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OPERATIONS AND SIMILARITY MEASURES BETWEEN (m, n)-FUZZY SETS

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Abstract. Recently Jun and Hur proposed (m, n)-fuzzy sets which can handle vagueness and uncertainty in information very efficiently in the process of solving complex problems. They defined basic operations over (m, n)-fuzzy sets. The present paper created some new operations over this super class of fuzzy sets and established many theorems related to the their properties. Further some distance and similarity measures of (m, n)-fuzzy sets are proposed and their properties are examined. Moreover, the proposed similarity measures are applied to the problem of pattern recognition.

Key words and Phrases: (m, n)-fuzzy sets, operations and similarity measure of (m, n)-fuzzy sets.

1. INTRODUCTION

The fusion of technology and generalized forms of classical sets is very useful to solve many real world complex problems which involve the vague and uncertain information. A classical set is defined by its characteristic function from universe of discourse to two point set {0,1}. Classical set theory is insufficient to handle the complex problems involving vague and uncertain information. To handle the vagueness and uncertainty of complex problems, Zadeh [19] in 1965, created fuzzy sets (FSs) as a generalization of classical sets which characterised by membership function from universe of discourse to closed interval [0,1]. FS theory is applicable in various areas such as control theory, artificial intelligence, pattern recognition,

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database system and medical diagnosis. Atanassov [2] created a super class of FSs called intuitionistic fuzzy sets (IFS). IFSs are widely used in the many fields of mathematics, computer science, management and medical sciences. Szmidt [15], Szmidt and Kacprzyk [16], and Wang and Xin [17] proposed various distances and similarity measures between IFS and studies applications of distance and similarity measures in pattern recognition and medical diagnosis. After the occurrence of Atanassov [2] paper, several generalizations of IFSs have been appeared in the literature. In the year 2013, Yager [18] presented a super class of IFSs called Pythagorean fuzzy set (PFS). Husain and Yang [5] and Zeng, Li and Yin [20] are initially contributed to similarity measures of PFSs. In 2019, Senapati and Yager [13] created Fermatean fuzzy sets (FFS) and defined basic operations over FFSs. In another paper Senapati and Yager [14] proposed some more operations over FFS and develped a FFWPM to solve MCDM problems. Recently Sahoo [12], Krisci [9], and Ejegwa and Onyeke [4] proposed various similarity measures for FFSs and studied their applications in MADM and MCDM problems. In 2020, a new notion called n-Pythagorean fuzzy sets (n-PFS) was created by Bryniarska [3] as a super class of FFSs and studied Yeger's aggregation operations for n-PFSs. The distance and similarity measures on n-PFS and their applications in MCDM problems are studied by Liu, Chen and Peng [10] and Peng and Liu [11]. Ibrahim and his coworkers [6, 7] initiated the study of (3,2)-Fuzzy sets and (3,4)-Fuzzy sets and proposed topological structures using these super classes of IFSs. The notion of (2,1)-fuzzy sets was created by AI-Shami [1] and presented their applications to MCDM methods. Recently Jun and hur[8] created the class of (m, n)-fuzzy sets ((m, n)-FSs) as a super class of n-PFSs. They defined some operations and properties on (m, n)-FSs and presented its applications in BCK-algebra.

The motivation of writing this research are first to redefined the complement of (m, n)-FSs which overcome the drawback of complement defined by Jun and Hur [8] in which "the complement of (m, n)-FS is not an (m, n)-FS but (n, m)-FS". Second, proposed some new operations over (m, n)-FSs which are not considered by Jun and Hur [8]. Third, to extend some distance and similarity measures of PFSs, FFSs to (m, n)-FSs and examine their applications in pattern recognition. Now the organization of paper is as follows. The second section of this paper reviews the notion of (m, n)-FSs and its relations to other generalized classes of IFSs. Third section proposed some new operations over (m, n)-FSs presents their properties. Section four proposed some distances and similarity measures over (m, n)-FSs and showed their validity using numerical examples. Section five gives an applications of similarity of (m, n)-FSs in pattern recognition.

2. PPRELIMINARIES

Throughout this paper $\mathbb{P}$ be a universe of discourse, $\mathbb{N}$ referred the set of all natural numbers and $m, n \in \mathbb{N}$.

Definition 2.1. *A structure* $\mathcal{M} = \{< p, \varrho_{\mathcal{M}}(p), \sigma_{\mathcal{M}}(p) >: p \in \mathbb{P}\}$ *where,* $\varrho_{\mathcal{M}} : \mathbb{P} \to [0,1]$ *and* $\sigma_{\mathcal{M}} : \mathbb{P} \to [0,1]$ *denotes the degree of membership and the degree of nonmembership of each* $p \in \mathbb{P}$ *to* $\mathcal{M}$ *is called:*

(a) Intuitionistic fuzzy set [2] *in* $\mathbb{P}$ *if* $0 \leq \varrho_{\mathcal{M}}(p) + \sigma_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(b) (2,1)-fuzzy set [1] *in* $\mathbb{P}$ *if* $0 \leq \varrho^2_{\mathcal{M}}(p) + \sigma^1_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(c) Pythagorean fuzzy set [18] *in* $\mathbb{P}$ *if* $0 \leq \varrho^2_{\mathcal{M}}(p) + \sigma^2_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(d) Fermatean fuzzy set [13] *in* $\mathbb{P}$ *if* $0 \leq \varrho^3_{\mathcal{M}}(p) + \sigma^3_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(e) (3,2)-fuzzy set [6] *in* $\mathbb{P}$ *if* $0 \leq \varrho^3_{\mathcal{M}}(p) + \sigma^2_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(f) (3,4)-fuzzy set [7] *in* $\mathbb{P}$ *if* $0 \leq \varrho^3_{\mathcal{M}}(p) + \sigma^4_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(g) n-Pythagorean fuzzy set [3] *where* $n \in \mathbb{N}$ *in* $\mathbb{P}$ *if* $0 \leq \varrho^n_{\mathcal{M}}(p) + \sigma^n_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.
(h) (m,n)*- fuzzy set* [8] *where* $m, n \in \mathbb{N}$ *in* $\mathbb{P}$ *if* $0 \leq \varrho^m_{\mathcal{M}}(p) + \sigma^n_{\mathcal{M}}(p) \leq 1$, $\forall p \in \mathbb{P}$.

Remark 2.2. [2] *Every fuzzy set* $\mathcal{M}$ *over* $\mathbb{P}$ *with membership function* $\varrho_{\mathcal{M}}$ *will be considered as an intuitionistic fuzzy set* $\mathcal{M} = \{< p, \varrho_{\mathcal{M}}(p), 1 - \varrho_{\mathcal{M}}(p) >: p \in \mathbb{P}\}$.

Remark 2.3. *Since* $a + b \leq 1 \Rightarrow a^2 + b \leq 1 \Rightarrow a^2 + b^2 \leq 1 \Rightarrow a^3 + b^2 \leq 1 \Rightarrow a^3 + b^3 \leq 1 \Rightarrow a^3 + b^4 \leq 1 \Rightarrow a^n + b^n \leq 1 \Rightarrow a^m + b^n \leq 1, \forall a, b \in [0,1]$ *and* $m, n \geq 4$, *from Definition 2.1 we obtain the following diagram of implications for the above generalizations of FSs:*

$$
\begin{array}{c}
\textbf{IFS} \\
\Downarrow \\
\textbf{(2,1)-FS} \\
\Downarrow \\
\textbf{PFS} \\
\Downarrow \\
\textbf{(3,2)-FS} \\
\Downarrow \\
\textbf{FFS} \\
\Downarrow \\
\textbf{(3,4)-FS} \\
\Downarrow \\
n\textbf{-PFS}(n \geq \mathbf{4}) \\
\Downarrow \\
(m,n)\textbf{-FS}(m,n \geq \mathbf{4}).
\end{array}
$$

The following example shows that the reverse implications are not true.

Example 2.4. *Let $\mathbb{P} = \{p_1, p_2\}$ and consider the following structures defined over $\mathbb{P}$:*

$$\mathcal{M}_1 = \{< p_1, 0.6, 0.5 >, < p_2, 0.4, 0.7 >\}.$$
$$\mathcal{M}_2 = \{< p_1, 0.6, 0.7 >, < p_2, 0.8, 0.5 >\}.$$
$$\mathcal{M}_3 = \{< p_1, 0.85, 0.6 >, < p_2, 0.9, 0.5 >\}.$$
$$\mathcal{M}_4 = \{< p_1, 0.8, 0.7 >, < p_2, 0.8, 0.75 >\}.$$
$$\mathcal{M}_5 = \{< p_1, 0.9, 0.7 >, < p_2, 0.9, 0.7 >\}.$$
$$\mathcal{M}_6 = \{< p_1, 0.85, 0.8 >, < p_2, 0.85, 0.8 >\}.$$
$$\mathcal{M}_7 = \{< p_1, 0.9, 0.85 >, < p_2, 0.85, 0.9 >\}.$$

Then

(a) $\mathcal{M}_1$ *is (2,1)-FS but not IFS.*
(b) $\mathcal{M}_2$ *is PFS but not (2,1)-FS.*
(c) $\mathcal{M}_3$ *is (3,2)-FS but not PFS.*
(d) $\mathcal{M}_4$ *is FFS but not (3,2)-FS.*
(e) $\mathcal{M}_5$ *is (3,4)-FS but not FFS.*
(f) $\mathcal{M}_6$ *is (4,4)-FS but not (3,4)-FS.*
(g) $\mathcal{M}_7$ *is (6,5)-FS but not (5,5)-FS.*

Definition 2.5. [8] *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}), \mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$. Then the basic operations over $\mathcal{F}^n_m(\mathbb{P})$ are defined as follows:*

(a) $\mathcal{M}_1 \Subset \mathcal{M}_2 \Leftrightarrow \varrho_{\mathcal{M}_1} \leq \varrho_{\mathcal{M}_2}$ *and* $\sigma_{\mathcal{M}_1} \geq \sigma_{\mathcal{M}_2}$.
(b) $\mathcal{M}_1 = \mathcal{M}_2 \Leftrightarrow \varrho_{\mathcal{M}_1} = \varrho_{\mathcal{M}_2}$ *and* $\sigma_{\mathcal{M}_1} = \sigma_{\mathcal{M}_2}$.
(c) $\mathcal{M}_1 \Cup \mathcal{M}_2 = (\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\}, \min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})$.
(d) $\mathcal{M}_1 \Cap \mathcal{M}_2 = (\min\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\}, \max\{\sigma_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}\})$.
(e) $\mathcal{M}^c = (\sigma_{\mathcal{M}}, \varrho_{\mathcal{M}})$.

Theorem 2.6. [8] *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1})$, $\mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2})$ and $\mathcal{M}_3 = (\varrho_{\mathcal{M}_3}, \sigma_{\mathcal{M}_3})$ be three (m, n)-FSs on $\mathbb{P}$. Then:*

(a) $\mathcal{M}_1 \Cup \mathcal{M}_2 = \mathcal{M}_2 \Cup \mathcal{M}_1$.
(b) $\mathcal{M}_1 \Cap \mathcal{M}_2 = \mathcal{M}_2 \Cap \mathcal{M}_1$.
(c) $(\mathcal{M}_1 \Cap \mathcal{M}_2) \Cap \mathcal{M}_3 = \mathcal{M}_1 \Cap (\mathcal{M}_2 \Cap \mathcal{M}_3)$.
(d) $(\mathcal{M}_1 \Cup \mathcal{M}_2) \Cup \mathcal{M}_3 = \mathcal{M}_1 \Cup (\mathcal{M}_2 \Cup \mathcal{M}_3)$.
(e) $(\mathcal{M}_1^c)^c = \mathcal{M}_1$.
(f) $(\mathcal{M}_1 \Cup \mathcal{M}_2)^c = \mathcal{M}_1^c \Cap \mathcal{M}_2^c$.
(g) $(\mathcal{M}_1 \Cap \mathcal{M}_2)^c = \mathcal{M}_1^c \Cup \mathcal{M}_2^c$.

Table 1: Notations and their descriptions

Notation	Description
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
(2,1)-FS	(2,1)-fuzzy set
PFS	Pythagorean fuzzy set
(3,2)-FS	(3,2)-fuzzy set
FFS	Fermatean fuzzy set
(3,4)-FS	(3,4)-fuzzy set
n-FS	n-Pythagorean fuzzy set
(m,n)-FS	(m,n)-fuzzy set
$\varrho_{\mathcal{M}}(p)$	Degree of membership of p in $\mathcal{M}$
$\sigma_{\mathcal{M}}(p)$	Degree of non-membership of p in $\mathcal{M}$
$\mathcal{F}_m^n(\mathbb{P})$	The family of all (m,n)-FSs defined over $\mathbb{P}$
$(\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}})$	(m,n)-FS $\mathcal{M} = \{< p, \varrho_{\mathcal{M}}(p), \sigma_{\mathcal{M}}(p) >: p \in \mathbb{P}\}$
$\mathcal{M}^c$	Complement of $\mathcal{M}$ in the sense of Jun and Hur [8]
$\complement\mathcal{M}$	Complement of $\mathcal{M}$ proposed in this paper
$\pi_{\mathcal{M}}(p)$	Degree of indeterminacy of p to $\mathcal{M}$
$\Box\mathcal{M}$	Necessity measure of $\mathcal{M}$
$\Diamond\mathcal{M}$	Possibility measure of $\mathcal{M}$

3. SOME NEW OPERATIONS OVER (m,n)-FUZZY SETS

Definition 3.1. *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}_m^n(\mathbb{P})$ and $p \in \mathbb{P}$. Then the expression $\pi_{\mathcal{M}}(p) = (1 - \varrho_{\mathcal{M}}^m(p) - \sigma_{\mathcal{M}}^n(p))^{\frac{2}{m+n}}$ is called the degree of indeterminacy of p to $\mathcal{M}$.*

Remark 3.2. *Clearly, $\pi_{\mathcal{M}}^{\frac{m+n}{2}}(p) + \varrho_{\mathcal{M}}^m(p) + \sigma_{\mathcal{M}}^n(p) = 1, \forall p \in \mathbb{P}$.*

Remark 3.3. *The degree of indeterminacy of $p \in \mathbb{P}$ to n-PFS (resp. FFS, PFS, IFS, (2,1)-FS, (3,2)-FS, (3,4)-FS) $\mathcal{M}$ is a special case of degree of indeterminacy of p to (m,n)-FS $\mathcal{M}$ for $m = n$ (resp. $m = n = 3; m = n = 2; m = n = 1; m = 2, n = 1; m = 3, n = 2; m = 3, n = 4$).*

Jun and Hur [8] pointed out that the complement of an (m,n)-FS is (n,m)-FS but not (m,n)-FS. We modified the definition of complement of (m,n)-FS as follows:

Definition 3.4. *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}_m^n(\mathbb{P})$. Then the complement of $\mathcal{M}$ denoted by $\complement\mathcal{M}$ is defined as follows:*

$$\complement\mathcal{M} = (\varrho_{\complement\mathcal{M}}, \sigma_{\complement\mathcal{M}}) = (\sigma_{\mathcal{M}}^{\frac{n}{m}}, \varrho_{\mathcal{M}}^{\frac{m}{n}}).$$

Theorem 3.5. *Let* $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}), \mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$. *Then:*

(i) $\complement\mathcal{M} \in \mathcal{F}^n_m(\mathbb{P})$.
(ii) $\complement\complement\mathcal{M} = \mathcal{M}$.
(iii) $\complement(\mathcal{M}_1 \Cup \mathcal{M}_2) = \complement\mathcal{M}_1 \Cap \complement\mathcal{M}_2$.
(iv) $\complement(\mathcal{M}_1 \Cap \mathcal{M}_2) = \complement\mathcal{M}_1 \Cup \complement\mathcal{M}_2$.

Proof. (i) Since,

$$\begin{aligned}\complement\mathcal{M} =&(\varrho_{\complement\mathcal{M}}, \sigma_{\complement\mathcal{M}})\\ =&(\sigma^{\frac{n}{m}}_{\mathcal{M}}, \varrho^{\frac{m}{n}}_{\mathcal{M}}),\end{aligned}$$

we have,

$$\begin{aligned}0 \leq \varrho^m_{\complement\mathcal{M}} + \sigma^n_{\complement\mathcal{M}} =&(\sigma^{\frac{n}{m}}_{\mathcal{M}})^m + (\varrho^{\frac{m}{n}}_{\mathcal{M}})^n\\ =&\sigma^n_{\mathcal{M}} + \varrho^m_{\mathcal{M}}\\ =&\varrho^m_{\mathcal{M}} + \sigma^n_{\mathcal{M}} \leq 1.\end{aligned}$$

Hence $\complement\mathcal{M} \in \mathcal{F}^n_m(\mathbb{P})$.

(ii) Easy and left to the readers.

(iii) We have,

$$\begin{aligned}\complement(\mathcal{M}_1 \Cup \mathcal{M}_2) =&\complement(\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\}, \min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})\\ =&(\min\{\sigma^{\frac{n}{m}}_{\mathcal{M}_1}, \sigma^{\frac{n}{m}}_{\mathcal{M}_2}\}, \max\{\varrho^{\frac{m}{n}}_{\mathcal{M}_1}, \varrho^{\frac{m}{n}}_{\mathcal{M}_2}\})\\ =&(\sigma^{\frac{n}{m}}_{\mathcal{M}_1}, \varrho^{\frac{m}{n}}_{\mathcal{M}_1}) \Cap (\sigma^{\frac{n}{m}}_{\mathcal{M}_2}, \varrho^{\frac{m}{n}}_{\mathcal{M}_2})\\ =&\complement\mathcal{M}_1 \Cap \complement\mathcal{M}_2.\end{aligned}$$

(iv) Follows on the similar process to (iii).

□

Definition 3.6. *Let* $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}^n_m(\mathbb{P})$. *Then the necessity and the possibility measures on* $\mathcal{M}$ *are defined as follows:*

(i) $\Box\mathcal{M} = (\varrho_{\mathcal{M}}, (1 - \varrho^m_{\mathcal{M}})^{\frac{1}{n}})$.
(ii) $\Diamond\mathcal{M} = ((1 - \sigma^n_{\mathcal{M}})^{\frac{1}{m}}, \sigma_{\mathcal{M}})$.

Example 3.7. *Let* $\mathbb{P} = \{p\}$ *and* $\mathcal{M} = \{< p, 0.85, 0.9 >\} \in \mathcal{F}^5_6(\mathbb{P})$. *Then:*

$$\pi_{\mathcal{M}}(p) = 0.5359121938.$$
$$\Box\mathcal{M} = \{< p, 0.85, 0.9096551054 >\}.$$
$$\Diamond\mathcal{M} = \{< p, 0.8593159181, 0.9 >\}.$$
$$\complement\mathcal{M} = \{< p, 0.9159436536, 0.8228159673 >\}.$$

Theorem 3.8. *If* $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}^n_m(\mathbb{P})$, *then*

(i) $\square\mathcal{M} \in \mathcal{F}_m^n(\mathbb{P})$.
(ii) $\diamond\mathcal{M} \in \mathcal{F}_m^n(\mathbb{P})$.

Proof. (i) Follows on noting that:

$$\begin{aligned}\varrho^m_{\square\mathcal{M}} + \sigma^n_{\square\mathcal{M}} =& \varrho^m_{\mathcal{M}} + ((1-\varrho^m_{\mathcal{M}})^{\frac{1}{n}})^n \\ =& \varrho^m_{\mathcal{M}} + (1-\varrho^m_{\mathcal{M}}) \\ =& 1.\end{aligned}$$

(ii) Follows on noting that:

$$\begin{aligned}\varrho^m_{\diamond\mathcal{M}} + \sigma^n_{\diamond\mathcal{M}} =& ((1-\sigma^n_{\mathcal{M}})^{\frac{1}{m}})^m + \sigma^n_{\mathcal{M}} \\ =& (1-\sigma^n_{\mathcal{M}}) + \sigma^n_{\mathcal{M}} \\ =& 1.\end{aligned}$$

□

Definition 3.9. *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}), \mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\kappa \in \mathbb{N}$. Then the operations, $\mathcal{M}_1 \oplus \mathcal{M}_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \kappa\mathcal{M}$ and $\mathcal{M}^\kappa$ are defined as follows:*

(i) $\mathcal{M}_1 \oplus \mathcal{M}_2 = (\varrho^m_{\mathcal{M}_1} + \varrho^m_{\mathcal{M}_2} - \varrho^m_{\mathcal{M}_1}\varrho^m_{\mathcal{M}_2},\ \sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})$.
(ii) $\mathcal{M}_1 \otimes \mathcal{M}_2 = (\varrho^m_{\mathcal{M}_1}\varrho^m_{\mathcal{M}_2},\ \sigma^n_{\mathcal{M}_1} + \sigma^n_{\mathcal{M}_2} - \sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})$.
(iii) $\kappa\mathcal{M} = (1-(1-\varrho^m_{\mathcal{M}})^\kappa,\ \sigma^{n\kappa}_{\mathcal{M}})$.
(iv) $\mathcal{M}^\kappa = (\varrho^{m\kappa}_{\mathcal{M}},\ 1-(1-\sigma^n_{\mathcal{M}})^\kappa)$.

Theorem 3.10. *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}), \mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\kappa \in \mathbb{N}$. Then:*

(i) $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathcal{F}_m^n(\mathbb{P})$.
(ii) $\mathcal{M}_1 \otimes \mathcal{M}_2 \in \mathcal{F}_m^n(\mathbb{P})$.
(iii) $\mathcal{M}_1 \Cup \mathcal{M}_2 \in \mathcal{F}_m^n(\mathbb{P})$.
(iv) $\mathcal{M}_1 \Cap \mathcal{M}_2 \in \mathcal{F}_m^n(\mathbb{P})$.
(v) $\kappa\mathcal{M} \in \mathcal{F}_m^n(\mathbb{P})$.
(vi) $\mathcal{M}^\kappa \in \mathcal{F}_m^n(\mathbb{P})$.

Proof. (i) Since,

$$\mathcal{M}_1 \oplus \mathcal{M}_2 = (\varrho^m_{\mathcal{M}_1} + \varrho^m_{\mathcal{M}_2} - \varrho^m_{\mathcal{M}_1}\varrho^m_{\mathcal{M}_2},\ \sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2}),$$

we have

$$\begin{aligned}\varrho^m_{\mathcal{M}_1\oplus\mathcal{M}_2} + \sigma^n_{\mathcal{M}_1\oplus\mathcal{M}_2} =& (\varrho^m_{\mathcal{M}_1} + \varrho^m_{\mathcal{M}_2} - \varrho^m_{\mathcal{M}_1}\varrho^m_{\mathcal{M}_2})^m + (\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^n \\ =& (\varrho^m_{\mathcal{M}_1}(1-\varrho^m_{\mathcal{M}_2}) + \varrho^m_{\mathcal{M}_2})^m + (\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^n \\ \geq& 0\end{aligned}$$

and

$$\begin{aligned}\varrho^m_{\mathcal{M}_1\oplus\mathcal{M}_2} + \sigma^n_{\mathcal{M}_1\oplus\mathcal{M}_2} =&(\varrho^m_{\mathcal{M}_1} + \varrho^m_{\mathcal{M}_2} - \varrho^m_{\mathcal{M}_1}\varrho^m_{\mathcal{M}_2})^m + (\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^n\\ \leq&((1-\sigma^n_{\mathcal{M}_1}) + (1-\sigma^n_{\mathcal{M}_2}) - (1-\sigma^n_{\mathcal{M}_1})(1-\sigma^n_{\mathcal{M}_2}))^m + (\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^n\\ =&(1-\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^m + (\sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2})^n\\ \leq&1,\end{aligned}$$

because $0 \leq \sigma^n_{\mathcal{M}_1}\sigma^n_{\mathcal{M}_2} \leq 1$ and $m, n \geq 1$. Hence $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathcal{F}^n_m(\mathbb{P})$.

(ii) Similar to (i).

(iii) Suppose $\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\} = \varrho_{\mathcal{M}_1}$. Since $\min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\} \leq \sigma_{\mathcal{M}_1}$, we have

$$\begin{aligned}0 \leq&\varrho^m_{\mathcal{M}_1\Cup\mathcal{M}_2} + \sigma^n_{\mathcal{M}_1\Cup\mathcal{M}_2}\\ =&(\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\})^m + (\min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})^n\\ \leq&\varrho^m_{\mathcal{M}_1} + \sigma^n_{\mathcal{M}_1}\\ \leq&1.\end{aligned}$$

Suppose now $\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\} = \varrho_{\mathcal{M}_2}$. Since $\min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\} \leq \sigma_{\mathcal{M}_2}$ we have

$$\begin{aligned}0 \leq&\varrho^m_{\mathcal{M}_1\Cup\mathcal{M}_2} + \sigma^n_{\mathcal{M}_1\Cup\mathcal{M}_2}\\ =&(\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\})^m + (\min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})^n\\ \leq&\varrho^m_{\mathcal{M}_2} + \sigma^n_{\mathcal{M}_2}\\ \leq&1.\end{aligned}$$

Proof of (iii) is complete.

(iv) Similar to (iii).

(v) Since $\mathcal{M} \in \mathcal{F}^n_m(\mathbb{P})$, we have $0 \leq \varrho^m_{\mathcal{M}} \leq 1, 0 \leq \sigma^n_{\mathcal{M}} \leq 1$ and $0 \leq \varrho^m_{\mathcal{M}} + \sigma^n_{\mathcal{M}} \leq 1$. Since,

$$\kappa\mathcal{M} = (1 - (1-\varrho^m_{\mathcal{M}})^\kappa,\ \sigma^{n\kappa}_{\mathcal{M}}),$$

we have

$$\begin{aligned}0 \leq&\varrho^m_{\kappa\mathcal{M}} + \sigma^n_{\kappa\mathcal{M}}\\ =&(1-(1-\varrho^m_{\mathcal{M}})^\kappa)^m + (\sigma^{n\kappa}_{\mathcal{M}})^n\\ \leq&(1-(\sigma^n_{\mathcal{M}})^\kappa)^m + (\sigma^{n\kappa}_{\mathcal{M}})^n\\ =&(1-\sigma^{n\kappa}_{\mathcal{M}})^m + (\sigma^{n\kappa}_{\mathcal{M}})^n\\ \leq&1,\end{aligned}$$

because $0 \leq \sigma^{n\kappa}_{\mathcal{M}} \leq 1, \forall n, m, k \geq 1$.

□

Theorem 3.11. *Let* $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}), \mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$ *and* $\kappa, \kappa_1, \kappa_2 \in \mathbb{N}$. *Then:*

(i) $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}_2 \oplus \mathcal{M}_1$.

(ii) $\mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{M}_2 \otimes \mathcal{M}_1$.

(iii) $\kappa(\mathcal{M}_1 \oplus \mathcal{M}_2) = \kappa\mathcal{M}_1 \oplus \kappa\mathcal{M}_2$.

(iv) $(\kappa_1+\kappa_2)\mathcal{M}=\kappa_1\mathcal{M}+\kappa_2\mathcal{M}$.
(v) $(\mathcal{M}_1\otimes\mathcal{M}_2)^\kappa=\mathcal{M}_1^\kappa\otimes\mathcal{M}_2^\kappa$.
(vi) $\mathcal{M}^{\kappa_1}\otimes\mathcal{M}^{\kappa_2}=\mathcal{M}^{(\kappa_1+\kappa_2)}$.

Proof. Proofs are easy and left to the readers. □

Theorem 3.12. *Let $\mathcal{M}=(\varrho_\mathcal{M},\sigma_\mathcal{M})\in\mathcal{F}_m^n(\mathbb{P})$ and $p\in\mathbb{P}$. If $\pi_\mathcal{M}(p)=0$, then $\pi_{\mathcal{M}^\kappa}(p)=0,\forall\kappa\in\mathbb{N}$.*

Proof. Since,

$$\pi_\mathcal{M}(p)=(1-\varrho_\mathcal{M}^m(p)-\sigma_\mathcal{M}^n(p))^{\frac{2}{m+n}},$$

we have

$$\begin{aligned}\pi_\mathcal{M}(p)=0 \Rightarrow&(1-\varrho_\mathcal{M}^m(p)-\sigma_\mathcal{M}^n(p))^{\frac{2}{m+n}}=0\\ \Rightarrow&\varrho_\mathcal{M}^m(p)+\sigma_\mathcal{M}^n(p)=1\\ \Rightarrow&\varrho_\mathcal{M}^m(p)=1-\sigma_\mathcal{M}^n(p).\end{aligned}$$

By using this result we have,

$$\begin{aligned}\mathcal{M}^\kappa=&(\varrho_\mathcal{M}^{m\kappa},\ 1-(1-\sigma_\mathcal{M}^n)^\kappa)\\ =&(\varrho_\mathcal{M}^{m\kappa},\ 1-(\varrho_\mathcal{M}^m)^\kappa)\\ =&(\varrho_\mathcal{M}^{m\kappa},\ 1-(\varrho_\mathcal{M}^{m\kappa})).\end{aligned}$$

Hence,

$$\begin{aligned}\pi_{\mathcal{M}^\kappa}(p)=&(1-(\varrho_\mathcal{M}^{m\kappa}(p))^m-(1-(\varrho_\mathcal{M}^{m\kappa}(p)))^n)^{\frac{2}{m+n}}\\ \Rightarrow&\pi_{\mathcal{M}^\kappa}^{\frac{m+n}{2}}(p)=1-(\varrho_\mathcal{M}^{m\kappa}(p))^m-(1-(\varrho_\mathcal{M}^{m\kappa}(p)))^n\\ \Rightarrow&\pi_{\mathcal{M}^\kappa}(p)=0.\end{aligned}$$

Theorem 3.13. *Let $\mathcal{M}=(\varrho_\mathcal{M},\sigma_\mathcal{M})\in\mathcal{F}_m^n(\mathbb{P})$ and $p\in\mathbb{P}$ and $\kappa,\kappa_1,\kappa_2\in\mathbb{N}$. Then:*

(i) $\kappa_1\geq\kappa_2\Rightarrow\mathcal{M}^{\kappa_1}\Subset\mathcal{M}^{\kappa_2}$.
(ii) $\kappa_1\geq\kappa_2\Rightarrow\kappa_2\mathcal{M}\Subset\kappa_1\mathcal{M}$.

Proof. (i) Since,

$$\begin{aligned}\mathcal{M}^{\kappa_1}&=(\varrho_\mathcal{M}^{m\kappa_1},\ 1-(1-\sigma_\mathcal{M}^n)^{\kappa_1})\\ \mathcal{M}^{\kappa_2}&=(\varrho_\mathcal{M}^{m\kappa_2},\ 1-(1-\sigma_\mathcal{M}^n)^{\kappa_2})\end{aligned}$$

we have,

$$\begin{aligned}\kappa_1\geq\kappa_2\Rightarrow&\varrho_\mathcal{M}^{\kappa_2}\geq\varrho_\mathcal{M}^{\kappa_1}\text{ and }(1-\sigma_\mathcal{M}^n)^{\kappa_1}\leq(1-\sigma_\mathcal{M}^n)^{\kappa_2}\\ \Rightarrow&\varrho_\mathcal{M}^{m\kappa_2}\geq\varrho_\mathcal{M}^{m\kappa_1}\text{ and }1-(1-\sigma_\mathcal{M}^n)^{\kappa_2}\leq1-(1-\sigma_\mathcal{M}^n)^{\kappa_1}\\ \Rightarrow&\varrho_{\mathcal{M}^{\kappa_2}}\geq\varrho_{\mathcal{M}^{\kappa_1}}\text{ and }\sigma_{\mathcal{M}^{\kappa_2}}\leq\sigma_{\mathcal{M}^{\kappa_1}}.\end{aligned}$$

Hence $\mathcal{M}^{\kappa_1} \Subset \mathcal{M}^{\kappa_2}$.

(ii) Similar to that of (i).

$\square$

Theorem 3.14. *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$ and $\kappa \in \mathbb{N}$. Then:*

(i) $\mathcal{M}_1 \Subset \mathcal{M}_2 \Rightarrow \kappa\mathcal{M}_1 \Subset \kappa\mathcal{M}_2$.

(ii) $\mathcal{M}_1 \Subset \mathcal{M}_2 \Rightarrow \mathcal{M}_1^\kappa \Subset \mathcal{M}_2^\kappa$.

(iii) $(\mathcal{M}_1 \Cup \mathcal{M}_2)^\kappa \Rightarrow \mathcal{M}_1^\kappa \Cup \mathcal{M}_2^\kappa$.

(iv) $\kappa(\mathcal{M}_1 \Cup \mathcal{M}_2) \Rightarrow \kappa\mathcal{M}_1 \Cup \kappa\mathcal{M}_2$.

(v) $(\mathcal{M}_1 \Cap \mathcal{M}_2)^\kappa \Rightarrow \mathcal{M}_1^\kappa \Cap \mathcal{M}_2^\kappa$.

(vi) $\kappa(\mathcal{M}_1 \Cap \mathcal{M}_2) \Rightarrow \kappa\mathcal{M}_1 \Cap \kappa\mathcal{M}_2$.

Proof. (i) Since $\mathcal{M}_1 \Subset \mathcal{M}_2$, we have

$$\begin{aligned}
\varrho_{\mathcal{M}_1} \le \varrho_{\mathcal{M}_2} \Rightarrow & \varrho^m_{\mathcal{M}_1} \le \varrho^m_{\mathcal{M}_2} \\
\Rightarrow & 1 - \varrho^m_{\mathcal{M}_2} \le 1 - \varrho^m_{\mathcal{M}_1} \\
\Rightarrow & (1 - \varrho^m_{\mathcal{M}_2})^\kappa \le (1 - \varrho^m_{\mathcal{M}_1})^\kappa \\
\Rightarrow & 1 - (1 - \varrho^m_{\mathcal{M}_1})^\kappa \le 1 - (1 - \varrho^m_{\mathcal{M}_2})^\kappa \\
\Rightarrow & \varrho_{\kappa\mathcal{M}_1} \le \varrho_{\kappa\mathcal{M}_2},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\mathcal{M}_1} \ge \varrho_{\mathcal{M}_2} \Rightarrow & \sigma^n_{\mathcal{M}_1} \ge \sigma^n_{\mathcal{M}_2} \\
\Rightarrow & \sigma^{n\kappa}_{\mathcal{M}_1} \ge \sigma^{n\kappa}_{\mathcal{M}_2} \\
\Rightarrow & \sigma_{\kappa\mathcal{M}_1} \ge \sigma_{\kappa\mathcal{M}_2}.
\end{aligned}$$

Hence $\kappa\mathcal{M}_1 \Subset \kappa\mathcal{M}_2$.

(ii) Similar to that of (i).

(iii) Follows since,

$$\mathcal{M}_1 \Cup \mathcal{M}_2 = (\max\{\varrho_{\mathcal{M}_1},\ \varrho_{\mathcal{M}_2}\}, \min\{\sigma_{\mathcal{M}_1},\ \sigma_{\mathcal{M}_2}\})$$

and

$$\begin{aligned}
(\mathcal{M}_1 \Cup \mathcal{M}_2)^\kappa = & (\varrho_{\mathcal{M}_1 \Cup \mathcal{M}_2}, \sigma_{\mathcal{M}_1 \Cup \mathcal{M}_2}) \\
= & ((\max\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\})^{m\kappa}, 1 - (1 - (\min\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})^n)^\kappa) \\
= & (\max\{\varrho^{m\kappa}_{\mathcal{M}_1}, \varrho^{m\kappa}_{\mathcal{M}_2}\}, 1 - (1 - \min\{\sigma^n_{\mathcal{M}_1}, \sigma^n_{\mathcal{M}_2}\})^\kappa) \\
= & (\max\{\varrho^{m\kappa}_{\mathcal{M}_1}, \varrho^{m\kappa}_{\mathcal{M}_2}\}, 1 - (\max\{1 - \sigma^n_{\mathcal{M}_1}, 1 - \sigma^n_{\mathcal{M}_2}\})^\kappa) \\
= & (\max\{\varrho^{m\kappa}_{\mathcal{M}_1}, \varrho^{m\kappa}_{\mathcal{M}_2}\}, 1 - (\max\{(1 - \sigma^n_{\mathcal{M}_1})^\kappa, (1 - \sigma^n_{\mathcal{M}_2})^\kappa\})) \\
= & (\max\{\varrho^{m\kappa}_{\mathcal{M}_1}, \varrho^{m\kappa}_{\mathcal{M}_2}\}, \min\{1 - (1 - \sigma^n_{\mathcal{M}_1})^\kappa, 1 - (1 - \sigma^n_{\mathcal{M}_2})^\kappa\}) \\
= & \mathcal{M}_1^\kappa \Cup \mathcal{M}_2^\kappa.
\end{aligned}$$

(iv) Follows since,

$$\mathcal{M}_1 \Cap \mathcal{M}_2 = (\min\{\varrho_{\mathcal{M}_1},\ \varrho_{\mathcal{M}_2}\},\ \max\{\sigma_{\mathcal{M}_1},\ \sigma_{\mathcal{M}_2}\})$$

and

$$
\begin{aligned}
(\mathcal{M}_1 \Cap \mathcal{M}_2)^\kappa =&(\varrho_{\mathcal{M}_1 \Cap \mathcal{M}_2}, \sigma_{\mathcal{M}_1 \Cap \mathcal{M}_2})\\
=&((\min\{\varrho_{\mathcal{M}_1}, \varrho_{\mathcal{M}_2}\})^{m\kappa}, 1-(1-(\max\{\sigma_{\mathcal{M}_1}, \sigma_{\mathcal{M}_2}\})^n)^\kappa)\\
=&(\min\{\varrho_{\mathcal{M}_1}^{m\kappa}, \varrho_{\mathcal{M}_2}^{m\kappa}\}, 1-(1-\max\{\sigma_{\mathcal{M}_1}^n, \sigma_{\mathcal{M}_2}^n\})^\kappa)\\
=&(\min\{\varrho_{\mathcal{M}_1}^{m\kappa}, \varrho_{\mathcal{M}_2}^{m\kappa}\}, 1-(\min\{1-\sigma_{\mathcal{M}_1}^n, 1-\sigma_{\mathcal{M}_2}^n\})^\kappa)\\
=&(\min\{\varrho_{\mathcal{M}_1}^{m\kappa}, \varrho_{\mathcal{M}_2}^{m\kappa}\}, 1-(\min\{(1-\sigma_{\mathcal{M}_1}^n)^\kappa, (1-\sigma_{\mathcal{M}_2}^n)^\kappa\}))\\
=&(\min\{\varrho_{\mathcal{M}_1}^{m\kappa}, \varrho_{\mathcal{M}_2}^{m\kappa}\}, \max\{1-(1-\sigma_{\mathcal{M}_1}^n)^\kappa, 1-(1-\sigma_{\mathcal{M}_2}^n)^\kappa\})\\
=&\mathcal{M}_1^\kappa \Cap \mathcal{M}_2^\kappa.
\end{aligned}
$$

(v) Similar to that of (iii).
(vi) Similar to that of (v).

□

Definition 3.15. *Let $\mathcal{M} = (\varrho_\mathcal{M}, \sigma_\mathcal{M}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\alpha \in [0,1]$ then the operator $\mathbb{D}_\alpha(\mathcal{M})$ is defined as follows:*

$$\mathbb{D}_\alpha(\mathcal{M}) = ((\varrho_\mathcal{M}^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_\mathcal{M}^n + (1-\alpha)\pi^{\frac{m+n}{2}})^{\frac{1}{n}}).$$

Theorem 3.16. *Let $\mathcal{M} = (\varrho_\mathcal{M}, \sigma_\mathcal{M}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\alpha, \beta \in [0,1]$. Then:*

(i) $\alpha \le \beta \Rightarrow \mathbb{D}_\alpha(\mathcal{M}) \Subset \mathbb{D}_\beta(\mathcal{M})$.
(ii) $\mathbb{D}_0(\mathcal{M}) = \Box\mathcal{M}$.
(iii) $\mathbb{D}_1(\mathcal{M}) = \Diamond\mathcal{M}$.

Proof. (i) The proof (i) is immediate.
(ii) Since,

$$\mathbb{D}_\alpha(\mathcal{M}) = ((\varrho_\mathcal{M}^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_\mathcal{M}^n + (1-\alpha)\pi^{\frac{m+n}{2}})^{\frac{1}{n}})$$

we have,

$$
\begin{aligned}
\mathbb{D}_0(\mathcal{M}) =&((\varrho_\mathcal{M}^m + (0)\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_\mathcal{M}^n + (1-0)\pi^{\frac{m+n}{2}})^{\frac{1}{n}})\\
=&(\varrho_\mathcal{M}, (\sigma_\mathcal{M}^n + \pi^{\frac{m+n}{2}})^{\frac{1}{n}})\\
=&(\varrho_\mathcal{M}, (\sigma_\mathcal{M}^n + 1 - \varrho_\mathcal{M}^m - \sigma_\mathcal{M}^n)^{\frac{1}{n}})\\
=&(\varrho_\mathcal{M}, (1-\varrho_\mathcal{M}^m)^{\frac{1}{n}})\\
=&\Box\mathcal{M}.
\end{aligned}
$$

This completes the proof.

(iii) It follows on noting that,

$$\begin{aligned}\mathbb{D}_1(\mathcal{M}) =&((\varrho_{\mathcal{M}}^m + (1)\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_{\mathcal{M}}^n + (1-1)\pi^{\frac{m+n}{2}})^{\frac{1}{n}})\\ =&((\varrho_{\mathcal{M}}^m + \pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_{\mathcal{M}}^n)^{\frac{1}{n}})\\ =&((\varrho_{\mathcal{M}}^m + (1-\varrho_{\mathcal{M}}^m - \sigma_{\mathcal{M}}^n)^{\frac{1}{m}}, \sigma_{\mathcal{M}})\\ =&((1-\sigma_{\mathcal{M}}^n)^{\frac{1}{m}}, \sigma_{\mathcal{M}})\\ =&\Diamond\mathcal{M}.\end{aligned}$$

□

Definition 3.17. *Let $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\alpha, \beta \in [0,1]$ where $\alpha + \beta \leq 1$. We define the operator $\mathbb{F}_{\alpha,\beta}(\mathcal{M})$ as:*

$$\mathbb{F}_{\alpha,\beta}(\mathcal{M}) = ((\varrho_{\mathcal{M}}^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_{\mathcal{M}}^n + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{n}}).$$

Theorem 3.18. *For any $\mathcal{M} = (\varrho_{\mathcal{M}}, \sigma_{\mathcal{M}}) \in \mathcal{F}_m^n(\mathbb{P})$ and $\alpha, \beta \in [0,1]$ where $\alpha+\beta \leq 1$. We have:*

(i) $\mathbb{F}_{\alpha,\beta}(\mathcal{M}) \in \mathcal{F}_m^n(\mathbb{P})$.
(ii) $0 \leq \gamma \leq \alpha \Rightarrow \mathbb{F}_{\gamma,\beta}(\mathcal{M}) \Subset \mathbb{F}_{\alpha,\beta}(\mathcal{M})$.
(iii) $0 \leq \gamma \leq \beta \Rightarrow \mathbb{F}_{\alpha,\beta}(\mathcal{M}) \Subset \mathbb{F}_{\alpha,\gamma}(\mathcal{M})$.
(iv) $\mathbb{D}_\alpha(\mathcal{M}) = \mathbb{F}_{\alpha,1-\alpha}(\mathcal{M})$.
(v) $\Box\mathcal{M} = \mathbb{F}_{0,1}(\mathcal{M})$.
(vi) $\Diamond\mathcal{M} = \mathbb{F}_{1,0}(\mathcal{M})$.
(vii) $\complement(\mathbb{F}_{\alpha,\beta}(\complement\mathcal{M})) = \mathbb{F}_{\beta,\alpha}(\mathcal{M})$.

Proof. (i) Follows since

$$\begin{aligned}\varrho_{\mathbb{F}_{\alpha,\beta}(\mathcal{M})}^m + \sigma_{\mathbb{F}_{\alpha,\beta}(\mathcal{M})}^n =&((\varrho_{\mathcal{M}}^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}})^m + ((\sigma_{\mathcal{M}}^n + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{n}})^n\\ =&\varrho_{\mathcal{M}}^m + \alpha(1-\varrho_{\mathcal{M}}^m - \sigma_{\mathcal{M}}^n) + \sigma_{\mathcal{M}}^n + \beta(1-\varrho_{\mathcal{M}}^m - \sigma_{\mathcal{M}}^n)\\ =&(\varrho_{\mathcal{M}}^m + \sigma_{\mathcal{M}}^n) + (\alpha+\beta)(1-\varrho_{\mathcal{M}}^m - \sigma_{\mathcal{M}}^n)\\ =&(\varrho_{\mathcal{M}}^m + \sigma_{\mathcal{M}}^n) + (\alpha+\beta) - (\alpha+\beta)(\varrho_{\mathcal{M}}^m + \sigma_{\mathcal{M}}^n)\\ \leq&1.\end{aligned}$$

The proofs of (ii) and (iii) are easy and left to the readers.

(iv) Follows on noting that:

$$\begin{aligned}\mathbb{F}_{\alpha,1-\alpha}(\mathcal{M}) =&((\varrho_{\mathcal{M}}^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_{\mathcal{M}}^n + (1-\alpha)\pi^{\frac{m+n}{2}})^{\frac{1}{n}})\\ =&\mathbb{D}_\alpha(\mathcal{M}).\end{aligned}$$

(v) Follows since,

$$\begin{aligned}\mathbb{F}_{\alpha,1-\alpha}(\mathcal{M}) = \mathbb{D}_\alpha(\mathcal{M}) \Rightarrow&\mathbb{F}_{0,1}(\mathcal{M}) = \mathbb{D}_0(\mathcal{M})\\ \Rightarrow&\mathbb{F}_{0,1}(\mathcal{M}) = \Box\mathcal{M}.\end{aligned}$$

(vi) Follows on noting that:

$$\begin{aligned}\mathbb{F}_{\alpha,1-\alpha}(\mathcal{M}) = \mathbb{D}_{\alpha}(\mathcal{M}) \Rightarrow & \mathbb{F}_{1,0}(\mathcal{M}) = \mathbb{D}_1(\mathcal{M}) \\ \Rightarrow & \mathbb{F}_{1,0}(\mathcal{M}) = \diamond\mathcal{M}.\end{aligned}$$

(vii) Since, $\complement\mathcal{M} = (\sigma_{\mathcal{M}}^{\frac{n}{m}}, \varrho_{\mathcal{M}}^{\frac{m}{n}})$, we have

$$\begin{aligned}\mathbb{F}_{\alpha,\beta}(\complement\mathcal{M}) =& (((\sigma_{\mathcal{M}}^{\frac{n}{m}})^m + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, ((\varrho_{\mathcal{M}}^{\frac{m}{n}})^n + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{n}}) \\ =& ((\sigma_{\mathcal{M}}^n + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\varrho_{\mathcal{M}}^m + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{n}}).\end{aligned}$$

And so,

$$\begin{aligned}\complement(\mathbb{F}_{\alpha,\beta}(\complement\mathcal{M})) =& (((\varrho_{\mathcal{M}}^m + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{n}})^{\frac{n}{m}}, ((\sigma_{\mathcal{M}}^n + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{m}})^{\frac{m}{n}}) \\ =& ((\varrho_{\mathcal{M}}^m + \beta\pi^{\frac{m+n}{2}})^{\frac{1}{m}}, (\sigma_{\mathcal{M}}^n + \alpha\pi^{\frac{m+n}{2}})^{\frac{1}{n}}) \\ =& \mathbb{F}_{\beta,\alpha}(\mathcal{M}).\end{aligned}$$

This completes the proof.

□

4. DISTANCES AND SIMILARITIES OVER $\mathcal{F}_m^n(\mathbb{P})$

Definition 4.1. *Let $\mathbb{P} = \{p_1, p_2, \dots, p_r\}$ be a universe of discourse and $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}), \mathcal{M}_3 = (\varrho_{\mathcal{M}_3}, \sigma_{\mathcal{M}_3}) \in \mathcal{F}_m^n(\mathbb{P})$, the distance function $d : \mathcal{F}_m^n(\mathbb{P}) \times \mathcal{F}_m^n(\mathbb{P}) \to [0, 1]$ is defined as:*

(i) $0 \leq d(\mathcal{M}_1, \mathcal{M}_2) \leq 1$ *(boundedness).*
(ii) $d(\mathcal{M}_1, \mathcal{M}_2) = 0 \Leftrightarrow \mathcal{M}_1 = \mathcal{M}_2$ *(separability).*
(iii) $d(\mathcal{M}_1, \mathcal{M}_2) = d(\mathcal{M}_2, \mathcal{M}_1)$ *(symmetric).*
(iv) $d(\mathcal{M}_1, \mathcal{M}_3) + d(\mathcal{M}_2, \mathcal{M}_3) \geq d(\mathcal{M}_1, \mathcal{M}_2)$ *(triangle inequality).*

Definition 4.2. *Let $\mathbb{P} = \{p_1, p_2, \dots, p_r\}$ be a universe of discourse. For any $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$, the Hamming distance is defined as:*

$$d^H_{F_m^n}(\mathcal{M}_1, \mathcal{M}_2) = \frac{1}{2}\sum_1^r \{|\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i)| + |\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i)| + |\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i)|\}.$$

Definition 4.3. *Let $\mathbb{P} = \{p_1, p_2, \dots, p_r\}$ be a universe of discourse. For any $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$, the normalize Hamming distance is defined as:*

$$d^{nH}_{F_m^n}(\mathcal{M}_1, \mathcal{M}_2) = \frac{1}{2r}\sum_1^r \{|\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i)| + |\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i)| + |\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i)|\}.$$

Definition 4.4. *Let $\mathbb{P} = \{p_1, p_2, \ldots, p_r\}$ be a universe of discourse. For any $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$, the Euclidean distance is defined as:*

$$d^E_{F^n_m}(\mathcal{M}_1, \mathcal{M}_2) = \sqrt{\frac{1}{2}\sum_1^r \{(\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i))^2 + (\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i))^2 + (\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i))^2\}}.$$

Definition 4.5. *Let $\mathbb{P} = \{p_1, p_2, \ldots, p_r\}$ be a universe of discourse. For any $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}^n_m(\mathbb{P})$, the normalized Euclidean distance is defined as:*

$$d^{nE}_{F^n_m}(\mathcal{M}_1, \mathcal{M}_2) = \sqrt{\frac{1}{2r}\sum_1^r \{(\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i))^2 + (\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i))^2 + (\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i))^2\}}.$$

Example 4.6. *Let $\mathbb{P} = \{p_1, p_2\}$ be a universe of discourse and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in \mathcal{F}^5_6(\mathbb{P})$, where*

$$\begin{aligned}
\mathcal{M}_1 &= \{< p_1, 0.7, 0.8 >, < p_2, 0.4.0.6 >\},\\
\mathcal{M}_2 &= \{< p_1, 0.9, 0.2 >, < p_2, 0.9.0.4 >\},\\
\mathcal{M}_3 &= \{< p_1, 0.5, 0.6 >, < p_2, 0.7.0.3 >\}.
\end{aligned}$$

Then

$$\begin{aligned}
d^H_{F^5_6}(\mathcal{M}_1, \mathcal{M}_2) =& \frac{1}{2}\{(|0.7 - 0.9| + |0.8 - 0.2| + |.9027842192 - 0.8501396989|)\\
&+(|0.4 - 0.9| + |0.6 - 0.4| + |0.989405842 - 0.848611401|)\}\\
=& \frac{1}{2}(0.8526445211 + 0.840794441)\\
=& 0.8467194811.
\end{aligned}$$

$$\begin{aligned}
d^E_{F^5_6}(\mathcal{M}_1, \mathcal{M}_2) =& (\frac{1}{2}\{((0.7 - 0.9)^2 + (0.8 - 0.2)^2 + (0.9027842192 - 0.8501396989)^2)\\
&+((0.4 - 0.9)^2 + (0.6 - 0.4)^2 + (0.989405842 - 0.848611401)^2)\})^{\frac{1}{2}}\\
=& (\frac{1}{2}(0.4027670223 + 0.5592645321))^{\frac{1}{2}}\\
=& 0.6935529091.
\end{aligned}$$

$$\begin{aligned}
d^{nH}_{F^5_6}(\mathcal{M}_1, \mathcal{M}_2) =& \frac{1}{2 \times 2}\{(|0.7 - 0.9| + |0.8 - 0.2| + |.9027842192 - 0.8501396989|)\\
&+(|0.4 - 0.9| + |0.6 - 0.4| + |0.989405842 - 0.848611401|)\}\\
=& \frac{1}{4}(0.8526445211 + 0.840794441)\\
=& 0.4233597405.
\end{aligned}$$

$$
\begin{aligned}
d^{nE}_{F^5_6}(\mathcal{M}_1,\mathcal{M}_2) =&(\frac{1}{2\times 2}\{((0.7-0.9)^2+(0.8-0.2)^2+(0.9027842192-0.8501396989)^2)\\
&+((0.4-0.9)^2+(0.6-0.4)^2+(0.989405842-0.848611401)^2)\})^{\frac{1}{2}}\\
=&(\frac{1}{4}(0.4027670223+0.5592645321))^{\frac{1}{2}}\\
=&0.4903395646.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&d^{H}_{F^5_6}(\mathcal{M}_1,\mathcal{M}_3)=0.46994264,\ d^{E}_{F^5_6}(\mathcal{M}_1,\mathcal{M}_3)=0.36559781.\\
&d^{nH}_{F^5_6}(\mathcal{M}_1,\mathcal{M}_3)=0.23497132,\ d^{nE}_{F^5_6}(\mathcal{M}_1,\mathcal{M}_3)=0.25851669.\\
&d^{H}_{F^5_6}(\mathcal{M}_2,\mathcal{M}_3)=0.44601147,\ d^{E}_{F^5_6}(\mathcal{M}_2,\mathcal{M}_3)=0.4484904468.\\
&d^{nH}_{F^5_6}(\mathcal{M}_2,\mathcal{M}_3)=0.22300573,\ d^{nE}_{F^5_6}(\mathcal{M}_2,\mathcal{M}_3)=0.31713063.
\end{aligned}
$$

Remark 4.7. *It follows from Example 4.6 that the four proposed distances satisfy all the conditions of Definition 4.1 in the case where the objects in (m,n)-FSs are equal.*

Example 4.8. *Let $\mathbb{P}=\{p_1,p_2,p_3,p_4\}$ and $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3\in\mathcal{F}^5_5(\mathbb{P})$, where*

$$
\begin{aligned}
\mathcal{M}_1&=\{<p_1,0.3,0.8>,<p_3,0.8.0.6>\},\\
\mathcal{M}_2&=\{<p_2,0.9,0.4>,<p_4,0.6.0.6>\},\\
\mathcal{M}_3&=\{<p_3,0.5,0.6>,<p_4,0.7.0.3>\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&d^{H}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_2)=3.811621212,\ d^{E}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_2)=1.719216928.\\
&d^{nH}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_2)=0.952905303,\ d^{nE}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_2)=0.859608464.\\
&d^{H}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_3)=2.761952567,\ d^{E}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_3)=1.221811482.\\
&d^{nH}_{F^5_5}(\mathcal{M}_1,\mathcal{M}_3)=0.690488141,\ d^{nE}_{F^5_5S}(\mathcal{M}_1,\mathcal{M}_3)=0.610905741.\\
&d^{H}_{F^5_5}(\mathcal{M}_2,\mathcal{M}_3)=2.230644175,\ d^{E}_{F^5_5}(\mathcal{M}_2,\mathcal{M}_3)=1.243517948.\\
&d^{nH}_{F^5_5}(\mathcal{M}_2,\mathcal{M}_3)=0.576610438,\ d^{nE}_{F^5_5}(\mathcal{M}_2,\mathcal{M}_3)=0.62175897.
\end{aligned}
$$

Remark 4.9. *In Example 4.8, $d^H_{F^n_m}$ and $d^E_{F^n_m}$ (for $m=n=5$) do not satisfy all the conditions of Definition 4.1 in the case where the objects in (m,n)-FSs are not equal. Therefore, they can not be adopted into finding the distance between (m,n)-FSs. Thus $d^{nH}_{F^n_m}$ and $d^{nE}_{F^n_m}$ are reliable distance measures for (m,n)-FSs.*

Definition 4.10. *Let $\mathbb{P} = \{p_1, p_2, \ldots, p_r\}$ be a universe of discourse and $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}), \mathcal{M}_3 = (\varrho_{\mathcal{M}_3}, \sigma_{\mathcal{M}_3}) \in \mathcal{F}_m^n(\mathbb{P})$, the similarity measure of $s : \mathcal{F}_m^n(\mathbb{P}) \times \mathcal{F}_m^n(\mathbb{P}) \to [0, 1]$ is defined as:*

(i) $0 \leq s(\mathcal{M}_1, \mathcal{M}_2) \leq 1$ *(boundedness).*
(ii) $s(\mathcal{M}_1, \mathcal{M}_2) = 1 \Leftrightarrow \mathcal{M}_1 = \mathcal{M}_2$ *(separability).*
(iii) $s(\mathcal{M}_1, \mathcal{M}_2) = s(\mathcal{M}_2, \mathcal{M}_1)$ *(symmetric).*
(iv) $s(\mathcal{M}_1, \mathcal{M}_3) + s(\mathcal{M}_2, \mathcal{M}_3) \geq s(\mathcal{M}_1, \mathcal{M}_2)$ *(triangle inequality).*

The following theorems can be easily proved by using Definitions 4.1 and 4.10.

Theorem 4.11. *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$. If $d(\mathcal{M}_1, \mathcal{M}_2)$ is a distance measure between (m, n)-FSs $\mathcal{M}_1$ and $\mathcal{M}_2$, then*

$$s(\mathcal{M}_1, \mathcal{M}_2) = 1 - d(\mathcal{M}_1, \mathcal{M}_2)$$

is a similarity measure of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Theorem 4.12. *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$. If $s(\mathcal{M}_1, \mathcal{M}_2)$ is a similarity measure between (m, n)-FSs $\mathcal{M}_1$ and $\mathcal{M}_2$, then*

$$d(\mathcal{M}_1, \mathcal{M}_2) = 1 - s(\mathcal{M}_1, \mathcal{M}_2)$$

is a distance measure of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Theorem 4.13. *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}),\ \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}), \mathcal{M}_3 = (\varrho_{\mathcal{M}_3}, \sigma_{\mathcal{M}_3}) \in \mathcal{F}_m^n(\mathbb{P})$. Suppose $\mathcal{M}_1 \Subset \mathcal{M}_2 \Subset \mathcal{M}_3$, then:*

(i) $d(\mathcal{M}_1, \mathcal{M}_3) \geq d(\mathcal{M}_1, \mathcal{M}_2)$ *and* $d(\mathcal{M}_1, \mathcal{M}_3) \geq d(\mathcal{M}_2, \mathcal{M}_3)$.
(ii) $s(\mathcal{M}_1, \mathcal{M}_3) \leq s(\mathcal{M}_1, \mathcal{M}_2)$ *and* $d(\mathcal{M}_1, \mathcal{M}_3) \leq d(\mathcal{M}_2, \mathcal{M}_3)$.

Theorem 4.14. *Let $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$. Then:*

(i) $d(\mathcal{M}_1, \mathcal{M}_2) = d(\mathcal{M}_1^c, \mathcal{M}_2^c)$.
(ii) $s(\mathcal{M}_1, \mathcal{M}_2) = s(\mathcal{M}_1^c, \mathcal{M}_2^c)$.

Definition 4.15. *Let $\mathbb{P} = \{p_1, p_2, \ldots, p_r\}$ be a universe of discourse. For any $\mathcal{M}_1 = (\varrho_{\mathcal{M}_1}, \sigma_{\mathcal{M}_1}), \mathcal{M}_2 = (\varrho_{\mathcal{M}_2}, \sigma_{\mathcal{M}_2}) \in \mathcal{F}_m^n(\mathbb{P})$. On the basis of Theorems 4.11, 4.12 and Definitions 4.3, 4.5 we define the two similarity measures as follows:*

$$s_1(\mathcal{M}_1, \mathcal{M}_2) = 1 - d_{F_m^n}^{nH}(\mathcal{M}_1, \mathcal{M}_2)$$

where,

$$d_{F_m^n}^{nH}(\mathcal{M}_1, \mathcal{M}_2) = \frac{1}{2r} \sum_1^r \{|\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i)| + |\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i)| + |\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i)|\},$$

and

$$s_2(\mathcal{M}_1, \mathcal{M}_2) = 1 - d^{nE}_{F^n_m}(\mathcal{M}_1, \mathcal{M}_2)$$

where,

$$d^{nE}_{F^n_m}(\mathcal{M}_1, \mathcal{M}_2) = \sqrt{\frac{1}{2r}\sum_1^r \{(\varrho_{\mathcal{M}_1}(p_i) - \varrho_{\mathcal{M}_2}(p_i))^2 + (\sigma_{\mathcal{M}_1}(p_i) - \sigma_{\mathcal{M}_2}(p_i))^2 + (\pi_{\mathcal{M}_1}(p_i) - \pi_{\mathcal{M}_2}(p_i))^2\}}.$$

Remark 4.16. *The similarity measures s_1 and s_2 satisfy all the conditions of Definition 4.10 in the case when the objects in (m, n)-FSs are equal. Consider $\mathbb{P} = \{p_1, p_2\}$ and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in \mathcal{F}^5_6(\mathbb{P})$ in Example 4.6. Then*

$$s_1(\mathcal{M}_1, \mathcal{M}_2) = 1 - d^{nH}_{F^5_6}(\mathcal{M}_1, \mathcal{M}_2) = 1 - 0.4233597405 = 0.5766402595.$$
$$s_1(\mathcal{M}_1, \mathcal{M}_3) = 1 - d^{nH}_{F^5_6}(\mathcal{M}_1, \mathcal{M}_3) = 1 - 0.23497132 = 0.760502868.$$
$$s_1(\mathcal{M}_2, \mathcal{M}_3) = 1 - d^{nH}_{F^5_6}(\mathcal{M}_2, \mathcal{M}_3) = 1 - 0.22300573 = 0.77699427.$$
$$s_2(\mathcal{M}_1, \mathcal{M}_2) = 1 - d^{nE}_{F^5_6}(\mathcal{M}_1, \mathcal{M}_2) = 1 - 0.4903395646 = 0.5096604354.$$
$$s_2(\mathcal{M}_1, \mathcal{M}_3) = 1 - d^{nE}_{F^5_6}(\mathcal{M}_1, \mathcal{M}_3) = 1 - 0.2585166 = 0.7414834.$$
$$s_2(\mathcal{M}_2, \mathcal{M}_3) = 1 - d^{nE}_{F^5_6}(\mathcal{M}_2, \mathcal{M}_3) = 1 - 0.31713063 = 0.68286937.$$

Clearly s_1 and s_2 satisfy all the conditions of similarity measures.

Remark 4.17. *The similarity measures s_1 and s_2 satisfy all the conditions of Definition 4.10 in the case when the objects in (m, n)-FSs are not equal. Let $\mathbb{P} = \{p_1, p_2, p_3, p_4\}$ and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in \mathcal{F}^5_5(\mathbb{P})$ are defined in Example 4.8. Then*

$$s_1(\mathcal{M}_1, \mathcal{M}_2) = 1 - d^{nH}_{F^5_5}(\mathcal{M}_1, \mathcal{M}_2) = 1 - 0.952905303 = 0.047094697.$$
$$s_1(\mathcal{M}_1, \mathcal{M}_3) = 1 - d^{nH}_{F^5_5}(\mathcal{M}_1, \mathcal{M}_3) = 1 - 0.690488141 = 0.309511859.$$
$$s_1(\mathcal{M}_2, \mathcal{M}_3) = 1 - d^{nH}_{F^5_5}(\mathcal{M}_2, \mathcal{M}_3) = 1 - 0.576610438 = 0.423389562.$$
$$s_2(\mathcal{M}_1, \mathcal{M}_2) = 1 - d^{nE}_{F^5_5}(\mathcal{M}_1, \mathcal{M}_2) = 1 - 0.859608464 = 0.140391536.$$
$$s_2(\mathcal{M}_1, \mathcal{M}_3) = 1 - d^{nE}_{F^5_5}(\mathcal{M}_1, \mathcal{M}_3) = 1 - 0.610905741 = 0.38909426.$$
$$s_2(\mathcal{M}_2, \mathcal{M}_3) = 1 - d^{nE}_{F^5_5}(\mathcal{M}_2, \mathcal{M}_3) = 1 - 0.62175897 = 0.37824103.$$

Clearly the similarity measures s_1 and s_2 satisfy all the conditions of Definition 4.10 in the case when the objects in (m, n)-FSs are not equal.

5. APPLICATIONS OF (m, n)-FUZZY SETS IN PATTERN RECOGNITION

The proposed similarity measures can be used in students selection in university, medical diagnosis of disease, plant leaf disease classifications, construction material selections and other MADM problems. The following example illustrate to recognized an unknown pattern using proposed similarity measures.

Example 5.1. *Let us consider three known patterns $\mathcal{M}_i (i = 1, 2, 3)$ which are represented by the (4,5)-fuzzy sets $\mathcal{M}_i (= 1, 2, 3)$ in the feature space as $\mathbb{P} = \{p_1, p_2, p_3\}$:*

$$\mathcal{M}_1 = \{< p_1, 0.5, 0.6 >, < p_2, 0.7, 0.7 >, < p_3, 0.4, 0.7 >\},$$
$$\mathcal{M}_2 = \{< p_1, 0.9, 0.8 >, < p_2, 0.8, 0.8 >, < p_3, 0.7, 0.6 >\},$$
$$\mathcal{M}_3 = \{< p_1, 0.5, 0.7 >, < p_2, 0.6, 0.6 >, < p_3, 0.7, 0.5 >\}.$$

Consider an unknown pattern $\mathcal{M} \in \mathcal{F}_4^5(\mathbb{P})$ that will be recognized, where

$$\mathcal{M} = \{< p_1, 0.9, 0.7 >, < p_2, 0.8, 0.7 >, < p_3, 0.6, 0.8 >\}.$$

Then, the proposed similarity measures s_1 and s_2 which have been computed from $\mathcal{M}$ to $\mathcal{M}_i (i = 1, 2, 3)$ are given in Table 2. From the numerical results presented in Table 1, we know that the similarity measures between $\mathcal{M}_2$ and $\mathcal{M}$ are the largest.

Table 2: Similarity measures between $\mathcal{M}_i (i = 1, 2, 3)$ and $\mathcal{M}$

Similarity Measure	$(\mathcal{M}_1, \mathcal{M})$	$(\mathcal{M}_2, \mathcal{M})$	$(\mathcal{M}_3, \mathcal{M})$
$s_1(\mathcal{M}_i, \mathcal{M})$	0.7780050828	0.8487350560	0.7588517561
$s_2(\mathcal{M}_i, \mathcal{M})$	0.7679178398	0.8382484863	0.7624882976

6. CONCLUDING REMARKS

The complement, necessity, possibility, and arithmetic operations over (m, n)-FSs are defined and several theorems related to properties of these operations have been established in this paper. Furthermore some distance and similarity measures over (m, n)-FSs are created and their validity are verified by taking suitable examples. An example of the applications of similarity measures proposed in pattern recognition is presented. In the future studied the hybrid structure of (m, n)-fuzzy sets with soft sets and rough sets can be created and thier applications in differnt filed of mathematics can be studied. Further more the dice, cosine and cotangent similarity measures and weighted aggradation operators of (m, n)-FSs can be created and their applications in MADM and MCDM problems can be examined.

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