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SOME RING STRUCTURES OF SKEW GENERALIZED POWER SERIES RINGS

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Abstract. In the current article, we are going to investigate a relationship between some abstract ring structures (weakly PP-rings, clean rings, uniquely clean rings and n-clean rings) along with the skew generalized power series rings $A[[N,\vartheta]]$, where A is one of the ring structures described above, $(N,\leq)$ represents a strictly ordered monoid while $\vartheta : N \to End(A)$ represents a monoid homomorphism. We shall propose unified extensions of the above-mentioned ring structures by employing specific conditions along with their proofs.

Key words and Phrases: Weakly PP-ring; Skew generalized power series ring; (N,ϑ)-Armendariz ring; clean ring; uniquely clean ring; n-clean ring.

1. Introduction

In the current article, A represents an associative ring with identity while N represents a monoid. A few definitions are needed for the formation of a skew generalized power series ring. Assuming that $(N,\leq)$ is a partially ordered set subsequently $(N,\leq)$ is termed artinian in case each decreasing arrangement of the elements of N elements are finite while $(N,\leq)$ is termed narrow if all subsets of pair-wise order-incomparable elements of N are finite. So, $(N,\leq)$ is artinian as well as narrow if each nonempty subset of the monoid N has atleast one but a limited number of least elements [8].

A pair $(N,\leq)$ represents an ordered monoid having a monoid N and an order $\leq$ on N so that for all $x,y,z \in N$, $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$. An ordered monoid $(N,\leq)$ is considered strictly ordered in case for all $x,y,z \in N$, $x < y$ implies $zx < zy$ and $xz < yz$ [8].

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Let A denotes a ring, $(N, \leq)$ denotes a strictly ordered monoid while $\vartheta : N \to End(A)$ represents a monoid homomorphism. Now, considering $s \in N$, let ϑ_s represents the image of s underneath ϑ which is, $\vartheta_s = \vartheta(s)$. Assume B being a set of all functions $\gamma : N \to A$ so that $supp(\gamma) = \{s \in N, : \gamma(s) \neq 0\}$ is artinian as well as narrow [10].

Thus for any $s \in N$ and $\gamma, \epsilon \in B$ the set

$$X_s(\gamma, \epsilon) = \{(a, b) \in supp(\gamma) \times supp(\epsilon) : s = ab\} \text{ is finite.}$$

Thus the product $\gamma\epsilon : N \to A$ of $\gamma, \epsilon \in B$ can be defined as follows:

$$(\gamma\epsilon)(s) = \sum_{(a,b) \in X_s(\gamma,\epsilon)} \gamma(a) \cdot \vartheta_a(\epsilon(b)).$$

The ring B is termed skew generalized power series ring having coefficients in A with exponents in N [4], represented by $A[[N, \vartheta, \leq]]$ (or by $A[[N, \vartheta]]$), and is formed by point-wise addition and multiplication. The compact generalization of skew polynomial rings, (skew) group rings, (skew) Laurent polynomial rings, (skew) power series rings, Mal'cev Neumann Laurent series rings, (skew) monoid rings along with the generalized power series rings leads to the skew generalized power series ring $A[[N, \vartheta]]$.

1 (symbol) signifies the identity element of a monoid N (when N is multiplicative), ring A and with ring $A[[N, \vartheta]]$ which is skew generalized power series.

For every $p \in A$ and $s \in N$, we do link elements $g_p, e_s \in A[[N, \vartheta]]$ which is defined by

$$g_p(a) = \begin{cases} p & \text{if } a = 1 \\ 0 & \text{if } a \in N \backslash \{1\}, \end{cases} \qquad e_s(a) = \begin{cases} 1 & \text{if } a = s \\ 0 & \text{if } a \in N \backslash \{s\} \end{cases} \tag{1}$$

It is obvious that $p \mapsto g_p$ is a ring embedding of A into $A[[N, \vartheta]]$, $s \mapsto e_s$ represents the monoid embedding of N to multiplicative monoid of $A[[N, \vartheta]]$ and $e_s g_p = g_{\vartheta_s(p)} e_s$. Furthermore, for each non-empty subset X of A we set $X[[N, \vartheta]] = \{\gamma \in A[[N, \vartheta]] : \gamma(s) \in X \cup \{0\}$ for every $s \in N\}$ which is a subset of $A[[N, \vartheta]]$, and for each non-empty subset Z of $A[[N, \vartheta]]$ we put $C_Z = \{\epsilon(t) : \epsilon \in Z, t \in N\}$ which is a subset of A.

Using the definition of a ring $A[[N, \vartheta]]$ which is skew generalized power series one, Marks et al. did introduce the idea of Armendariz ring as a way to skew generalized power series while providing a coherent methodology to each of the Armendariz rings class. Let A stands for a ring, $(N, \leq)$ represents one strictly ordered monoid and $\vartheta : N \to End(A)$ is a monoid homomorphism. Then A is (N, ϑ)-Armendariz when $\gamma\epsilon = 0$ for $\gamma, \epsilon \in A[[N, \vartheta]]$, then $\gamma(\epsilon) \cdot \vartheta_s(\epsilon(t)) = 0$ for all $s, t \in N$. If $N = \{1\}$, here each and every ring is (N, ϑ)-Armendariz [4].

The article is divided into four sections. The first section consists of some introduction which is important to understand the further sections.

Section 2 talks about the extension of weakly PP-rings. Ring A is weakly PP if each principal left ideal ${}_AAep$ is projective for every $p \in A$ and every primitive idempotent $e \in A$. Liu proved that if A exists as a commutative ring with $(N, \leq)$

a strictly totally ordered monoid that fulfills the order $0 \leq s$ for all $s \in N$ then ring $[[A^{N,\leq}]]$ is a weakly PP provided that A itself is one weakly PP-ring [2]. In Theorem 2.6 we study the above result of Liu and Ahsan [3] for non-commutative ring A for the instance of skew generalized power series by making use of (N, ϑ)-Armendariz ring.

Section 3 is primarily about studying the results of clean and uniquely clean ring for a more generalized structure. Element c of a ring A is termed clean when $c = u + e$, wherever $u \in U(A)$ and $e \in Id(A)$ (The set of all idempotents of A). A ring A can be termed clean if its every element is clean. Nicholson presented the view of clean ring and showed that the ring A is clean provided that $A[[c]]$ is clean [6]. Further, Liu proved that if $(N, \leq)$ a strictly ordered monoid fulfilling the conditional requirement $0 \leq s$ for every $s \in N$, subsequently $[[A^{N,\leq}]]$ stands as a clean ring provided A too is a clean ring [2]. Afterwards, Salem showed that in case $(N, \leq)$ is totally ordered artinian monoid, thenceforth $[[A^{N,\leq}]]$ is clean provided A itself is clean [11].

Here, we prove all the above results in the case of ring which is skew generalised power series. Please recall that ring A is considered uniquely clean if each of it's element is distinctively the sum of an idempotent and one unit. Anderson and Camillo [1] concentrated on uniquely clean rings in the commutative case while Nicholson and Zhou [7] talked about them in the non-commutative circumstances. Liu showed that for a reduced ring A, $[[A^{N,\leq}]]$ is a uniquely clean ring provided A too is uniquely clean, where N is a torsion-free as well as cancellative monoid and $\leq$ strict order on N so that $(N, \leq)$ fulfills the condition $0 \leq s$ for every $s \in N$ [2]. In Theorem 3.4 we prove the above result for skew generalized power series and hence we have a combined result.

Section 4 deals with n-clean rings. Recently, Xiao and Tong introduced the concept of n-clean ring which is a generalization of clean ring. Element c in A is called n-clean if $c = e + u_1 + u_2 + \cdots + u_n$, where $u_i \in U(A)$ for all $i = 1, 2, \ldots, n$ and $e \in Id(A)$. A ring A can be considered as n-clean provided each component of A is n-clean, where n is any positive integer [13]. Xiao and Tong proved that for a positive integer n, the ring $A[[c]]$ is n-clean if and proved if A is n-clean [13]. Here, we study the above result to the most extended ring structure skew generalized power series $A[[N, \vartheta]]$.

2. Weakly PP-rings of Skew Generalized Power series

Assuming A as a commutative ring, $(N, \leq)$ one strictly totally ordered monoid fulfilling the prerequisite $0 \leq s$ for all $s \in N$ thenceforth ring $[[A^{N,\leq}]]$ stands as a weakly PP provided that ring A too represents a weakly PP-ring [3]. In Theorem 2.6 we study the above result of Liu and Ahsan for non-commutative ring A for skew generalized power series through the use of (N, ϑ)-Armendariz ring. For proving this Theorem, we must prove Lemma 2.3 which is an extension of [3, Lemma 3.2] and also need some definitions.

Definition 2.1. *Two non-zero idempotents a and b are called orthogonal if $ab = ba = 0$.*

Definition 2.2. *A non-zero idempotent ' a ' of a ring A is primitive if it can not be written as $a = e + f$ where e and f are two non-zero orthogonal idempotents in A.*

Lemma 2.3. *Assume A to be a (N,ϑ)-Armendariz ring, $(N,\leq)$ representing one strictly ordered monoid fulfilling the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. If $\gamma \in A[[N,\vartheta]]$ is a primitive idempotent, $\gamma(1)$ too is a primitive idempotent of A so that $\gamma = g_{\gamma(1)}$.*

Proof. Assume $\gamma \in A[[N,\vartheta]]$ is a primitive idempotent, by [4, Proposition 4.10] there exists an idempotent $\gamma(1) \in A$ such that $\gamma = g_{\gamma(1)}$. Assume that $\gamma(1)$ is not primitive, $\gamma(1) = f + f'$, where f and f' are non-zero orthogonal idempotents of A. Let $\epsilon_1, \epsilon_2 \in A[[N,\vartheta]]$ be defined via

$$\epsilon_1(s) = \begin{cases} 0 & \text{if } 1 < s \\ f & \text{if } 1 = s, \end{cases} \qquad \epsilon_2(s) = \begin{cases} 0 & \text{if } 1 < s \\ f' & \text{if } 1 = s. \end{cases}$$

Thus $\gamma(1) = f + f' = \epsilon_1(1) + \epsilon_2(1)$ implies that $\gamma = \epsilon_1 + \epsilon_2$. Now, we have

$$(\epsilon_1\epsilon_2)(s) = \sum_{(a,b)\in X_s(\epsilon_1,\epsilon_2)} \epsilon_1(a)\cdot\vartheta_a(\epsilon_2(b))$$

where $X_s(\epsilon_1,\epsilon_2) = \{(a,b) \in supp(\epsilon_1) \times supp(\epsilon_2) : ab = s\}$.

Therefore $(\epsilon_1\epsilon_2)(s) = \epsilon_1(1)\epsilon_2(1) = f.f' = 0$.

Thus ϵ_1 and ϵ_2 are nonzero orthogonal idempotents of $A[[N,\vartheta]]$. It follows that γ is not primitive, which is a contradiction. Hence $\gamma(1)$ is primitive.

Let us recollect the definition of compatible endomorphism from [4].

Definition 2.4. *The endomorphism τ of ring A is considered compatible if for each $x, y \in A$, $xy = 0 \Leftrightarrow x\tau(y) = 0$ [4].*

Definition 2.5. *Let A be a ring, $(N,\leq)$ a strictly ordered monoid plus $\vartheta : N \to End(A)$ a monoid homomorphism. In this case, A is N-compatible if ϑ_s is compatible for every $s \in N$ [4].*

Here, we are able to establish the section's primary theorem.

Theorem 2.6. *Let A be a (N,ϑ)-Armendariz ring, $(N,\leq)$ a strictly ordered monoid fulfilling the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. Let's suppose that A is N-compatible, then $A[[N,\vartheta]]$ denotes a weakly PP-ring provided ring A too is weakly PP.*

Proof. Let's assume that the ring A represents one weakly PP-ring, $\gamma \in A[[N,\vartheta]]$ and ϕ a primitive idempotent of $A[[N,\vartheta]]$. Then by Lemma 2.3, $\phi(1)$ is a primitive idempotent of A such that $g_{\phi(1)} = \phi$. Thus $ann_l^A(\phi(1)p) = Ae$, where $p \in A$ and $e^2 = e \in A$. To prove the result, we need only to show that $ann_l^{A[[N,\vartheta]]}(\phi\gamma) = A[[N,\vartheta]]g_e$, where g_e is an idempotent of $A[[N,\vartheta]]$. Let $\epsilon \in ann_l^{A[[N,\vartheta]]}(\phi\gamma)$ which implies $\epsilon\phi\gamma = 0$. Since A be (N,ϑ)-Armendariz and N-compatible, we have $\epsilon(s)\phi(1)\gamma(t) = 0$ for all $s, t \in N$. It follows that $\epsilon(s) \in ann_l^A(\phi(1)\gamma(t)) = Ae$

which gives $\epsilon \in A[[N,\vartheta]]g_e$. Thus $ann_l^{A[[N,\vartheta]]}(\phi\gamma) \subseteq A[[N,\vartheta]]g_e$. Again consider $\epsilon = \rho g_e \in A[[N,\vartheta]]g_e$, so for any $s_0 \in N$, $\epsilon(s_0) = \rho(s_0)e$ which implies $\epsilon(s_0) \in Ae$. It follows that $\epsilon(s_0) \in Ae = ann_l^A(\phi(1)\gamma(t))$ for any $p \in A$, where $\phi(1)$ is a primitive idempotent of A since A is weakly PP-ring. Therefore $\epsilon(s_0)\phi(1)p$ implies that $\epsilon(s_0)\phi(1)g_p(1) = 0$. Now define $\gamma \in A[[N,\vartheta]]$, via

$$\gamma(t) = \begin{cases} g_p(t) & \text{if } t = 1 \\ 0 & \text{if } t > 1. \end{cases}$$

Thus we have $\epsilon\phi\gamma = 0$ which implies that $\epsilon \in ann_l^{A[[N,\vartheta]]}(\phi\gamma)$. Therefore $A[[N,\vartheta]]g_p \subseteq ann_l^{A[[N,\vartheta]]}(\phi\gamma)$. Hence $A[[N,\vartheta]]$ is established as a weakly PP-ring.

Contrarily, let's assume that $A[[N,\vartheta]]$ is weakly PP-ring, $p \in A$ and e is a primitive idempotent of A. Then g_e too is primitive idempotent of $A[[N,\vartheta]]$ and $g_p \in A[[N,\vartheta]]$ such that $supp(g_e) = 1$ and $supp(g_p) = 1$, respectively. Since $A[[N,\vartheta]]$ is weakly PP-ring, $ann_l^{A[[N,\vartheta]]}(g_e g_p) = A[[N,\vartheta]]\phi$, where $\phi^2 = \phi \in A[[N,\vartheta]]$ then by [4, Proposition 4.10], $\phi(1)$ is an A idempotent such that $\phi = g\phi(1)$. To prove the result, we need only to show that $ann_l^A(ep) = A\phi(1)$. Let $q \in ann_l^A(ep)$. Then $qep = 0$. Then $(g_q g_e g_p)(s) = 0$ implies that $g_q \in ann_l^{A[[N,\vartheta]]}(g_e g_p) = A[[N,\vartheta]]\phi$. It follows that $g_q \in A[[N,\vartheta]]\phi$ implies $g_q = \epsilon\phi$ for any $\epsilon \in A[[N,\vartheta]]$. Thus $q = g_q(1) = (\epsilon\phi)(1) = \epsilon(1)\phi(1) \in A\phi(1)$, therefore $ann_l^A(ep) \subseteq A\phi(1)$. Now, suppose any $q \in A\phi(1)$. This implies that $q = r\phi(1)$, where $r \in \mathcal{R}$. Then $g_q(1) = g_r(1)\phi(1) = (g_r\phi)(1)$ implies $g_q = g_r\phi \in A[[N,\vartheta]]\phi = ann_l^{A[[N,\vartheta]]}(g_e g_p)$ since $A[[N,\vartheta]]$ is weakly PP-ring. Thus $g_q g_e g_p = 0$ implies $q_e p = 0$. It follows that $A\phi(1) \subseteq ann_l^A(geg_p)$. Hence A is a weakly PP-ring. Recall from [4] that an ordered monoid $(N,\leq)$ is left naturally ordered provided for all $s,t \in N$, $s \leq t$ implies that $t \in Ns$. □

The following two corollaries are obtained from the above theorem.

Corollary 2.7. *Suppose A is a ring without any non-zero divisors, $(N,\leq)$ a strictly ordered monoid that meets condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. Suppose the order $\leq$ may be improved to a strict total order of monoid $\preceq$ suchlike that monoid $(N,\preceq)$ is naturally left ordered as well as A is N-compatible. Then $A[[N,\vartheta]]$ is considered as weakly PP-ring provided ring A too is weakly PP.*

Proof. By [4, Proposition 2.8] and Theorem 2.6, proof is established and complete. □

Corollary 2.8. *Let A be a (N,ϑ)-Armendariz ring, $(N,\leq)$ a strictly totally ordered monoid satisfying the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. Let's us assume that A is N-compatible, then $A[[N,\vartheta]]$ is considered a weakly PP-ring provided A too is weakly PP-ring.*

Proof. Because $(N,\leq)$ is a strictly totally ordered monoid, then $(N,\leq)$ is a strictly ordered monoid. Hence by virtue of Theorem 2.6, proof is complete. □

3. Clean Rings of Skew Generalized Power Series

Nicholson characterized the idea of a clean ring and demonstrated that ring A is clean provided $A[[c]]$ too is clean [6]. Further, Liu proved that if $(N, \leq)$ a strictly ordered monoid fulfilling the condition $0 \leq s$ for every $s \in N$, subsequently $[[A^{N,\leq}]]$ is a clean ring provided that ring A is also clean [2]. Afterwards, Salem showed if $(N, \leq)$ is totally ordered artinian monoid, then $[[A^{N,\leq}]]$ is clean provided A is also clean [11]. Here, we prove all the above results for skew generalized power series ring. We should prove the below Lemma to establish the outcomes:

Lemma 3.1. *Suppose A being a ring, $(N, \leq)$ a strictly ordered monoid satisfying the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ be a monoid homomorphism. Suppose that $\gamma \in A[[N, \vartheta]]$ such that $\gamma \in U(A[[N, \vartheta]])$ then $\gamma(1) \in U(A)$.*

Proof. Suppose $\gamma \in U(A[[N, \vartheta]])$ then there exists $\epsilon \in U(A[[N, \vartheta]])$ such that $\gamma\epsilon = \epsilon\gamma = g_1$ which implies $1 = \gamma(1)\epsilon(1) = \epsilon(1)\gamma(1)$. Hence $\gamma(1) \in U(A)$. □

Theorem 3.2. *Assume $(N, \leq)$ to be a strictly ordered monoid satisfying the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. The following statements hold:*

1. *if ring A is a clean then $A[[N, \vartheta]]$ too is a clean ring*
2. *if A be a (N, ϑ)-Armendariz ring $A[[N, \vartheta]]$ is a clean ring then A too is a clean ring.*

Proof. Assume that A is a clean ring and let any $0 \neq \gamma \in A[[N, \vartheta]]$. Then $supp(\gamma)$ is nonempty. Assume that minimal element of $supp(\gamma) = 1$, then from [5, Lemma 2.5], $\gamma(1) \in A$. As A is clean ring, we have $\gamma(1) = u + e$, where $u \in U(A)$ and $e_2 = e \in Id(A)$. It follows that $\gamma(1) - g_e(1) \in A$ implies $\gamma - g_e \in A[[N, \vartheta]]$ such that $supp(\gamma - g_e) = 1$. So $(\gamma - g_e)(1) = \gamma(1) - g_e(1) = \gamma(1) - e = u \in U(A)$. Thus by [5, Proposition 2.2], $\gamma - g_e \in U(A[[N, \vartheta]])$. Now since $e^2 = e$ we have $g_e^2 = g_e \in A[[N, \vartheta]]$. Therefore $\gamma = (\gamma - g_e) + g_e$ for any $\gamma \in A[[N, \vartheta]]$. So $A[[N, \vartheta]]$ is a clean ring.

Conversely, let's suppose that $A[[N, \vartheta]]$ is a clean ring and $p \in A$. Then $g_p \in A[[N, \vartheta]]$ and $g_p = \gamma + \phi$, where $\gamma \in U(A[[N, \vartheta]])$ and $\phi^2 = \phi \in A[[N, \vartheta]]$, $A[[N, \vartheta]]$ is a clean ring. From [Equation 1] it follows that $p = \gamma(1) + \phi(1)$. By Lemma 3.1, $\gamma(1) \in U(A)$ and by [4, Proposition 4.10] $\phi(1)$ is an idempotent of A such that $\phi = g\phi(1)$. □

The following corollary is obtained from the above Theorem.

Corollary 3.3. *Let A be a (N, ϑ)-Armendariz ring, $(N, \leq)$ a strictly totally ordered monoid satisfying the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. Then $A[[N, \vartheta]]$ is clean provided A too is clean.*

Proof. Since $(N, \leq)$ is a monoid which is strictly totally ordered, then $(N, \leq)$ is a strictly ordered monoid. Therefore by virtue of Theorem 3.2, proof can be established.

Liu showed that for a reduced ring A, ring $[[A^{N,\leq}]]$ is a uniquely clean provided A too is uniquely clean, where N is a torsion-free and cancellative monoid and $\leq$ strict order on N such that $(N,\leq)$ fulfills the condition $0 \leq s$ for every $s \in N$ [2]. In following Theorem we prove the above result for ring which is a skew generalized power series and hence we have a unified result. □

Theorem 3.4. *Assume A be a (N,ϑ)-Armendariz ring, $(N,\leq)$ a strictly ordered monoid satisfying the condition $1 \leq s$ for every $s \in N$ and $\vartheta : N \to End(A)$ a monoid homomorphism. Then ring $A[[N,\vartheta]]$ is considered uniquely clean provided that ring A too is a uniquely clean one.*

Proof. Let us assume A is uniquely clean, then by Theorem 3.2, $A[[N,\vartheta]]$ is clean. Now, we have only to show that $A[[N,\vartheta]]$ is uniquely clean. Assume that $A[[N,\vartheta]]$ is not uniquely clean, then for any $\gamma \in A[[N,\vartheta]]$, $\gamma = \epsilon_1 + \phi_1 = \epsilon_2 + \phi_2$, where ϕ_1, ϕ_2 are idempotents of $A[[N,\vartheta]]$ and $\epsilon_1 \neq \epsilon_2 \in U(A[[N,\vartheta]])$. Since A is (N,ϑ)-Armendariz, by [4, Proposition 4.10], $\phi_1(1)$ and $\phi_2(1)$ are idempotents of A such that $\phi_1 = g_{\phi 1(1)}$ and $\phi_2 = g_{\phi 2(1)}$, respectively. Thus

$$(\epsilon_1 - \epsilon_2)(s) = \begin{cases} \phi_2(1) - \phi_1(1) & \text{if } s = 1 \\ \epsilon_1(s) - \epsilon_2(s) & \text{if } s > 1. \end{cases}$$

It follows that $\gamma(1) = \epsilon_1(1) + \phi_1(1) = \epsilon_2(1) + \phi_2(1)$. By Lemma 3.1, $\epsilon_1(1), \epsilon_2(1) \in U(A)$. Since A is uniquely clean, $\phi_1(1) = \phi_2(1)$ and $\epsilon_1(1) = \epsilon_2(1)$. Thus $\epsilon_1 = \epsilon_2 \in U(A[[N,\vartheta]])$ which is a contradiction. Therefore $A[[N,\vartheta]]$ is uniquely clean.

Proof of the converse is straight forward. □

Now, we get some crucial corollaries.

Corollary 3.5. *Let us say that A a ring without non-zero zero divisors, $(N,\leq)$ a strictly ordered monoid fulfilling the condition $1 \leq s$ for every $s \in N$, $\vartheta : N \to End(A)$ a monoid homomorphism and $A[[N,\vartheta]]$, the skew generalized power series ring. Assume that $\leq$ can be refined to a strict total order $\preceq$ in a way that the monoid $(N,\preceq)$ is left naturally ordered. Then $A[[N,\vartheta]]$ is considered as uniquely clean provided that ring A too is uniquely clean.*

Proof. Since A is a ring without non-zero zero divisors then from [4, Proposition 2.8], A is (N,ϑ)-Armendariz ring and with reference to Theorem 3.4, the proof is fulfilled.

Corollary 3.6 (Liu [2, Theorem 5.6])**.** *Assume A being a reduced ring, N a torsion-free cancellative monoid and $\leq$ strict order on N so that $(N,\leq)$ meets the condition $0 \leq s$ for every $s \in N$. Then ring $[[A^{N,\leq}]]$ is an uniquely clean provided that ring A too is an uniquely clean.*

Proof. Since N a torsion-free and cancellative monoid and $\leq$ strict order on N such that $(N,\leq)$ fulfills the condition $0 \leq s$ for every $s \in N$ then $(N,\leq)$ is a strictly ordered monoid and A be a (N,ϑ)-Armendariz ring. Thus by the Theorem 3.2, proof is fulfilled. □

4. n-Clean Rings of Skew Generalized Power Series

Xiao and Tong proved that for a positive integer n, the ring $A[[c]]$ is n-clean if and provided that A is n-clean [13]. Here, we study the above result to the most extended structure skew generalized power series ring $A[[N,\vartheta]]$.

Theorem 4.1. *Assume that $(N,\leq)$ a strictly ordered monoid satisfying the condition $1\leq s$ for every $s\in N$ and $\vartheta: N\to End(A)$ a monoid homomorphism. The following statements hold:*

1. *if A is n-clean ring then $A[[N,\vartheta]]$ is also n-clean ring*
2. *if A be a (N,ϑ)-Armendariz ring and $A[[N,\vartheta]]$ is n-clean ring then A too is n-clean ring.*

Proof. Suppose that A be an n-clean ring and let any $0\neq\gamma\in A[[N,\vartheta]]$. Then $supp(\gamma)$ is nonempty. Assume that minimal element of $supp(\gamma)=1$, then $\gamma(1)\in A$. Thus $\gamma(1)=e+u_1+u_2+\cdots+u_n$, where $e\in Id(A)$ and $u_i\in U(A)$ for each $i=1,2,\ldots,n$ as A is a n-clean ring. Define $\epsilon i\in A[[N,\vartheta]]$ for each $i=1,2,\ldots,n$ via

$$\epsilon_i(s)=\begin{cases}0 & \text{if } 1<s\\ u_i & \text{if } s=1.\end{cases}$$

Then $\gamma(1)=g_e(1)+\epsilon_1(1)+\epsilon_2(1)+\cdots+\epsilon_n(1)$ and by [5, Proposition 2.2], $\epsilon_i\in U(A[[N,\vartheta]])$ for each $i=1,2,\ldots,n$. Thus $\gamma=g_e+\epsilon_1+\epsilon_2+\cdots+\epsilon_n$. Since $e_2=e$ implies $g_e(1)(g_e(1)-1)=0$ which implies $g_e(1)\vartheta 1(g_e(1)-1)=0$. Thus $g_e^2=g_e\in A[[N,\vartheta]]$. Hence $A[[N,\vartheta]]$ proves to be a n-clean ring.

Conversely, $A[[N,\vartheta]]$ is a n-clean ring and suppose $p\in A$. Then $g_p\in A[[N,\vartheta]]$ and $g_p=\gamma_1+\gamma_2+\cdots+\gamma_n+\phi$, where $\gamma_i\in U(A[[N,\vartheta]])$ for each $i=1,2,\ldots,n$ and $\phi^2=\phi\in A[[N,\vartheta]]$. It follows that $p=\gamma_1(1)+\gamma_2(1)+\cdots+\gamma_n(1)+\phi(1)$. Thus by Lemma 3.1, $\gamma_i(1)\in U(A)$ for each $i=1,2,\ldots,n$ and by [4, Proposition 4.10] $\phi(1)$ is one idempotent of A such that $\phi=g\phi(1)$. Hence A is an n-clean ring. □

We get the resulting corollary.

Corollary 4.2. *Let A be one (N,ϑ)-Armendariz ring, $(N,\leq)$ a strictly totally ordered monoid fulfilling the condition $1\leq s$ for every $s\in N$, $\vartheta: N\to End(A)$ a monoid homomorphism. Then $A[[N,\vartheta]]$ is n-clean ring provided that ring A also is a n-clean.*

Proof. Since $(N,\leq)$ is a monoid which is strictly totally ordered, then $(N,\leq)$ can be considered strictly ordered monoid. Hence by the Theorem 4.1, proof is complete. □

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