdots, f_n\}$ is an orthogonal set respect to $\langle \cdot, \cdot\rangle_A$.*

2.2. **The Completeness of subspaces of n-Normed Space.** We have that $(Y, \langle \cdot, \cdot\rangle_A)$ is an inner product space. Using (10), all of the properties of $(\mathbb{R}^n, \langle \cdot, \cdot\rangle_{\mathbb{R}^n})$ can be delivered to $(Y, \langle \cdot, \cdot\rangle_A)$. Note that each norm definition in $\mathbb{R}^n$ is equivalent. Meanwhile, $\mathbb{R}^n$ as a normed space is a complete space. The completness of Y of n-normed Space X as follows.

Theorem 2.8. *The subspace $(Y, \|\cdot\|_A)$ is a complete space.*

PROOF. Let (w_k) be a Cauchy sequence in Y. Hence,

$$w_k = a_{1w_k} f_1 + \cdots + a_{nw_k} f_n.$$

It means that for any $\epsilon > 0$, there is an $n'_\epsilon \in \mathbb{N}$ such that for every $k, l > n'_\epsilon$, we have

$$\|w_k - w_l\|_A = \|a_{w_k} - a_{w_l}\|_{\mathbb{R}^n} \|f_1, \cdots, f_n\| < \epsilon.$$

As consequence, we obtain

$$\|a_{w_k} - a_{w_l}\|_{\mathbb{R}^n} < \frac{\epsilon}{\|f_1, \cdots, f_n\|} = \epsilon'.$$

We say that $a_{w_k} \in \mathbb{R}^n$ is also a Cauchy sequence. Because $\mathbb{R}^n$ is a complete space respect to $\|\cdot\|_{\mathbb{R}^n}$, then $a_{w_k} \in \mathbb{R}^n$ is a convergence sequence. Clearly, (w_k) in Y is also a convergence sequence. Hence, $(Y, \|\cdot\|_A)$ is a complete space. □

By Theorem 2.8, we also conclude that $(Y, \|\cdot\|_A)$ is a Banach space. Moreover, here $\|\cdot\|_A$ is induced by $\langle \cdot, \cdot\rangle_A$, so $(Y, \langle \cdot, \cdot\rangle_A)$ is a Hilbert space. On $(Y, \langle \cdot, \cdot\rangle_A)$, several functionals can be defined (see some books of functional analysis). In particular, to define the m-inner product, we have to use $m < n$.

Here, we note that $Y = \text{span }(A)$ and an orthogonal set $A = \{f_1, \ldots, f_n\}$ on $(X, \|\cdot, \ldots, \cdot\|)$ respect to $\langle \cdot, \cdot \rangle_A$ (see Corollary 2.7). Now, we give a linearly independent set $B = \{g_1, \ldots, g_n\}$ on Y. One may form

$$g_1 = \sum_{i=1}^{n} a_{i\,g_1} f_i$$

$$g_2 = \sum_{i=1}^{n} a_{i\,g_2} f_i$$

$$\vdots$$

$$g_n = \sum_{i=1}^{n} a_{i\,g_n} f_i$$

or $\mathcal{B} = K\mathcal{A}$, where $\mathcal{B} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$, $K = \begin{bmatrix} a_{1\,g_1} & a_{2\,g_1} & \cdots & a_{n\,g_1} \\ a_{1\,g_2} & a_{2\,g_2} & \cdots & a_{n\,g_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\,g_n} & a_{2\,g_n} & \cdots & a_{n\,g_n} \end{bmatrix}$ and $\mathcal{A} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$.

It is obvious that K is an invertible matrix, so $K^{-1}\mathcal{B} = \mathcal{A}$ holds. It means that set A can also be developed by set B. With an initial $\bar{g_1} = \frac{g_1}{\|g_1\|_A}$, we obtain an orthonormal set $\bar{B} = \{\bar{g_1}, \bar{g_2}, \cdots, \bar{g_n}\}$ by using the Gram-Schmidt process respect to $\langle \cdot, \cdot \rangle_A$. Consequently,

$$Y = \text{span }(A) = \text{span }(B) = \text{span }(\bar{B}).$$

3. FURTHER RESULTS

Here, we recall $(Y, \langle \cdot, \cdot \rangle_A)$. For any $p, q \in Y$, we write $p = \sum_{j=1}^{n} a_{j\,p} f_j$ and $q = \sum_{j=1}^{n} a_{j\,q} f_j$ where $a_p = (a_{1\,p}, \cdots, a_{n\,p}), a_q = (a_{1\,q}, \cdots, a_{n\,q}) \in \mathbb{R}^n$. Next, we have the angle between p and q $(\phi(p,q))$ of $(Y, \langle \cdot, \cdot \rangle_A)$ as follows.

$$\cos^2 \phi(p,q) = \frac{\langle p, q \rangle_A^2}{\|p\|_A^2 \, \|q\|_A^2} = \frac{\langle a_p, a_q \rangle_{\mathbb{R}^n}^2 \, \|f_1, \cdots, f_n\|^4}{\|a_p\|_{\mathbb{R}^n}^2 \, \|a_q\|_{\mathbb{R}^n}^2 \, \|f_1, \cdots, f_n\|^4} = \frac{\langle a_p, a_q \rangle_{\mathbb{R}^n}^2}{\|a_p\|_{\mathbb{R}^n}^2 \, \|a_q\|_{\mathbb{R}^n}^2}. \quad (12)$$

Hence, the angle ϕ between two vectors p and q in $(Y, \langle \cdot, \cdot \rangle_A)$ is equivalen to the angle between two vectors a_p and a_q in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$.

Moreover, we can formulate angle ϕ between two subspaces on Y. The result is shown as follows.

See in [12] and now let $P = \text{span}\{p_1, \cdots, p_{n_1}\}$ and $Q = \text{span}\{q_1, \cdots, q_{n_2}\}$ be subspaces of Y, with $1 \leq n_1 \leq n_2$ and $\{p_1, \cdots, p_{n_1}\}$, $\{q_1, \cdots, q_{n_2}\}$ are orthogonal sets. Now, we have the angle ϕ between P dan Q

$$\cos^2 \phi(P, Q) = \det(M^T M) \tag{13}$$

where

$$M^T = \begin{bmatrix} \frac{\langle p_1, q_1 \rangle_A}{\|p_1\|_A \|q_1\|_A} & \frac{\langle p_1, q_2 \rangle_A}{\|p_1\|_A \|q_2\|_A} & \cdots & \frac{\langle p_1, q_{n_2} \rangle_A}{\|p_1\|_A \|q_{n_2}\|_A} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\langle p_{n_1}, q_1 \rangle_A}{\|p_{n_1}\|_A \|q_1\|_A} & \frac{\langle p_{n_1}, q_2 \rangle_A}{\|p_{n_1}\|_A \|q_2\|_A} & \cdots & \frac{\langle p_{n_1}, q_{n_2} \rangle_A}{\|p_{n_1}\|_A \|q_{n_2}\|_A} \end{bmatrix}.$$

Since $\frac{\langle p_j, q_k \rangle_A}{\|p_j\|_A \|q_k\|_A} = \frac{\langle a_{p_j}, a_{q_k} \rangle_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2}{\|a_{p_j}\|_{\mathbb{R}^n} \|a_{q_k}\|_{\mathbb{R}^n} \|f_1, \cdots, f_n\|^2} = \frac{\langle a_{p_j}, a_{q_k} \rangle_{\mathbb{R}^n}}{\|a_{p_j}\|_{\mathbb{R}^n} \|a_{q_k}\|_{\mathbb{R}^n}}$ where $j : 1, \cdots, n_1$ and $k : 1, \cdots, n_2$, then

$$M^T = \begin{bmatrix} \frac{\langle a_{p_1}, a_{q_1} \rangle_{\mathbb{R}^n}}{\|a_{p_1}\|_{\mathbb{R}^n} \|a_{q_1}\|_{\mathbb{R}^n}} & \frac{\langle a_{p_1}, a_{q_2} \rangle_{\mathbb{R}^n}}{\|a_{p_1}\|_{\mathbb{R}^n} \|a_{q_2}\|_{\mathbb{R}^n}} & \cdots & \frac{\langle a_{p_1}, a_{q_{n_2}} \rangle_{\mathbb{R}^n}}{\|a_{p_1}\|_{\mathbb{R}^n} \|a_{q_{n_2}}\|_{\mathbb{R}^n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\langle a_{p_{n_1}}, a_{q_1} \rangle_{\mathbb{R}^n}}{\|a_{p_{n_1}}\|_{\mathbb{R}^n} \|a_{q_1}\|_{\mathbb{R}^n}} & \frac{\langle a_{p_{n_1}}, a_{q_2} \rangle_{\mathbb{R}^n}}{\|a_{p_{n_1}}\|_{\mathbb{R}^n} \|a_{q_2}\|_{\mathbb{R}^n}} & \cdots & \frac{\langle a_{p_{n_1}}, a_{q_{n_2}} \rangle_{\mathbb{R}^n}}{\|a_{p_{n_1}}\|_{\mathbb{R}^n} \|a_{q_{n_2}}\|_{\mathbb{R}^n}} \end{bmatrix}.$$

We know that (12) is a spesial case of (13).

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