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TWO IMPORTANT EXTENSION THEOREMS FOR THE GAP-INTEGRAL

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Abstract. The concept of the GAP-integral was introduced by the authors [4]. In this paper Cauchy and Harnack extensions for the GAP-integral on the real line are presented.

Key words and Phrases: Approximate full cover, density point, Δ-division, δ-fine partition, GAP-integral, Henstock integral, Saks-Henstock lemma, approximate strong Lusin condition.

Abstrak. Konsep tentang GAP-integral telah diperkenalkan oleh penulis [4]. Pada paper ini dibahas perluasan Cauchy dan Harnack untuk GAP-integral pada bilangan riil.

Kata kunci: Pendekatan selimut penuh, titik padatan, pembagian-Δ, partisi halus-δ, GAP-integral, integral Henstock, lema Saks-Henstock, pendekatan kondisi Lusin kuat.

1. Introduction

The Approximately Continuous Perron integral was introduced by Burkill [2] and its Riemann-type definition was given by Bullen [1]. Schwabik [8] presented a generalized version of the Perron integral leading to the new approach to a Generalized Ordinary Differential Equation. The authors introduced the concept of the Generalized Approximately Continuous Perron (GAP) integral and established some significant results of this integral in several papers [3, 4, 5].

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It is well-known that the Cauchy extension was first used on the Riemann integral to integrate some functions unbounded in the neighbourhood of a finite number of points. Another extension, first given by Harnack, extends the integral to an open set if the integral exists over each component interval of the open set and if some other conditions are satisfied.

Attempt has been made in this paper to establish Cauchy and Harnack extensions for the GAP-integral together with some significant results.

2. Preliminaries

Definition 2.1. [6] *Let E be a measurable set and let c be a real number. The density of E at c is defined by*

$$d_cE = \lim_{h\to 0^+} \frac{\mu(E\cap(c-h,c+h))}{2h},$$

provided the limit exists. It is clear that $0 \le d_cE \le 1$ when it exists. The point c is a point of density of E if $d_cE = 1$.

Definition 2.2. *A collection Δ of closed subintervals of $[a,b]$ is called an approximate full cover (AFC) if for every $x \in [a,b]$ there exists a measurable set $D_x \subset [a,b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u,v] \in \Delta$ whenever $u,\ v \in D_x$ and $u \le x \le v$.*

For all approximate full covers that occur in this paper the sets $D_x \subset [a,b]$ are the same. Then the relations $\Delta_1 \subseteq \Delta_2$ or $\Delta_1 \cap \Delta_2$ for approximate full covers $\Delta_1,\ \Delta_2$ are clear.

A division of $[a,b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is called a Δ-division if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, if we have $x_{i-1} \le \xi_i \le x_i$ and $x_{i-1},\ x_i \in D_{\xi_i}$ for all i. We call ξ_i the associated point of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \ldots, n)$ the division points.

A division of $[a,b]$ given by $a \le y_1 < z_1 < y_2 < z_2 \cdots < y_m < z_m \le b$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ is called a Δ-partial division if

$$\bigcup_{i=1}^{m} [y_i, z_i] \subset [a,b]$$

and Δ is an approximate full cover with $[y_i, z_i]$ coming from Δ or more precisely, $y_i \le \zeta_i \le z_i$ and $y_i,\ z_i \in D_{\zeta_i}$ for all i.

Let $E \subset [a,b]$. Given a family of measurable sets $D_x \subset [a,b]$ for each $x \in E$ such that $x \in D_x$ and D_x has density 1 at x, if $u,\ v \in D_\zeta$ with $\zeta \in [u,v]$, we say that ζ is an associated point of $[u,v]$ and u, v the division points. The set of all

$([u,v],\zeta)$, where $\zeta \in E$, is called an approximate full cover on E.

We say that a finite set of non-overlapping $([u_i,v_i],\zeta_i) \in \Delta$, where $\zeta_i \in E$ for all $i = 1,\ldots,p$, is a Δ-partial division on E.

In [4], the GAP-integral is defined as follows :

Definition 2.3. *A function $U : [a,b] \times [a,b] \to R$ is said to be Generalized Approximate Perron (GAP)-integrable to a real number A if for every $\epsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have*

$$|(D)\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - A| < \epsilon$$

and we write $A = (GAP)\int_a^b U$.

The set of all functions U which are GAP-integrable on $[a,b]$ is denoted by GAP$[a,b]$. We use the notation

$$S(U,D) = (D)\sum\{U(\tau,\beta) - U(\tau,\alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$.

Note that the integral is uniquely determined.

Remark 2.4. *If the AFC Δ in Definition 2.3. is replaced by an ordinary full cover that is, the family of all $([\alpha,\beta],\tau)$ which are δ-fine for some $\delta(\tau) > 0$, i.e. $\tau \in [\alpha,\beta]$, $[\alpha,\beta] \subset [\tau-\delta(\tau),\tau+\delta(\tau)]$, then we have a general definition of Henstock integral [7]. Setting $U(\tau,t) = f(\tau)t$ and $U(\tau,t) = f(\tau)g(t)$ where $f,g : [a,b] \to R$ and $\tau, t \in [a,b]$, we obtain Rienmann-type and Rienmann-Stieltjes type integrals respectively for the functions f, g and a given Δ-division D of $[a,b]$. Considering $U(\tau,t) = f(\tau)t$ in Definition 2.3., we obtain the classical approximately continuous Perron integral. This definition is given in a more general form because of the general form of the function U.*

With the notion of partial division we have proved the following theorem in [4].

Theorem 2.5. (Saks-Henstock Lemma) *Let $U : [a,b] \times [a,b] \to R$ be GAP-integrable over $[a,b]$. Then, given $\epsilon > 0$, there is an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D = \{([\alpha_{j-1},\alpha_j],\tau_j); j = 1,2,\ldots,q\}$ of $[a,b]$ we have*

$$\left|\sum_{j=1}^{q}\{U(\tau_j,\alpha_j) - U(\tau_j,\alpha_{j-1})\} - (GAP)\int_a^b U\right| < \epsilon.$$

Then, if $\{([\beta_j,\gamma_j],\zeta_j); j = 1,2,\ldots,m\}$ represents a Δ-partial division of $[a,b]$, we have

$$\left|\sum_{j=1}^{m}[\{U(\zeta_j,\gamma_j) - U(\zeta_j,\beta_j)\} - (GAP)\int_{\beta_j}^{\gamma_j} U]\right| < \epsilon.$$

In [4], the indefinite GAP-integral is defined as follows :

Definition 2.6. *Let $U \in GAP[a,b]$. The function $\phi : [a,b] \to R$ defined by $\phi(s) = (GAP)\int_a^s U,\ a < s \leq b,\ \ \phi(a) = 0$ is called the indefinite GAP-integral of U. For $[\alpha,\beta] \subset [a,b]$, we put $\phi(\alpha,\beta) = \phi(\beta) - \phi(\alpha) = (GAP)\int_\alpha^\beta U$.*

We need the following definitions to establish some results in the consequence:

Definition 2.7. [7] *A number A is said to be the approximate limit of a function f at x_0 if for every $\epsilon > 0$ there exists a set D of density 1 at x_0 such that*

$$|f(x) - A| < \epsilon \text{ for every } x \in D.$$

We write $\lim\limits_{x \to x_0} ap\ f(x) = A$.

Definition 2.8. [7] *Let $X \subset [a,b]$. A function F is said to be $AC(X)$ if for every $\epsilon > 0$ there is a $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i,b_i]\}$ satisfying $\sum\limits_i |b_i - a_i| < \eta$ we have $\sum\limits_i |F(b_i) - F(a_i)| < \epsilon$ with the end points $a_i,\ b_i \in X$ for all i.*

Definition 2.9. *A function F is said to satisfy the approximate strong Lusin condition (ASL), on a subset X of $[a,b]$, if for every set E of measure zero and for every $\epsilon > 0$ there exists an approximate full cover Δ on X such that for any Δ-partial division $D = ([\alpha,\beta],\tau)$ on $E \cap X$ i.e. $\tau \in E \cap X$, we have $(D)\sum |F(\beta) - F(\alpha)| < \epsilon$.*

3. Main Results

Theorem 3.1. *Let $U : [a,b] \times [a,b] \to R$ be such that $U \in GAP[a,c]$ for every $c \in [a,b)$ and let there exist a finite limit*

$$\lim_{c \longrightarrow b^-} \ ap\ \left[(GAP)\int_a^c U - U(b,c) + U(b,b)\right] = A. \tag{1}$$

Then $U \in GAP[a,b]$ and $(GAP)\int_a^b U = A$.

Proof. Let $\epsilon > 0$ be given. Let $a = a_0 < a_1 < \dots$ be an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of points $a_n \in [a,b)$ with $\lim\limits_{n \longrightarrow \infty} a_n = b$. By the assumption, we have $U \in GAP[a,a_n]$ for every $n = 1,2,\dots$. Therefore, for every $n = 1,2,\dots$ there exists an approximate full cover Δ_n of $[a,a_n]$ such that for any Δ_n-division $D = ([\alpha,\beta],\tau)$ of $[a,a_n]$ we have

$$\left|\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - (GAP)\int_a^{a_n} U\right| < \epsilon/2^{n+1},\ n = 1,2,\dots.$$

Again, since,

$$\lim_{c \to b^-} ap\quad [(GAP)\int_a^c U - U(b,c) + U(b,b)] = A,$$

for every $\epsilon > 0$ there exists a set D_b of density 1 at b with $b \in D_b$ and a positive integer N such that $|(GAP)\int_a^c U - U(b,c) + U(b,b) - A| < \epsilon$ whenever $c \in D_b - \{b\}$ and $b > c \geq a_N$. Let Δ_0 be an approximate full cover of $(c,b]$. Now for any $\tau \in [a,b)$ there is exactly one $m(\tau) = 1,2,\ldots$ such that $\tau \in [a_{m(\tau)-1}, a_{m(\tau)})$. Let Δ' be an approximate full cover of $[a, a_{m(\tau)})$ for given $\tau \in [a,b)$ such that $\Delta' \subseteq \Delta_{m(\tau)}$.

Let $c \in [a,b)$ be given and that $D' = \{([\alpha_{j-1}, \alpha_j], \tau_j); j = 1,2,\ldots,k-1\}$ be a Δ'-division of $[a,c]$. If $m(\tau_j) =$ n, then $[\alpha_{j-1}, \alpha_j] \subset [a, a_n]$ and also $([\alpha_{j-1}, \alpha_j], \tau_j) \in \Delta_n$. Let

$$\sum_{j=1,\ m(\tau_j)=n}^{k-1} \left[\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_{\alpha_{j-1}}^{\alpha_j} U \right]$$

be the sum of those terms in the corresponding total sum

$$\sum_{j=1}^{k-1} \left[\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_{\alpha_{j-1}}^{\alpha_j} U \right]$$

such that the associated points τ_j satisfy the relation $\tau_j \in [a_{n-1}, a_n)$. By the Saks-Henstock lemma, we obtain

$$\left| \sum_{j=1,\ m(\tau_j)=n}^{k-1} \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_{\alpha_{j-1}}^{\alpha_j} U \right| < \epsilon/2^{n+1}.$$

Then

$$\begin{aligned}
&\left| \sum_{j=1}^{k-1} \{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_a^c U \right| \\
&\quad = \left| \sum_{j=1}^{k-1} [\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_{\alpha_{j-1}}^{\alpha_j} U] \right| \\
&\quad \leq \sum_{n=1}^{\infty} \left| \sum_{j=1,\ m(\tau_j)=n}^{k-1} [\{U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})\} - (GAP)\int_{\alpha_{j-1}}^{\alpha_j} U] \right| \\
&\quad \leq \sum_{n=1}^{\infty} \epsilon/2^{n+1} = \epsilon
\end{aligned}$$

We now define an approximate full cover Δ of $[a,b]$ as follows: When $\tau \in [a,b)$, we choose the interval-point pair $([\alpha,\beta],\tau)$ from Δ' and we choose the interval-point pair $([\alpha,\beta],\tau)$ from Δ_0 if $\tau = b$. Set $\Delta = \Delta' \cup \Delta_0$. Then Δ is an approximate full cover of $[a,b]$. Let $D = \{([\alpha_{j-1}, \alpha_j], \tau_j); j = 1,2,\ldots,k\}$ be an arbitrary Δ-division of $[a,b]$. Then by the choice of the approximate full cover Δ we have $\tau_k = \alpha_k = b$ and $\alpha_{k-1} \in (a_N, b)$. Furthermore, $([\alpha_{k-1}, b], b) \in \Delta_0$ since $\alpha_{k-1} \in D_b - \{b\}$ and $\alpha_{k-1} < b$. Then we have

$$
\begin{aligned}
|S(U,D)-A| &= \left|\sum_{j=1}^{k-1}\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}+\{U(\tau_k,\alpha_k)-U(\tau_k,\alpha_{k-1})\}-A\right| \\
&\leq \left|\sum_{j=1}^{k-1}\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int_a^{\alpha_{k-1}} U\right| \\
&\quad + \left|(GAP)\int_a^{\alpha_{k-1}} U - U(b,\alpha_{k-1})+U(b,b)-A\right| \\
&< \epsilon+\epsilon = 2\epsilon.
\end{aligned}
$$

Hence $U \in GAP[a,b]$ and $(GAP)\int_a^b U = A$.

Remark 3.2. *A similar theorem also holds when the left hand limit is replaced by right hand limit as stated below :*

Theorem 3.3. *Let $U : [a,b]\times[a,b] \to R$ be such that $U \in GAP[c,b]$ for every $c \in (a,b]$ and let there exist a finite limit*

$$
\lim_{c\longrightarrow a^+} \ ap \ \left[(GAP)\int_c^b U - U(a,a)+U(a,c)\right] = A. \tag{2}
$$

Then $U \in GAP[a,b]$ and $(GAP)\int_a^b U = A$.

Theorem 3.1 and Theorem 3.3 given here represent the Cauchy extension of the GAP-integral.

Theorem 3.4. (Harnack extension) *Let X be a closed set in $[a,b]$ with H the set of all points of density in X. Suppose $(a,b)-X = \bigcup_{k=1}^{\infty}(a_k,b_k)$ with $a,\ b \in X$. If the conditions*
(i) U is GAP-integrable on X i.e. $U\chi_X$ is GAP-integrable on $[a,b]$ where $U\chi_X(\tau,t) = U(\tau,t)$ when $\tau,\ t \in X$ and 0 otherwise and U is GAP-integrable on each of the intervals $[a_k,b_k],\ k = 1,2,\ldots$
(ii) the series $\sum_{k=1}^{\infty}\left|(GAP)\int_{a_k}^{b_k} U\right|$ converges,
(iii) for any given $\epsilon > 0$ there exists an approximate full cover Δ_0 of $X-H$ such that for any Δ_0-partial division $D = ([\alpha,\beta],\tau)$ of $X-H$ we have,

$$
(D)\sum_{\beta\in(a_k,b_k)}\left|(GAP)\int_{a_k}^{\beta} U\right| < \epsilon
$$

and

$$(D)\sum_{\alpha\in(a_k,b_k)}\left|(GAP)\int_{\alpha}^{b_k}U\right|<\epsilon$$

where $\sum$ *runs over* β *or* α *whenever* $(\tau,\beta]\subset(a_k,b_k)$ *or* $[\alpha,\tau)\subset(a_k,b_k)$ *respectively such that* $\tau\in X-H$,
are satisfied, then $U\in GAP[a,b]$ *with*

$$(GAP)\textstyle\int_a^b U=(GAP)\int_a^b U\chi_X+\displaystyle\sum_{k=1}^{\infty}(GAP)\int_{a_k}^{b_k}U.$$

Proof. It is sufficient to prove that $U-U\chi_X$ is GAP-integrable on $[a,b]$. By (ii), for all $\epsilon>0$ there exists a positive integer n_0 such that

$$\sum_{k=n_0+1}^{\infty}\left|(GAP)\int_{a_k}^{b_k}U\right|<\epsilon.$$

By (i), for each $k\in N$ there exists an approximate full cover Δ_k of $[a_k,b_k]$ such that for all Δ_k-division $D=([\alpha,\beta],\tau)$ of $[a_k,b_k]$ we have

$$\left|\sum\{U(\tau,\beta)-U(\tau,\alpha)\}-(GAP)\int_{a_k}^{b_k}U\right|<\epsilon/2^k$$

We may suppose that for each $([\alpha,\beta],\tau)\in\Delta_k$, we have

(a) $[\alpha,\beta]\subset(a_k,b_k)$ when $\tau\in(a_k,b_k)$, $k=1,2,\ldots$
(b) $(\tau,\beta]\subset(a_k,b_k)$ when $\tau=a_k$ for some k, $k=1,2,\ldots$
(c) $[\alpha,\tau)\subset(a_k,b_k)$ when $\tau=b_k$ for some k, $k=1,2,\ldots$.

At each $\tau\in H$, there exists a set D_τ of density 1 at τ such that for $\alpha,\ \beta\in D_\tau$ with $\alpha\le\tau\le\beta$ we may suppose that $[a_k,b_k]\subset[\alpha,\beta]$ for some $k>n_0$. The set $\{[\alpha,\beta]:\alpha\le\tau\le\beta\ and\ [a_k,b_k]\subset[\alpha,\beta]\ for\ some\ k>n_0\}$ defines an approximate full cover Δ_H of H. We now define an approximate full cover Δ of $[a,b]$ as follows: When $\tau\in[a_k,b_k]$ for some $k=1,2,\ldots$, we choose the interval-point pair $([\alpha,\beta],\tau)$ from Δ_k, when $\tau\in H$, we choose the interval-point pair $([\alpha,\beta],\tau)$ from Δ_H, when $\tau\in X-H$, we choose the interval-point pair $([\alpha,\beta],\tau)$ from Δ_0 with α or $\beta\in(a_k,b_k)$ for some $k>n_0$. Consider $\Delta=\bigcup_{k\in N}\Delta_k\cup\Delta_H\cup\Delta_0$. Then Δ is an approximate full cover of $[a,b]$. Then for any Δ-division $D=([\alpha,\beta],\tau)$ of $[a,b]$ we split the sum $\sum_D$ into $\sum_{D_k}$, $\sum_{D_H}$ and $\sum_{D_0}$ in which $\tau\in[a_k,b_k]$, $k=1,2,\ldots$, $\tau\in H$ and $\tau\in X-H$ respectively and D_k is a Δ_k-division of $[a_k,b_k]$, $k=1,2,\ldots$,
D_H is a Δ_H-division of H and D_0 is a Δ_0-partial division on $X-H$. Then we have

$$\left|(D)\sum\left[\{U(\tau,\beta)-U(\tau,\alpha)\}-\{U\chi_X(\tau,\beta)-U\chi_X(\tau,\alpha)\}\right]-\sum_{k=1}^{\infty}(GAP)\int_{a_k}^{b_k}U\right|$$

$$\leq (D_k)\sum_{k=1}^{n_0}\left|\sum\{U(\tau,\beta)-U(\tau,\alpha)\}-(GAP)\int_{a_k}^{b_k}U\right|$$
$$+\ (D_0)\sum_{\substack{\alpha\,\in\,(a_k,b_k)\\ k>n_0}}\left|(GAP)\int_{\alpha}^{b_k}U\right|$$
$$+(D_0)\sum_{\substack{\beta\,\in\,(a_k,b_k)\\ k>n_0}}\left|(GAP)\int_{a_k}^{\beta}U\right|$$
$$+(D_H)\sum_{\substack{(a_k,b_k)\,\subset\,(\alpha,\beta)\\ k>n_0}}\left|(GAP)\int_{a_k}^{b_k}U\right|$$
$$<\sum_{k=1}^{\infty}\epsilon/2^k+\epsilon+\epsilon+\sum_{k=n_0+1}^{\infty}\left|(GAP)\int_{a_k}^{b_k}U\right|$$
$$<4\epsilon.$$

Thus $U-U\chi_X$ is GAP-integrable on $[a,b]$ and so is U. Also

$$GAP)\int_a^b U=(GAP)\int_a^b U\chi_X+\sum_{k=1}^{\infty}(GAP)\int_{a_k}^{b_k}U.$$

Note 3.5. *In case of real line, Cauchy extension for the Henstock integral is not a special case of Harnack extension* [7]*. So we can conclude that Cauchy extension for the GAP-integral is not a special case of Harnack extension because our integral can be reduced to Henstock integral by taking $U(\tau,t)=f(\tau)t$ and full cover (i.e.δ-fine partition) in stead of approximate full cover.*

Theorem 3.6. *Let X be a closed subset of $[a,b]$ with H the set of all points of density in X. Set*

$$(a,b)-X=\bigcup_{k=1}^{\infty}(a_k,b_k).$$

Suppose that $U\in GAP[a,b]$ with its primitive ϕ being $AC(X)$ and satisfying ASL on X, then for any given $\epsilon>0$ there exists an approximate full cover Δ on $X-H$ such that for any Δ-partial division $D=([\alpha,\beta],\tau)$ on $X-H$ we have,

$$(D)\sum_{\beta\in(a_k,b_k)}|(GAP)\int_{a_k}^{\beta}U|<\epsilon$$

and

$$(D)\sum_{\alpha\in(a_k,b_k)}|(GAP)\int_{\alpha}^{b_k}U|<\epsilon$$

Proof. Since ϕ is $AC(X)$, for every $\epsilon > 0$ there is a $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[\alpha, \beta]\}$ with $\alpha, \beta \in X$ satisfying $\sum|\beta - \alpha| < \eta$ we have, $\sum|\phi(\alpha, \beta)| < \epsilon/2$. By Lebesgue density theorem, $m(X - H) = 0$, so we may choose an open set G such that $G \supset X - H$ and $m(G) < \eta$. Since the primitive ϕ satisfies ASL on X, given $\epsilon > 0$ there exists an approximate full cover Δ on X such that for any Δ-partial division $D = ([\alpha, \beta], \tau)$ on $X - H$ we have $(D)\sum|\phi(\alpha, \beta)| < \epsilon/2$. We may modify Δ, if necessary, so that $[\alpha, \beta] \subset G$ whenever $\tau \in X - H$.
Then for any Δ-partial division $D = ([\alpha, \beta], \tau)$ on $X - H$ we have,

$$\begin{aligned}
(D) \sum_{\beta \in (a_k, b_k)} |(GAP) \int_{a_k}^{\beta} U| &\leq (D) \sum_{\beta \in (a_k, b_k)} |(GAP) \int_{\tau}^{\beta} U| \\
&\quad + (D) \sum_{\beta \in (a_k, b_k)} |(GAP) \int_{a_k}^{\tau} U| \\
&= (D) \sum_{\beta \in (a_k, b_k)} |\phi(\tau, \beta)| \\
&\quad + (D) \sum_{\beta \in (a_k, b_k)} |\phi(a_k, \tau)| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Similarly we can prove the second inequality.
The proof is now complete.

Corollary 3.7. *If $U \in GAP[a, b]$ with its primitive ϕ being $AC(X)$ and satisfying ASL on X where X is a closed subset of $[a, b]$ then $U\chi_X$ is GAP-integrable on $[a, b]$ and*

$$(GAP) \int_a^b U\chi_X = (GAP) \int_a^b U - \sum_{k=1}^{\infty} (GAP) \int_{a_k}^{b_k} U$$

where $(a, b) - X = \bigcup_{k=1}^{\infty} (a_k, b_k)$.

Proof: It follows from Theorems (3.4) and (3.6).

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