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On Generalized Space Matter Tensor

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Abstract. Extending the concept of Petrov tensor, in this article we attempt to introduce generalised space matter tensor [1],[2], [3], [4]. In the Riemannian manifold, it is found that the second Bianchi identity for the generalized space-matter tensor is satisfied if the energy-momentum tensor is of Codazzi type [5]. We study the nature of Riemannian manifolds by imposing curvature restrictions like symmetry, recurrent, weakly symmetry [6], [7], [8] etc. on this generalized Petrov space-matter tensor. We obtain the eigen values of the Ricci tensor S corresponding to the vector fields associated with the various $1-$ forms.

Key words and Phrases: Generalized Space-Matter Tensor, Scalar Curvature:

1. INTRODUCTION

A tensor P of type $(0, 4)$ satisfying all the algebraic properties of the Riemannian curvature tensor was introduced by A. Z. Petrov [9] in 1949. It is defined by

$$P = \frac{k}{2}\, g \wedge T + R - \sigma G, \tag{1}$$

where R is the Riemann curvature tensor of type $(0, 4)$, T is the energy-momentum tensor of type (0, 2), k is a cosmological constant, σ is the energy density (scalar), G is a tensor of type $(0, 4)$ given by

$$G(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U)$$

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for all $X, Y, Z, U \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M and the Kulkarni-Nomizu product $u \wedge v$ of two (0, 2) tensors u and v is defined by

$$\begin{aligned}(u \wedge v)(X_1, X_2, X_3, X_4) &= u(X_1, X_4)v(X_2, X_3) + u(X_2, X_3)v(X_1, X_4) \\ &- u(X_1, X_3)v(X_2, X_4) - u(X_2, X_4)v(X_1, X_3),\end{aligned}$$

$X_i \in \chi(M)$, $i = 1, 2, 3, 4$. The tensor P is known as the space-matter tensor of type (0, 4) of the manifold M. The first part of the tensor represents the curvature of the space and the second part represents the distribution and motion of the matter.

A tensor field $\tilde{P}$ of type $(0, 4)$ is said to be generalized space matter tensor [1] if it satisfies the following equation

$$\tilde{P} = aR + \frac{k}{2} b\, g \wedge T + cG, \tag{2}$$

where a, b, c are non-zero scalars. If we take $a = 1$, $b = 1$ and $c = -\sigma$ then $\tilde{P}$ reduces to P.
Einstein's field equation is given by

$$kT = S + \left(\lambda - \frac{r}{2}\right) g, \tag{3}$$

where λ is a cosmological constant, r is the scalar curvature, and S is the Ricci tensor of type $(0, 2)$. By virtue of (3), (2) takes the form

$$\tilde{P} = aR + \frac{b}{2}\, g \wedge S + \left(c + b\lambda - \frac{br}{2}\right) G. \tag{4}$$

The paper is categorized into two major sections. The immediate section contains some of the interesting properties of the generalized space-matter tensor $\tilde{P}$. In the other section, we have considered few limitations on the generalized space-matter tensor $\tilde{P}$ like symmetric, reccurant, weakly symmetric and studied the behavior of the manifold under consideration.We obtained the eigen values of the Ricci tensor S corresponding to the vector fields associated with the various 1-forms.

2. PRELIMINARIES

In this section we deal with some fundamental properties of $\tilde{P}$ under certain curvature conditions. The Ricci tensor S of type $(0, 2)$ and the scalar curvature r can be obtained from the curvature tensor by the following relations

$$S(X, Y) = g(QX, Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i) \quad \text{and} \quad r = \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} g(Qe_i, e_i),$$

where $\{e_i : i = 1, 2, ..., n\}$ is an orthonormal basis for the tangent space at any point of the manifold and Q is the symmetric endomorphism corresponding to the Ricci tensor S.

Differentiating (2) covariantly and then taking cyclic sum with respect to X, Y, Z; we obtain by the view of Bianchi identity that

$$
\begin{aligned}
&(\nabla_X \tilde{P})(Y,Z,U,V) + (\nabla_Y \tilde{P})(Z,X,U,V) + (\nabla_Z \tilde{P})(X,Y,U,V) \qquad (5)\\
= \; & \frac{k}{2} b[\{(\nabla_X T)(Z,U) - (\nabla_Z T)(X,U)\}g(Y,V) + \{(\nabla_Y T)(X,U) \\
& -(\nabla_X T)(Y,U)\}g(Z,V) + \{(\nabla_Z T)(Y,U) - (\nabla_Y T)(Z,U)\}g(X,V) \\
& +\{(\nabla_X T)(Y,V) - (\nabla_Y T)(X,V)\}g(Z,U) + \{(\nabla_Z T)(X,V) \\
& -(\nabla_X T)(Z,V)\}g(Y,U) + \{(\nabla_Y T)(Z,V) - (\nabla_Z T)(Y,V)\}g(X,U)] \\
& +\frac{k}{2}\{db(X)\, g \wedge T(Y,Z,U,V) + db(Y)\, g \wedge T(Z,X,U,V) \\
& +db(Z)\, g \wedge T(X,Y,U,V)\} + dc(X)\{g(Z,U)g(Y,V) - g(Z,V)g(Y,U)\} \\
& +dc(Y)\{g(Z,V)g(X,U) - g(Z,U)g(X,V)\} \\
& +dc(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}.
\end{aligned}
$$

If we consider the energy-momentum tensor to be Codazzi type [5] in a Riemannian manifold, then we obtain

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z) = (\nabla_Z T)(X,Y)$$

for all vector fields X, Y, Z on the manifold provided that b, c are constants. Hence (5) takes the form

$$(\nabla_X \tilde{P})(Y,Z,U,V) + (\nabla_Y \tilde{P})(Z,X,U,V) + (\nabla_Z \tilde{P})(X,Y,U,V) = 0. \qquad (6)$$

Thus we infer that:

Theorem 2.1. *In a Riemannian manifold the second Bianchi identity for the generalized space-matter tensor is given by* (*6*) *provided that the energy-momentum tensor is of Codazzi type and* b, c *are constants.*

Using (3) in (5) we get

$$
\begin{aligned}
&(\nabla_X \tilde{P})(Y,Z,U,V) + (\nabla_Y \tilde{P})(Z,X,U,V) + (\nabla_Z \tilde{P})(X,Y,U,V) \qquad (7)\\
= \; & \frac{1}{2} b[\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\}g(Y,V) + \{(\nabla_Y S)(X,U) \\
& -(\nabla_X S)(Y,U)\}g(Z,V) + \{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}g(X,V) \\
& +\{(\nabla_X S)(Y,V) - (\nabla_Y S)(X,V)\}g(Z,U) + \{(\nabla_Z S)(X,V) \\
& -(\nabla_X S)(Z,V)\}g(Y,U) + \{(\nabla_Y S)(Z,V) - (\nabla_Z S)(Y,V)\}g(X,U)] \\
& +\frac{1}{2}\{db(X)\, g \wedge S(Y,Z,U,V) + db(Y)\, g \wedge S(Z,X,U,V) \\
& +db(Z)\, g \wedge S(X,Y,U,V)\}] + \{dc(X) + (\lambda - \frac{r}{2})db(X) \\
& -\frac{b}{2}dr(X)\}\{g(Z,U)g(Y,V) - g(Z,V)g(Y,U)\} + \{dc(Y) + (\lambda - \frac{r}{2})db(Y) \\
& -\frac{b}{2}dr(Y)\}\{g(Z,V)g(X,U) - g(Z,U)g(X,V)\} + \{dc(Z) + (\lambda - \frac{r}{2})db(Z)
\end{aligned}
$$

$$-\frac{b}{2}dr(Z)\}\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}.$$

Now if we assume that the Ricci tensor is of Codazzi type [5],then $dr(X)=0$ for all vector fields X, which reduces (7) to (6), provided $b,\ c$ are constants. Also if in a Riemannian manifold admitting Einstein's field equation where $b,\ c$ are constants, the relation (6) holds then (7) transforms into

$$\begin{aligned}&\{(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)\}g(Y,V)+\{(\nabla_Y S)(X,U)\\ &-(\nabla_X S)(Y,U)\}g(Z,V)+\{(\nabla_Z S)(Y,U)-(\nabla_Y S)(Z,U)\}g(X,V)\\ &+\{(\nabla_X S)(Y,V)-(\nabla_Y S)(X,V)\}g(Z,U)+\{(\nabla_Z S)(X,V)\\ &-(\nabla_X S)(Z,V)\}g(Y,U)+\{(\nabla_Y S)(Z,V)-(\nabla_Z S)(Y,V)\}g(X,U)\\ &-[dr(X)\{g(Z,U)g(Y,V)-g(Z,V)g(Y,U)\}+dr(Y)\{g(Z,V)g(X,U)\\ &-g(Z,U)g(X,V)\}+dr(Z)\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}]=0,\end{aligned}$$

since $b\neq 0$. Contracting with respect to Y and V, we obtain

$$2(n-3)[(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)]=(2n-5)[dr(X)g(Z,U)-dr(Z)g(X,U)],$$

which gives, on contraction over Z and U that $dr(X)=0$ for all vector fields X and consequently the last relation reduces to

$$(\nabla_X S)(Z,U)=(\nabla_Z S)(X,U)\quad \text{for all } X,\ Y,\ Z\in\chi(M).$$

Therefore the Ricci tensor as well as energy-momentum tensor is of Codazzi type. Hence we can state the following:

Theorem 2.2. *In a Riemannian manifold admitting Einstein's field equation with cosmological constant, the second Bianchi identity for the generalized space-matter tensor is given by* (6) *if and only if the Ricci tensor is of Codazzi type, whenever* $b,\ c$ *are constants.*

3. GENERALIZED SPACE-MATTER TENSOR SATISFYING EINSTEIN EQUATION UNDER SOME RESTRICTIONS

We now intend to study Riemannian manifolds by imposing different restrictions on the on the generalized space-matter tensor.

3.1. **Riemannian manifold with vanishing generalized space-matter tensor.** First of all we consider a Riemannian manifold $(M^n,\ g)$ $(n>3)$, in which the generalized space-matter tensor of type $(0,4)$ vanishes identically. Then the equation (4) leads to the following

$$aR+\frac{b}{2}g\wedge S+\left(c+b\lambda-\frac{br}{2}\right)G=0. \tag{8}$$

Contraction of (8) yields

$$uS+[2(n-1)(c+b\lambda)-(n-2)br]g=0, \tag{9}$$

where $u = 2a + (n-2)b$. Further contraction gives

$$vr + 2n(n-1)(c + b\lambda) = 0, \tag{10}$$

where $v = 2a - (n-1)(n-2)b$. Thus we can state the following:

Theorem 3.1. *If in a Riemannian manifold* (M^n, g) $(n > 3)$*, admitting Einstein's field equation and with vanishing generalized space-matter tensor,* $2a \neq (n-1)(n-2)b$*, then the scalar curvature is given by the relation* (10)*.*

3.2. **Riemannian manifold with symmetric generalized space-matter tensor.** Next we assume that in a Riemannian manifold (M^n, g) $(n > 3)$ the generalized space-matter tensor $\tilde{P}$ of type $(0, 4)$ is symmetric, i. e.

$$(\nabla_X \tilde{P})(Y, Z, U, V) = 0. \tag{11}$$

In view of (4) and (11), we attain

$$\begin{aligned} &2a(\nabla_X R)(Y, Z, U, V) + b[(\nabla_X S)(Y, V)g(Z, U) \\ &+(\nabla_X S)(Z, U)g(Y, V) - (\nabla_X S)(Y, U)g(Z, V) \\ &-(\nabla_X S)(Z, V)g(Y, U)] + 2[dc(X) + \lambda db(X) \\ &-\frac{1}{2}\{bdr(X) + rdb(X)\}]G(Y, Z, U, V) + 2da(X)R(Y, Z, U, V) \\ &+db(X)[S(Y, V)g(Z, U) + S(Z, U)g(Y, V) \\ &-S(Y, U)g(Z, V) - S(Z, V)g(Y, U)] = 0. \end{aligned} \tag{12}$$

Contraction of (12) over Y and V gives

$$\begin{aligned} u(\nabla_X S)(Z, U) \quad &+ \quad du(X)S(Z, U) + [2(n-1)\{dc(X) + \lambda db(X)\} \\ &- \quad (n-2)\{bdr(X) + rdb(X)\}]g(Z, U) = 0. \end{aligned} \tag{13}$$

By further contraction and then putting $u = 0$ we obtain

$$dc(X) = -\lambda db(X). \tag{14}$$

This leads to the following:

Theorem 3.2. *In a Riemannian manifold* (M^n, g) $(n > 3)$*, admitting Einstein's field equation and with symmetric generalized space-matter tensor;* b*,* c *are connected by the relation* $dc(X) = -\lambda db(X)$*, whenever* $u = 0$*.*

Setting $X = U = e_i$ in the relation (13) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} [2a - (n-2)b]dr(Z) + 2du(QZ) \quad &+ \quad 4(n-1)[dc(Z) + \lambda db(Z)] \\ &= \quad 2(n-2)rdb(Z). \end{aligned} \tag{15}$$

By the help of (14) and (15) it follows that

$$(n-2)udr(Z) + 2ndu(QZ) = 2du(Z)r. \tag{16}$$

If r is constant then from the above relation we get that either u is also constant or

$$J_1(QX) = \frac{r}{n} J_1(X),$$

which gives

$$S(X, \tau_1) = \frac{r}{n} g(X, \tau_1),$$

where $g(X, \tau_1) = J_1(X) = du(X)$ for all vector fields X. Hence we have the following:

Theorem 3.3. *If in a Riemannian manifold (M^n, g) $(n > 3)$, admitting Einstein's field equation and with symmetric generalized space-matter tensor, the scalar curvature r is constant, then either u is constant or $\frac{r}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_1 defined by $g(X, \tau_1) = J_1(X) = du(X)$, for all $X \in \chi(M)$.*

Let us consider that a, b are constants and u is non-zero. Then (16) takes the form

$$dr(X) = 0 \text{ for all } X \in \chi(M). \tag{17}$$

By the virtue of (17) and (14) it follows that

$$dc(X) = 0 \text{ for all } X \in \chi(M). \tag{18}$$

Using (18), (17) in (13); we have

$$(\nabla_X S)(Z, U) = 0 \text{ for all } X, Z, U \in \chi(M). \tag{19}$$

Finally in the view of (17), (18) and (19) the equation (12) reduces to the following equation

$$\nabla R = 0. \tag{20}$$

Further if (20) holds, then the relations (19) and (17) also hold and consequently differentiating (4) covariantly, we obtain by the virtue of (17), (19) and (20)

$$(\nabla_X \tilde{P})(Y, Z, U, V) = dc(X) G(Y, Z, U, V).$$

Hence we can state the following:

Theorem 3.4. *A Riemannian manifold (M^n, g) $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor is symmetric whenever a, b are constants and u is non-zero and conversely provided c is also constant.*

3.3. **Riemannian manifold with recurrent generalized space-matter tensor.** Again if in a Riemannian manifold (M^n, g) $(n > 3)$ admitting Einstein's field equation, the generalized space-matter tensor $\tilde{P}$ of type $(0, 4)$ is recurrent then we have

$$(\nabla_X \tilde{P})(Y, Z, U, V) = L(X) \tilde{P}(Y, Z, U, V), \tag{21}$$

where L is the non-zero 1-form of recurrence such that $L(X) = g(X, \rho)$ for all vector fields X and ρ be the unit vector field associated with L. By the virtue of the above relation, the relation (6) leads to the following relation

$$L(X)\tilde{P}(Y, Z, U, V) + L(Y)\tilde{P}(Z, X, U, V) + L(Z)\tilde{P}(X, Y, U, V) = 0.$$

Contracting the above relation with respect to Y, V and applying (4), we obtain

$$\{2(n-1)(c + \lambda b) - (n-2)br\}[L(X)g(Z, U) - L(Z)g(X, U)]$$

$$+u[L(X)S(Z,U) - L(Z)S(X,U)] + 2L(\tilde{P}(Z,X)U) = 0.$$

Now setting $X = U = e_i$ in the above relation and taking summation over i, $1 \leq i \leq n$, it follows that

$$S(Z,\rho) = \frac{r_0}{2u} g(Z,\rho), \tag{22}$$

where $r_0 = 2(n-1)(n-2)(c+\lambda b) + \{2a - (n-2)(n-3)b\}r$ and $g(Z,\rho) = L(Z)$ for all vector fields Z. Thus we get the following:

Theorem 3.5. *If in a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation with recurrent generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{r_0}{2u}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $g(X,\rho) = L(X)$ for all vector fields X, whenever $b,\ c$ are constants.*

Now in the view of (21), (4) converts into

$$\begin{aligned} & 2a(\nabla_X R)(Y,Z,U,V) + b[(\nabla_X S)(Y,V)g(Z,U) \\ & +(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) \\ & -(\nabla_X S)(Z,V)g(Y,U)] + 2[dc(X) + \lambda db(X) \\ & -\frac{1}{2}\{bdr(X) + rdb(X)\}]G(Y,Z,U,V) + 2da(X)R(Y,Z,U,V) \\ & +db(X)[S(Y,V)g(Z,U) + S(Z,U)g(Y,V) \\ & -S(Y,U)g(Z,V) - S(Z,V)g(Y,U)] \\ = \ & L(X)[2aR(Y,Z,U,V) + b\{S(Y,V)g(Z,U) \\ & +S(Z,U)g(Y,V) - S(Y,U)g(Z,V) \\ & -S(Z,V)g(Y,U)\} + 2(c + \lambda b - \frac{br}{2})G(Y,Z,U,V)]. \end{aligned}$$

Let us set $Y = V = e_i$ in the above equation and take summation over i, $1 \leq i \leq n$ to obtain

$$\begin{aligned} & u(\nabla_X S)(Z,U) + du(X)S(Z,U) + [2(n-1)\{dc(X) \\ & +\lambda db(X)\} - (n-2)\{bdr(X) + rdb(X)\}]g(Z,U) \\ = \ & L(X)[uS(Z,U) + \{2(n-1)(c+\lambda b) - (n-2)br\}g(Z,U)]. \end{aligned} \tag{23}$$

Taking contraction with respect to Z and U, we get

$$\begin{aligned} vdr(X) + dv(X)r \ & + \ 2n(n-1)[dc(X) + \lambda db(X)] \\ & = \ [vr + 2n(n-1)(c+\lambda b)]L(X). \end{aligned} \tag{24}$$

which reduces to

$$\begin{aligned} vdr(\rho) + dv(\rho)r \ & + \ 2n(n-1)[dc(\rho) + \lambda db(\rho)] \\ & = \ [vr + 2n(n-1)(c+\lambda b)]L(\rho). \end{aligned} \tag{25}$$

by substituting $X = \rho$. Thus we conclude that:

Theorem 3.6. *In a Riemannian manifold* (M^n, g) $(n > 3)$ *admitting Einstein's field equation and recurrent generalized space-matter tensor, the generator of recurrence* ρ *is given by the relation* (25).

If $v = 0$ then from the equation (24) it can be obtained that

$$L(X) = \frac{dc(X) + \lambda db(X)}{(c + \lambda b)}. \tag{26}$$

Further if $v \neq 0$ and $r,\ a,\ b,\ c$ are constants then from the equation (24) we also have

$$r = -2n(n-1)\frac{(c + \lambda b)}{v}, \tag{27}$$

which leads to the following:

Theorem 3.7. *In a Riemannian manifold* (M^n, g) $(n > 3)$ *admitting Einstein's field equation and recurrent generalized space-matter, if* $v = 0$ *then the 1-form of recurrence* L *is given by the relation* (26) *otherwise the scalar curvature* r *is given by* (27) *whenever the scalar curvature* r *itself and* $a,\ b,\ c$ *are constants.*

Again setting $Y = V = e_i$ in (23) and taking summation over i, $1 \leq i \leq n$, we have

$$\begin{aligned} &[2a - (n-2)b]dr(Z) + 2du(QZ) + 4(n-1)[dc(Z) \\ &+\lambda db(Z)] - 2(n-2)r db(Z) \\ &= 2uL(QZ) + 2[2(n-1)(c + \lambda b) - (n-2)br]L(Z). \end{aligned} \tag{28}$$

By the virtue of (24) and (28) it follows that

$$(n-2)u dr(Z) + 2n[du(QZ) - uL(QZ)] = 2r[du(Z) - uL(Z)]. \tag{29}$$

If r is constant then the last relation yields

$$J_2(QZ) = \frac{r}{n} J_2(Z), \tag{30}$$

which gives

$$S(Z, \tau_2) = \frac{r}{n} g(Z, \tau_2),$$

where $g(Z, \tau_2) = J_2(Z) = du(Z) - uL(Z)$ for all vector fields Z. Let us consider $r,\ a,\ b$ be constants. Then we have from (29)

$$L(QZ) = \frac{r}{n} L(Z), \tag{31}$$

which yields

$$S(Z, \rho) = \frac{r}{n} g(Z, \rho).$$

Thus we can state the following:

Theorem 3.8. *If in a Riemannian manifold* (M^n, g) $(n > 3)$ *admitting Einstein's field equation and recurrent generalized space-matter tensor, the scalar curvature* r *is constant then,*
(i) $\frac{r}{n}$ *is an eigen value of the Ricci tensor* S *corresponding to the eigen vector* τ_2 *defined by* $g(X, \tau_2) = J_2(X) = du(X) - uL(X)$ *for all vector fields* X.

(ii) $\frac{r}{n}$ *is an eigen value of the Ricci tensor* S *corresponding to the eigen vector* ρ *defined by* $L(X) = g(X, \rho)$ *for all vector fields* X*, whenever* a *and* b *are constants.*

3.4. **Riemannian manifold with weakly symmetric generalized space-matter tensor.** Lastly, let in a Riemannian manifold (M^n, g) $(n > 3)$ admitting Einstein's field equation, we take the generalized space-matter tensor $\tilde{P}$ of type (0, 4) as weakly symmetric [6], [7]. Then we have

$$\begin{aligned}(\nabla_X \tilde{P})(Y, Z, U, V) &= A(X)\tilde{P}(Y, Z, U, V) + B(Y)\tilde{P}(X, Z, U, V) \\ &+ B(Z)\tilde{P}(Y, X, U, V) + E(U)\tilde{P}(Y, Z, X, V) \\ &+ E(V)\tilde{P}(Y, Z, U, X),\end{aligned} \tag{32}$$

where A, B and E are 1-forms (not simultaneously zero) such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$, $E(X) = g(X, \rho_3)$ for all vector fields X and ρ_1, ρ_2, ρ_3 be the unit vector fields associated with A, B, E respectively. In the view of (32), the equation (6) takes the form

$$J_3(X)\tilde{P}(Y, Z, U, V) + J_3(Y)\tilde{P}(Z, X, U, V) + J_3(Z)\tilde{P}(X, Y, U, V) = 0.$$

The contraction of the above relation over Y , V and the equation (4) gives

$$\begin{aligned}&\{2(n-1)(c + \lambda b) - (n-2)br\}[J_3(X)g(Z, U) - J_3(Z)g(X, U)] \\ &+u[J_3(X)S(Z, U) - J_3(Z)S(X, U)] + 2J_3(\tilde{P}(Z, X)U) = 0,\end{aligned}$$

where $J_3(X) = A(X) - 2B(X)$ for all vector fields X. Now contracting the above relation with respect to X and U, we obtain

$$J_3(QX) = \frac{r_0}{2u} J_3(X) \tag{33}$$

i.e.

$$S(X, \tau_3) = \frac{r_0}{2u} g(X, \tau_3),$$

where $g(X, \tau_3) = J_3(X)$ for all vector fields X. This leads to the following:

Theorem 3.9. *If in a Riemannian manifold* (M^n, g) $(n > 3)$ *admitting Einstein's field equation with weakly symmetric generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then* $\frac{r_0}{2u}$ *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* τ_3*, defined by* $g(X, \tau_3) = J_3(X) = A(X) - 2B(X)$ *for all vector fields* X*, whenever* b*,* c *are constants.*

Now in the view of (32), we obtain

$$\begin{aligned}&\sum(\nabla_X \tilde{P})(e_i, Z, U, e_i) \\ =\ & A(X)\sum \tilde{P}(e_i, Z, U, e_i) + B(Z)\sum \tilde{P}(e_i, X, U, e_i) \\ &+E(U)\sum \tilde{P}(e_i, Z, X, e_i) + \tilde{P}(X, Z, U, \rho_2) + \tilde{P}(X, U, Z, \rho_3).\end{aligned}$$

Applying (4) in the above relation, it follows that

$$\begin{aligned}&u(\nabla_X S)(Z, U) + du(X)S(Z, U) + [2(n-1)\{dc(X) + \lambda db(X)\} \\ &-(n-2)\{bdr(X) + rdb(X)\}]g(Z, U)\end{aligned} \tag{34}$$

$$
\begin{aligned}
= \; & A(X)[uS(Z,U)+\{2(n-1)(c+\lambda b) \\
& -(n-2)br\}g(Z,U)]+[2aR(X,Z,U,\rho_2)+b\{B(QX)g(Z,U) \\
& +B(X)S(Z,U)-B(Z)S(X,U)-B(QZ)g(X,U)\}+2(c+\lambda b \\
& -\frac{br}{2})G(X,Z,U,\rho_2)]+B(Z)[2aS(X,U)+b\{rg(X,U) \\
& +(n-2)S(X,U)\}+2(n-1)(c+\lambda b-\frac{br}{2})g(X,U)] \\
& +E(U)[2aS(X,Z)+b\{rg(X,Z)+(n-2)S(X,Z)\}+2(n-1)(c+\lambda b \\
& -\frac{br}{2})g(X,Z)]+[2aR(\rho_3,Z,U,X)+b\{E(QX)g(Z,U) \\
& +E(X)S(Z,U)-E(U)S(X,Z)-E(QU)g(X,Z)\} \\
& +2(c+\lambda b-\frac{br}{2})G(\rho_3,Z,U,X)].
\end{aligned}
$$

Setting $Z=U=e_i$ in (34) and taking summation over i, $1\le i\le n$, we find

$$
\begin{aligned}
vdr(X) \quad + \quad & rdv(X)+2n(n-1)[dc(X)+\lambda db(X)] \\
= \quad & [vr+2n(n-1)(c+\lambda b)]A(X)+2[uJ_4(QX) \\
+ \quad & \{br+2(n-1)(c+\lambda b-\frac{br}{2})\}J_4(X)],
\end{aligned}
\tag{35}
$$

where $J_4(X)=B(X)+E(X)$ for all vector fields X. Contracting (34) with respect to X and U, we have

$$
\begin{aligned}
& \frac{1}{2}[2a-(n-2)b]dr(Z)+du(QZ) \\
& +2(n-1)[dc(Z)+\lambda db(Z)]-(n-2)rdb(Z) \\
= \; & uA(QZ)+[2(n-1)(c+\lambda b)-(n-2)br]A(Z) \\
& +[\{2a+(2n-3)b\}r+2(n-1)^2(c+\lambda b-\frac{br}{2})]B(Z) \\
& -uJ_5(QZ)+[br+2(n-1)(c+\lambda b-\frac{br}{2})]E(Z),
\end{aligned}
\tag{36}
$$

where $J_5(Z)=B(Z)-E(Z)$ for all vector fields Z. Finally contracting (34) with respect to $X,\ Z$ and replacing U by Z, we get

$$
\begin{aligned}
& \frac{1}{2}[2a-(n-2)b]dr(Z)+du(QZ) \\
& +2(n-1)[dc(Z)+\lambda db(Z)]-(n-2)rdb(Z) \\
= \; & uA(QZ)+[2(n-1)(c+\lambda b)-(n-2)br]A(Z) \\
& +uJ_5(QZ)+[br+2(n-1)(c+\lambda b-\frac{br}{2})]B(Z) \\
& +[\{2a+(2n-3)b\}r+2(n-1)^2(c+\lambda b-\frac{br}{2})]E(Z).
\end{aligned}
\tag{37}
$$

Now (36) and (37) yield

$$
J_5(QX)=\frac{1}{u}[\{a+(n-2)b\}r+(n-1)(n-2)\{c-(r-2\lambda)\frac{b}{2}\}]J_5(X), \tag{38}
$$

provided that u is non-zero. Which gives

$$S(X,\tau_5) = \frac{1}{u}[\{a+(n-2)b\}r+(n-1)(n-2)\{c-(r-2\lambda)\frac{b}{2}\}]g(X,\tau_5),$$

where $g(X,\tau_5) = J_5(X) = B(X) - E(X)$ for all vector fields X. And if $u = 0$, then we can obtain

$$\text{either} \quad r = \frac{2(n-1)(c+\lambda b)}{(n-2)b} \quad \text{or} \quad B(X) = E(X). \tag{39}$$

Again (36) and (37) yield

$$\begin{aligned} & [2a-(n-2)b]dr(Z) + 2du(QZ) \\ & +4(n-1)[dc(Z)+\lambda db(Z)] - 2(n-2)rdb(Z) \\ = \; & 2uA(QZ) + 2[2(n-1)(c+\lambda b)-(n-2)br]A(Z) \\ & +2[\{a+(n-1)b\}r + n(n-1)(c+\lambda b - \frac{br}{2})]J_4(Z), \end{aligned} \tag{40}$$

Now by the help of (35) and (40), we have

$$\begin{aligned} (n-2)udr(Z) \; & + \; 2ndu(QZ) - 2du(Z)r \\ & = \; 2u[nA(QZ) - rA(Z) - 2J_4(QZ)] \\ & + \; 4[\{2na-(n-2)(n^2-n-4)b\}r \\ & + \; 2(n+2)(n-1)(n-2)(c+\lambda b)]J_4(Z). \end{aligned} \tag{41}$$

Suppose $u = 0$. Then from the above relation we get

$$\text{either} \quad r = \frac{2(n-1)(c+\lambda b)}{(n-2)b} \quad \text{or} \quad B(X) = -E(X). \tag{42}$$

Hence from (39) and (42) we have, if $u = 0$ then the only possible case is $r = \frac{2(n-1)(c+\lambda b)}{(n-2)b}$. Now by the virtue of (35) and (40), we have

$$\begin{aligned} & [2da(Z)-(n-2)(n-3)db(Z)]r + 2(n-1)(n-3)[dc(Z)+\lambda db(Z)] \\ & -(n-2)^2bdr(Z) - [2ar-(n-2)\{(n-3)br-2(n-1)(c+\lambda b)\}]J_6(Z) \\ = \; & 2du(QZ) - 2uJ_6(QZ), \end{aligned} \tag{43}$$

where $J_6(Z) = A(Z) - B(Z) - E(Z)$ for all vector fields Z. If r, a, b, c are constants and u is non-zero then the above relation becomes

$$J_6(QZ) = \frac{1}{2u}[\{2a-(n-3)(n-2)b\}r + 2(n-2)(n-1)(c+\lambda b)]J_6(Z),$$

which implies

$$S(Z,\tau_6) = \frac{1}{2u}[\{2a-(n-3)(n-2)b\}r + 2(n-2)(n-1)(c+\lambda b)]g(Z,\tau_6),$$

where $g(Z,\tau_6) = J_6(Z) = A(Z) - B(Z) - E(Z)$ for all vector fields Z. Further using (35) and (40), we can obtain

$$\begin{aligned} & [4a-n(n-2)b]dr(Z) + 2du(QZ) \\ & +[2da(Z)-(n+1)(n-2)db(Z)]r \\ & +2(n+2)(n-1)[dc(Z)+\lambda db(Z)] \end{aligned} \tag{44}$$

$$= 2uJ_7(QZ) + [2ar - (n+1)(n-2)br + 2(n+2)(n-1)(c+\lambda b)]J_7(Z),$$

where $J_7(Z) = A(Z)+B(Z)+E(Z)$ for all vector fields Z. If $r,\ a,\ b,\ c$ are constants and u is non-zero then above relation reduces to

$$J_7(QZ) = \frac{-1}{2u}[\{2a - (n+1)(n-2)b\}r + 2(n+2)(n-1)(c+\lambda b)]J_7(Z).$$

which implies

$$S(Z,\tau_7) = \frac{-1}{2u}[\{2a - (n+1)(n-2)b\}r + 2(n+2)(n-1)(c+\lambda b)]g(Z,\tau_7),$$

where $g(Z,\tau_7) = J_7(Z) = A(Z)+B(Z)+E(Z)$ for all vector fields Z. Thus we can state the following:

Theorem 3.10. *In a Riemannian manifold $(M^n,\ g)$ $(n>3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if u is zero then the scalar curvature r is given by $r = \frac{2(n-1)(c+\lambda b)}{(n-2)b}$ otherwise,*
(i) $\frac{1}{u}[\{a+(n-2)b\}r+(n-1)(n-2)\{c-(r-2\lambda)\frac{b}{2}\}]$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_5 defined by $g(X,\tau_5) = J_5(X) = B(X)-E(X)$ for all $X \in \chi(M)$.
(ii) $\frac{1}{2u}[\{2a-(n-3)(n-2)b\}r+2(n-2)(n-1)(c+\lambda b)]$ and $\frac{-1}{2u}[\{2a-(n+1)(n-2)b\}r+2(n+2)(n-1)(c+\lambda b)]$ are eigen values of the Ricci tensor S corresponding to the eigen vector τ_6 defined by $g(X,\tau_6) = J_6(X) = A(X)-B(X)-E(X)$ for all $X \in \chi(M)$ and the eigen vector τ_7 defined by $g(X,\tau_7) = J_7(X) = A(X)+B(X)+E(Z)$ for all $X \in \chi(M)$ respectively, whenever $r,\ a,\ b,\ c$ are constants.

The forms of the Ricci tensor and scalar curvature is obtained for Riemannian manifold with generalized space matter tensor satisfying Einstein field equation under the restrictions that:

- the generalized space matter tensor is vanishing,
- the generalized space matter tensor is symmetric,
- the generalized space matter tensor is recurrent,
- the generalized space matter tensor is weakly symmetric.

The most obvious idea is to create examples of the generalized space matter tensor. Moreover, the generalized space matter tensor may be studied as a tool in soliton theory like generalized Ricci soliton, Yamabe soliton and others. The tensor may also be used as a tool in the theory of relativity and cosmology.

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