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HAMILTONICITY AND EULERIANITY OF SOME BIPARTITE GRAPHS ASSOCIATED TO FINITE GROUPS

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Abstract.
Let G be a finite group. Associate a simple undirected graph Γ_G with G, called a bipartite graph associated to elements and cosets of subgroups of G, as follows : Take $G \cup S_G$ as the vertices of Γ_G, with S_G is the set of all subgroups of group G and $a \in G$ and $H \in S_G$ if and only if $aH = Ha$. In this paper, hamiltonicity and Eulerianity of Γ_G for some finite groups G are studied. In particular, the results obtained that for any cyclic group G, Γ_G is hamiltonian if and only if $|G| = 2$ and Γ_G is Eulerian if and only if $|G|$ is an even non-perfect square number. Also, we prove that Γ_{D_n} is Eulerian if k is even and $n = 2^k$ and $\Gamma(D_n)$ is not Eulerian for some other cases of n.

Key words and Phrases: bipartite graph, hamiltonian graph, Eulerian graph, semi-Eulerian graph, finite group.

1. INTRODUCTION

Let Γ be a connected graph. We denote the sets of vertices and edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree of a vertex a in Γ is the number of edges incident to a and it is denoted by $\deg(a)$. A graph Γ is bipartite graph if its vertices can be split into two independent sets so that no two vertices within the same set are adjacent. Bipartite graph Γ is said to be complete if every vertex in one set is adjacent to each vertex in other. A connected graph Γ is hamiltonian if there exists a cycle containing every vertex in Γ exactly once. Such a cycle is called hamiltonian cycle. A closed trail that meets every edge of Γ is called Eulerian trail. A graph Γ that contains Eulerian trail is called Eulerian graph. Non Eulerian graph is semi-Eulerian if there exists a trail that contains every edge of Γ.

Throughout this paper, for any subgroup H of G and $a \in G$, the left and the right coset of H containing a is $aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$,

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respectively. If $aH = Ha$ for every $a \in G$, then H is called normal subgroup of G. Moreover, D_n denotes a dihedral group of degree n. For other elemental definitions in graph and group theory, we refer to [1],[2],[3],[4],[5].

Algebraic graph theory has become a substantial attention in the last several decades. The first notion of such interplay is the concept of Cayley graph [6], which is connecting graph theory and group theory. Other different concept of a graph defined on group theory can be found in [7],[8],[9]. In 2021, Al-Kaseasbeh and Erfanian [10] defined a bipartite graph associated to elements and cosets of subgroups of a finite graph. Moreover, they gave some basic properties of Γ_G including diameter, girth, connectivity, completeness, dominating number, planarity and outer planarity. Also, they shed light on the hamiltonicity of Γ_{D_n}.

In section 2, we recall some definitions, examples, and basic characteristics of Γ_G for arbitrary group G including connectivity, completeness, and hamiltonicity. Also, we give the chromatic index and Eulerianity of the graph. In section 3, we give a necessary and sufficient condition of Γ_G connected to hamiltonicity and Eulerianity for cyclic group G. In section 4, we recall some definitions and properties of dihedral group D_n and determine Eulerianity of Γ_{D_n} for many cases of n.

2. SOME PROPERTIES OF Γ_G

In this section, we recall a definition of Γ_G and give some examples in order to give a perspective of this graph. Also, we recall some basic properties of Γ_G from [10] and give some other characteristics of this graph.

Definition 2.1. [10] *Let G be a finite group. A bipartite graph associated to elements and cosets of subgroups of G denoted by Γ_G is the simple undirected graph with the vertex set $V(\Gamma_G) = G \cup S_G$ and two vertices $a \in G$ and $H \in S_G$ are adjacent if and only if $aH = Ha$.*

In following example, we give Γ_G for a non abelian group G.

Example 2.2. *Consider the dihedral group $D_3 = \{e, a, b, b^2, ab, ab^2\}$. Then, we have $S_{D_3} = \{\{e\}, \langle e, a\rangle, \langle e, ab\rangle, \langle e, ab^2\rangle, \langle b\rangle, \langle b, a\rangle\}$. It it clear that $e \in D_3$ is adjacent with all vertices in S_{D_3}. It is also obvious that the subgroups $\{e\}, \langle b\rangle$, and $\langle b, a\rangle$ are normal in D_3. Therefore, $\{e\}, \langle b\rangle$, and $\langle b, a\rangle$ are adjacent with all vertices in the set D_3. By definition, the vertices $a, ab, ab^2 \in D_3$ are adjacent with $\langle e, a\rangle, \langle e, ab\rangle$, and $\langle e, ab^2\rangle$, respectively. The graph $\Gamma(D_3)$ is the graph as shown in Figure 1.*

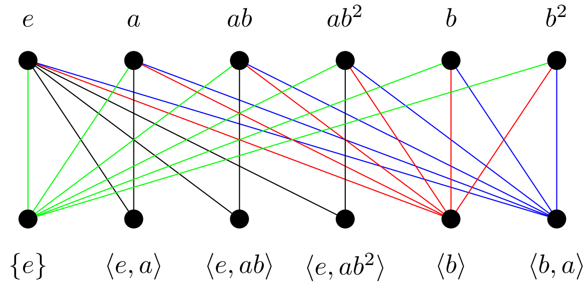

Figure 1. $\Gamma(D_3)$

In the following example we give Γ_G for an abelian group G.

Example 2.3. *Let $G = \mathbb{Z}_6$. We have $S_{\mathbb{Z}_6} = \{\{e\}, \langle 2\rangle, \langle 3\rangle, \mathbb{Z}_6\}$. Since all subgroup of $\mathbb{Z}_6$ are normal, every vertex in $S_{\mathbb{Z}_6}$ is adjacent with all vertices in S_3. Hence, $\Gamma(\mathbb{Z}_6)$ is a complete bipartite graph as shown in Figure 2.*

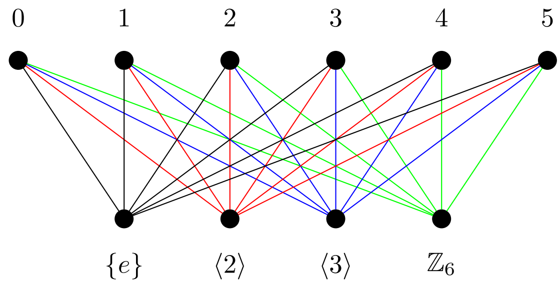

Figure 2. $\Gamma(\mathbb{Z}_6)$

In 2021, Al-Kaseasbeh and Erfanian determined the connectivity of Γ_G and gave a necessary and sufficient condition of Γ_G to be a complete bipartite graph for any group G as follows.

Theorem 2.4. [10] *The graph Γ_G is connected with diam$(\Gamma_G) \leq 3$.*

Theorem 2.5. [10] *The finite group G is Dedekind group if and only if Γ_G is a complete bipartite graph.*

The chromatic index of Γ is the minimum number of colors coloring edges of Γ such that no two adjacent edges have the same color. Note that the chromatic index of a bipartite graph Γ is the largest vertex degree of Γ [5]. Based on this fact, we give the following proposition.

Proposition 2.6. *Let G be a group. Then the chromatic index of Γ_G is* $\max\{|G|, |S_G|\}$.

Al-Kaseasbeh and Erfanian gave a necessary condition of Γ_G to be a hamiltonian graph.

Theorem 2.7. [10] *Let Γ_G be a hamiltonian graph. Then $|G| = |S_G|$.*

Recall that a connected graph Γ is Eulerian if and only if the degree of each vertex of Γ is even [5]. By this fact, we determine a necessary condition of Γ_G to be Eulerian graph.

Theorem 2.8. *If Γ_G is a Eulerian graph, then both $|G|$ and $|S_G|$ are even.*

Proof. Suppose that Γ_G is Eulerian and $V(\Gamma_G) = G \cup S_G$. Assume that $|G|$ or $|S_G|$ is odd. Note that the vertex $\{e\} \in S_G$ is adjacent to all vertices in G. Therefore, if $|G|$ is odd, then the vertex $\{e\} \in S_G$ has odd degree which is a contradiction to Eulerianity of Γ_G. Also, note that the vertex $e \in G$ is adjacent to all vertices in S_G. Again, if $|S_G|$ is odd, then the vertex $e \in G$ has odd degree which implies a contradiction. □

3. HAMILTONICITY AND EULERIANITY OF Γ_G WITH G IS CYCLIC

In this section, we examine hamiltonicity and Eulerianity of Γ_G for arbitrary finite cyclic group G. Let G be a finite cyclic group of order n. Recall that the number of all subgroups of G is $\tau(n)$, with $\tau(n)$ is the number of divisors of n. Hence, $|S_G| = \tau(n)$.

In the following theorem, we give the necessary and sufficient condition of Γ_G to be a hamiltonian graph.

Theorem 3.1. *Let G be a nontrivial finite cyclic group. The graph $\Gamma(G)$ is hamiltonian if and only if $|G| = 2$.*

Proof. Consider that G is a cyclic group of order $n \geq 3$. Note that every natural number $n > 1$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ for some different prime numbers $p_1, p_2, \dots, p_3$ and $\alpha_i \in \mathbb{N}$ for every $i = 1, 2, \dots, m$. Therefore, we have

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1).$$

Moreover, for $p_i \neq 2$ and $\alpha_i \neq 1$ we have

$$p_i^{\alpha_i} > \alpha_i + 1$$

for every $i = 1, 2, \dots, m$. Thus, we have

$$\begin{aligned} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m} &> (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1) \\ n &> \tau(n) \\ |G| &> |S_G|. \end{aligned}$$

In other words, $|G| \neq |S_G|$ implies that Γ_G is not hamiltonian.
($\Leftarrow$) Suppose $G = \{e, x\}$. We know that the set of all subgroups of G is $S_G = \{\{e\}, G\}$. It is obvious that both $e \in G$ and $x \in G$ are adjacent to every vertex in S_G. Thus, the graph Γ_G is a cyclic graph which is obviously hamiltonian. □

Also, we give the necessary and sufficient condition of Γ_G to be a Eulerian graph.

Theorem 3.2. *Let G be a cyclic group of order n. The graph Γ_G is Eulerian if and only if n is an even non-square number.*

Proof. ($\Rightarrow$) Assume that Γ_G is Eulerian. Therefore, the degree of each vertex of Γ_G is even. Note that the group G is a Dedekind group since G is an abelian group. Thus, the graph Γ_G is a complete bipartite graph. Based on those two facts, both $|G| = n$ and $|S_G| = \tau(n)$ are even. Note that n is a square number if and only if it has odd number of positive divisors. Hence, n is an even non-square number.
($\Leftarrow$) Since n is an even non-square number, n has even number of positive divisors. Hence, $|S_G| = \tau(n)$ is even. Note that Γ_G is a complete bipartite group, since G is an abelian group. Therefore, every vertex in G is adjacent to all vertices in S_G and every vertex in S_G is adjacent to all vertices in G. Hence, the degree of every vertex $a \in G$ and $H \in S_G$ is $|S_G|$ and $|G|$, respectively. Since both $|G|$ and $|S_G|$ are even, the degree of every vertex of Γ_G are even implying that Γ_G is Eulerian. □

4. EULERIANITY OF BIPARTITE GRAPH Γ_{D_n}

A group generated by two elements a and b such that $a^2 = b^n = e$ and $ba = ab^{-1}$ is dihedral group of order $2n$ and denoted by D_n. In this section, we examine the Eulerianity of the graph Γ_{D_n}. We prove that for even number k, $\Gamma_{D_{2k}}$ is Eulerian and we show that Γ_{D_n} is not Eulerian for several cases. First, we start with some properties of dihedral group D_n.

Next, we recall some properties of dihedral group D_n in the following lemmas. For simplicity, we write $D_n = \langle a, b : a^2 = b^n = e, ba = ab^{-1}\rangle$ to define this group.

Lemma 4.1. *Every subgroup $\langle b^d \rangle$ of dihedral group D_n is normal, with $d|n$.*

Lemma 4.2. *If n is odd, then $Z(D_n) = \{e\}$. If n is even, then $Z(D_n) = \{e, b^{\frac{n}{2}}\}$.*

Lemma 4.3. [2] *The number of subgroups of dihedral group D_n is $\tau(n) + \sigma(n)$, with $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n.*

Also, we give the following lemma.

Lemma 4.4. *Let $D_n = \langle a, b : a^2 = b^n = e, ba = ab^{-1}\rangle$ be a dihedral group of order $2n$ and $m = \frac{n}{d}$ such that $d|n$. Let $\langle b^d, ab^i \rangle$ be a subgroup of D_n with $0 \leq i < d$.*

Then, $x\langle b^d, ab^i\rangle = \langle b^d, ab^i\rangle x$ *if and only if*

$$x \in \{b^j, ab^t : j = \frac{dp}{2}, t = \frac{dp}{2} + i, 0 \leq p < 2m\},$$

for every $x \in D_n$.

Proof. For $\langle b^d, ab^i\rangle$ with $d|n$ and $n = md$, we have

$$\begin{aligned}\langle b^d, ab^i\rangle &= \{b^{dk} : 0 \leq k < m\} \cup \{ab^{i+kd} : 0 \leq k < m\}\\ &= \langle b^d\rangle \cup \{ab^{i+kd} : 0 \leq k < m\}.\end{aligned}$$

By Lemma 4.1, we know that $\langle b^d\rangle$ for $d|n$ is a normal subgroup of D_n. Then, it is obvious that $x\langle b^d\rangle = \langle b^d\rangle x$ for every $x \in D_n$. Hence, we only examine the set

$$\{ab^{i+kd} : 0 \leq k < m\} \subset \langle b^d, ab^i\rangle.$$

Then, we may consider the following two cases.

(1) For $x = b^j$ with $0 \leq j < n$, we have

$$b^j\{ab^{i+kd} : 0 \leq k < m\} = \{ab^{i+kd} : 0 \leq k < m\}b^j$$

$$\{ab^{(i+kd)-j} : 0 \leq (i+kd) - j < n, 0 \leq k < m\} = \{ab^{(i+kd)+j} : 0 \leq (i+kd) + j < n, 0 \leq k < m\},$$

if and only if $i + kd - j \equiv i + kd + j \mod n$. Therefore, by some properties of modular arithmetic we get $j \equiv \frac{d}{2} \mod n$. Hence, we have $j \in \{\frac{dp}{2} : 0 \leq p < 2m\}$.

(2) For $x = ab^t$ with $0 \leq t < n$, we have

$$ab^t\{ab^{i+kd} : 0 \leq k < m\} = \{ab^{i+kd} : 0 \leq k < m\}ab^t$$

$$\{b^{-t+(i+kd)} : 0 \leq -t + (i+kd) < n, 0 \leq k < m\} = \{b^{t-(i+kd)} : 0 \leq t - (i+kd) < n, 0 \leq k < m\},$$

if and only if $-t+i+kd \equiv t-i-kd \mod n$. Therefore, by some properties of modular arithmethic we get $t \equiv \frac{d}{2} + i \mod n$. Hence, $t \in \{\frac{dp}{2} + i : 0 \leq p < 2m\}$.

By two cases above, we have considered all cases of $x \in D_n$. Thus, the proof is complete. □

As a consequence of Lemma 4.4, we give the following corollaries.

Corollary 4.5. *Let* $b^j \in D_n$, *with* $0 \leq j < n$. *Then,* $b^j\langle b^d, ab^i\rangle = \langle b^d, ab^i\rangle b^j$ *if and only if* $d|2j$.

Corollary 4.6. *Let* $ab^t \in D_n$, *with* $0 \leq t < n$. *Then,* $ab^t\langle b^d, ab^i\rangle = \langle b^d, ab^i\rangle ab^t$ *if and only if* $i \equiv t \mod d$.

Now, we want to show for what values of n, Γ_{D_n} is Eulerian or not. Obviously $|D_n| = 2n$ is even. Therefore, by Theorem 2.8, if $|S_{D_n}|$ is odd, then Γ_{D_n} cannot be Eulerian. We give a value of n where $|S_{D_n}|$ is odd in Theorem 4.8. We also show that for that n, Graph Γ_{D_n} is not semi-Eulerian based on the fact that a connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree [5].

Lemma 4.7. *Let $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ for some different odd prime numbers $p_1, p_2, \dots, p_n$. If k is odd and α_i is even for every $i = 1, 2, \dots, m$, then $\tau(n) + \sigma(n)$ is odd.*

Proof. Since k is odd, $\tau(n) = (k+1)(\alpha_1+1)(\alpha_2+1)\dots(\alpha_m+1)$ is even. Now, note that every factor of $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ is odd since p_i for $i = 1, 2, \dots, m$ are different odd prime numbers, and $\tau(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}) = (\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_m + 1)$ is odd since α_i is even for every $i = 1, 2, \dots, m$. Thus, $\sigma(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m})$ is odd. Also, note that all odd factors of n are all factors of $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. Consequently, $\sigma(n)$ is odd. Thus, $\tau(n) + \sigma(n)$ is odd. □

Theorem 4.8. *Let $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ with p_i for $i = 1, 2, \dots n$ are different odd prime numbers, k is odd, and α_i for every $i = 1, 2, \dots, m$ are even. Then, Γ_{D_n} is not Eulerian. Moreover, Γ_{D_n} is not semi-Eulerian.*

Proof. By Lemma 4.7, we know that $|S_{D_n}| = \tau(n) + \sigma(n)$ is odd. Therefore, Γ_{D_n} is not Eulerian. Next, we are going to show that Γ_{D_n} is not semi-Eulerian. It is obvious that n is even. Therefore, by Lemma 4.2, we have $Z(D_n) = \{e, b^{\frac{n}{2}}\}$. Thus, the vertices $e, b^{\frac{n}{2}} \in D_n$ are adjacent to every vertex in S_{D_n}. Consequently, $\deg(e) = \deg(b^{\frac{n}{2}}) = |S_{D_n}|$ is odd. Moreover, by Lemma 4.1 and Corollary 4.5, the vertex $b \in D_n$ is adjacent to every vertex in $\{\langle b^d \rangle : d|n\} \cup \{\langle b^d, ab^i \rangle : d|2, 0 \le i < d\}$. Hence, $\deg(b) = |\{\langle b^d \rangle : d|n\}| + |\{\langle b^d, ab^i \rangle : d|2, 0 \le i < d\}| = \tau(n) + 3$. Since $\tau(n)$ is even, $\deg(b)$ is odd. Thus, at least we have three vertices with odd degree. In other words, Γ_{D_n} is not semi-Eulerian. □

For some cases of n such that $|S_{D_n}|$ is even, we should investigate the degree of each vertex in $V(\Gamma_{D_n})$ to determine whether the graph is Eulerian or not. We start with the following simple lemmas.

Lemma 4.9. *For every $0 \le j < n$, degree of vertex $b^j \in D_n$ is $\tau(n) + \sum_{d|2j} d$.*

Proof. By Lemma 4.1, every vertex in D_n is adjacent to every vertex $\langle b^d \rangle \in S_{D_n}$, with $d|n$. By Corollary 4.5, every $b^j \in D_n$ is adjacent to every vertex in $\{\langle b^d, ab^i \rangle : 0 \le i < d, d|2j\}$. Hence, $\deg(b^j) = \tau(n) + |\{\langle b^d, ab^i \rangle : 0 \le i < d, d|2j\}| = \tau(n) + \sum_{d|2j} d$. □

Lemma 4.10. *For every $0 \le t < n$, degree of vertex $ab^t \in D_n$ is $2\tau(n)$.*

Proof. By Corollary 4.6, every vertex $ab^t \in D_n$ is adjacent to every vertex in $\{\langle b^d, ab^i \rangle : d|n, 0 \le i < d, \text{ and } i \equiv t \mod n\}$. Also, by Lemma 4.1, every vertex in D_n is adjacent to every vertex in $\{\langle b^d \rangle : d|n\}$. Hence, $\deg(ab^t) = |\{\langle b^d, ab^i \rangle : d|n, 0 \le i < d, \text{ and } i \equiv t \mod n\}| + |\{\langle b^d \rangle : d|n\}| = \tau(n) + \tau(n) = 2\tau(n)$. □

Lemma 4.11. *Every vertex $H \in S_{D_n}$ has even degree.*

Proof. By Lemma 4.1, every vertex $\langle b^d \rangle \in S_{D_n}$ is adjacent to every vertex in D_n, with $d|n$. Hence, $\deg(\langle b^d \rangle) = |D_{2^k}|$, which is obviously even. Moreover, by Lemma

4.4, every vertex $\langle b^d, ab^i\rangle$ is adjacent to every vertex in $\{b^j, ab^t : j = \frac{dp}{2}, t = \frac{dp}{2} + i, 0 \le p < 2m\}$, with $d|n, m = \frac{n}{d}$, and $0 \le i < d$. Hence,

$$\begin{aligned}\deg(\langle b^d, ab^i\rangle) &= |\{b^j, ab^t : j = \frac{dp}{2}, t = \frac{dp}{2} + i, 0 \le p < 2m\}| \\ &= |\{b^j : j = \frac{dp}{2}, 0 \le p < 2m\}| + |\{ab^t : t = \frac{dp}{2} + i, 0 \le p < 2m\}| \\ &= 2|\{\frac{p}{2} : 0 \le p < 2m\}|,\end{aligned}$$

which is also obviously even. □

By Lemma 4.9, 4.10, and 4.11, we know that to determine whether the graph Γ_{D_n} is Eulerian or not, it is enough to investigate the degree of each vertex $b^j \in D_n$. In the following theorem, we show that Γ_{D_n} for $n = 2^k$ is Eulerian, with k is even.

Theorem 4.12. *For every even number k, $\Gamma_{D_{2^k}}$ is Eulerian.*

Proof. Since k is even, both $\tau(2^k) = k + 1$ and $\sigma(2^k) = 1 + 2 + 2^2 + \cdots + 2^k$ are odd. Let us start by investigating degree of every vertex $b^j \in D_{2^k}$. By Lemma 4.9, we have

$$\begin{aligned}\deg(b^j) &= |\{\langle b^d, ab^i\rangle : 0 \le i < d, d|2j\}| + \tau(2^k) \\ &= |\{\langle b, a\rangle\}| + |\{\langle b^d, ab^i\rangle : 0 \le i < d, d|2j, d \ne 1\}| + \tau(2^k) \\ &= 1 + \sum_{d|2j, d\ne 1} d + \tau(2^k).\end{aligned}$$

Since k is even, d is even whenever $d|2^k$ and $d \ne 1$. Therefore, $\sum_{d|2j,d\ne1} d$ is even. Thus, $\deg(b^j)$ is even for every $b^j \in D_{2^k}$. Moreover, by Lemma 4.10, it is obvious that the degree of every vertex $ab^t \in D_{2^k}$ is even. Thus, the degree of every vertex in D_{2^k} is even. Also, by Lemma 4.11, every vertex in $S_{D_{2^k}}$ is even. Hence, the degree of every vertex in $V(\Gamma_{D_{2^k}})$ is even which implies $\Gamma_{D_{2^k}}$ is Eulerian. □

In the next remaining theorems, we give several cases of n such that $|S_{D_n}|$ is even, but Γ_{D_n} is not Eulerian.

Theorem 4.13. *The graph $\Gamma_{D_{3^k}}$ is semi-Eulerian, for $k = 1, 2$.*

Proof. For $k = 1$, we have Γ_{D_3} represented in Figure 1. From the figure, we know that every vertex of Γ_{D_3} is of even degree except b and b^2. Hence, Γ_{D_3} is semi-Eulerian. For $k = 2$, we should investigate the degree of every vertex $b^j \in \Gamma_{D_9}$. By Lemma 4.9, we have $\deg(b^j) = \tau(9) + \sum_{d|2j} d = 3 + \sum_{d|2j} d$. For every $0 \le j < 9$, the degree of b^j is in the following table.

j	$d\|2j$	$\sum_{d\|2j} d$	$\deg(b^j)$
1	1	1	4
2	1	1	4
3	1,3	4	7
4	1	1	4
5	1	1	4

j	$d\|2j$	$\sum_{d\|2j} d$	$\deg(b^j)$
6	1,3	4	7
7	1	1	4
8	1	1	4
9	1,3,9	13	16

Table 1. Degree of $b^j \in D_9$

From the table, it is clear that Γ_{D_9} has two vertices in the form of b^j of odd degree. Note that by Lemma 4.10 and Lemma 4.11, the degree of each vertex $ab^t \in D_9$ and $H \in S_{D_9}$ is even, respectively. Thus, the graph Γ_{D_9} has exactly two vertices of odd degree which implies that the graph is semi-Eulerian. □

Theorem 4.14. *For every $k \geq 3$, the graph $\Gamma_{D_{3^k}}$ is neither Eulerian nor semi-Eulerian.*

Proof. To prove that $\Gamma_{D_{3^k}}$ is neither Eulerian nor semi-Eulerian, it is enough to show that at least there are three vertices in $V(\Gamma_{D_{3^k}})$ that have odd degree, for every $k \geq 3$. We may consider the following two cases.

(1) For k is odd, $\tau(3^k)$ is even. Since $k \geq 3$, we have $3^k \geq 27$. Therefore, vertices $b, b^2, b^3 \in D_{3^k}$. On the other hand, by Lemma 4.9 we have $\deg(b^j) = \tau(3^k) + \sum_{d|2j,d|3^k} d = \tau(3^k) + 1$, which is odd for every $j = 1, 2, 4$.
(2) For k is even, $\tau(3^k)$ is odd. Since $k \geq 3$, we have $3^k \geq 27$. Therefore, vertices $b^3, b^6, b^{12} \in D_{3^k}$. On the other hand, by Lemma 4.9 we have $\deg(b^j) = \tau(3^k) + \sum_{d|2j,d|3^k} d = \tau(3^k) + (1 + 3) = \tau(3^k) + 4$, which is odd for every $j = 3, 6, 12$.

Since for all $k \geq 3$ we can find three vertices of odd degree, it is proved that $\Gamma_{D_{3^k}}$ is neither Eulerian nor semi-Eulerian. □

Theorem 4.15. *For all $k \geq 1$ and prime numbers $p \geq 5$, the graph Γ_{D_n} is neither Eulerian nor semi-Eulerian if $n = p^k$.*

Proof. We consider the following cases.

(1) For k is odd, $\tau(p^k)$ is even. Note that for every $k \geq 1$ and $p \geq 5$, we have $p^k \geq 5$. Therefore, the vertices $b, b^2, b^3 \in \Gamma_{D_{p^k}}$. Moreover, by Lemma 4.9 we have $\deg(b^j) = \tau(p^k) + \sum_{d|2j,d|p^k} d = \tau(p^k) + 1$ for every $j = 1, 2, 3$. Since $\tau(p^k)$ is even, $\deg(b^j)$ is odd for every $j = 1, 2, 3$.
(2) For k is even, $\tau(p^k)$ is odd. Note that for every $k \geq 2$ and $p \geq 5$, we have $p^k \geq p^2 > 3p$. Therefore, the vertices $b^p, b^2p, b^3p \in D_{p^k}$. Moreover, by Lemma 4.9 we have $\deg(b^j) = \tau(p^k) + \sum_{d|2j,d|p^k} d = \tau(p^k) + (p + 1)$ for every $j = p, 2p, 3p$. Since $\tau(p^k)$ is odd $\deg(b^j)$ is odd for every $j = p, 2p, 3p$.

By the two cases above, we know that at least there are three vertices in $V(\Gamma_{D_{p^k}})$ that have odd degree, which implies that $\Gamma_{D_{p^k}}$ is neither Eulerian nor semi-Eulerian. □

Theorem 4.16. *Let $n = p_1p_2\dots p_m$ for some different odd prime numbers $p_1, p_2, \dots, p_m$ and $m \geq 2$. The graph Γ_{D_n} is neither Eulerian nor semi-Eulerian.*

Proof. For $n = p_1p_2\dots p_m$ with $p_1, p_2, \dots, p_m$ are some different odd prime numbers and $m \geq 2$, we have $\tau(n) = 2^m$ and $n \geq 15$. Therefore, the vertices $b, b^2, b^4 \in D_n$. By Lemma 4.9, we have $\deg(b^j) = \tau(n) + \sum_{d|j,d|n} d = 2^m + 1$, for every $j = 1, 2, 4$. Note that 2^m is even. Thus, the degree of b^j for every $j = 1, 2, 4$ is odd. In other words, there are three vertices in $V(\Gamma_{D_n})$ that have odd degree which implies that Γ_{D_n} is neither Eulerian nor semi-Eulerian. □

5. Conclusions

We have already studied a bipartite graph Γ_G which is especially connected to hamiltonicity and Eulerianity for some finite groups. In this paper, we have obtained the necessary and sufficient condition for hamiltonicity and Eulerianity of Γ_G with G is a finite cyclic group. However, we have not obtained a condition for graph Γ_G to be semi-Eulerian.

The hamiltonicity of the graph Γ_{D_n} has been discussed in [10]. For the Eulerianity of Γ_{D_n}, we have shown that $\Gamma_{D_{2^k}}$ is Eulerian for every even number k and $\Gamma_{D_{3^k}}$ is semi Eulerian for $k = 1, 2$. Also, we have shown that Γ_{D_n} is neither Eulerian nor semi-Eulerian for some cases of n. However, all cases of n given in this paper do not cover all the existing cases. One may continue further research for the remaining cases.

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