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$L_{3,2,1}$ LABELING OF FIRECRACKER GRAPH

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Abstract. Let $G = (V, E)$ be a graph. An L(3,2,1) labeling of G is a function $f : V \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V$, $|f(u) - f(v)| \geq 3$ if $d(u, v) = 1$, $|f(u) - f(v)| \geq 2$ if $d(u, v) = 2$, and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 3$. Let $k \in \mathbb{N}$, a $k - L(3, 2, 1)$ labeling is a labeling L(3,2,1) where all labels are not greater than k. An L(3,2,1) number of G, denoted by $\lambda_{3,2,1}(G)$, is the smallest non negative integer k such that G has a $k - L$(3,2,1) labeling. In this paper, we determine $\lambda_{3,2,1}$ of firecracker graphs.

Key words and Phrases: $L(3, 2, 1)$ number, firecraker graph

1. INTRODUCTION

Graph theory has numerous real-world applications. One of them is used to govern the use of FM frequencies in the world of technology, particularly radio. Each transmitter must be assigned a frequency in order to send a signal. The earliest frequency assignment issues came when it was discovered that transmitters designated to the same or closely similar frequencies could interfere with one another [5]. To minimize interference, assigning various transmitters to different non-interfering frequencies, or coming as near to this as feasible, given the limits, was a straightforward solution. Another issue is determining how to select a new

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frequency and minimizing the frequency range used in order to decrease expenditures.

In 2004, Liu and Shao [6] defined a L(3,2,1) labeling. They get results about $\lambda_{3,2,1}$ in several classes of graphs, with determining boundaries for $\lambda_{3,2,1}$ on irregular graphs, Halin Graphs, and Planar Graphs with a maximum degree Δ. Then in 2005, Clipperton cs [2] studied $\lambda_{3,2,1}$ on classes of graphs like path graphs, cycle graphs, n-ary tree graphs, and regular caterpillar graphs.

In 2011, Chia cs [1] studied some general concepts regarding labeling L(3,2,1) and gave an upper bound of $\lambda_{3,2,1}$ for any graph with a maximum degree Δ. Furthermore, they also described the labeling L(3,2,1) on a tree, rooted tree, and Cartesian product of path graphs and cycle graphs. In this paper, the author determines $\lambda_{3,2,1}$ on the firecracker graph.

Definition 1.1. [2] *Let $G = (V, E)$ be a graph and f be a mapping $f : V \to \mathbb{N}$. f is an $L(3,2,1)$-labeling of G if, for all $x, y \in V$,*

$$|f(x) - f(y)| \geq \begin{cases} 3, & \text{if } d(x,y) = 1; \\ 2, & \text{if } d(x,y) = 2; \\ 1, & \text{if } d(x,y) = 3 \end{cases}$$

Definition 1.2. [2] *Let $k \in \mathbb{N} \cup \{0\}$. A k-L(3,2,1) labeling is labeling L(3,2,1) such that every label used is not greater than k. The L(3,2,1) number on G graph, denoted as $\lambda_{3,2,1}(G)$, is the smallest number k so that G graph has the labeling k-L(3,2,1).*

Lemma 1.3. [1] *If G' subgraph of G, then $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$.*

Proof. By contradiction, suppose $\lambda_{3,2,1}(G') > \lambda_{3,2,1}(G)$. Let $\lambda_{3,2,1}(G') = k_1$ and $\lambda_{3,2,1}(G) = k_2$, meaning that k_1 is the smallest number so that graph G has the labeling k_1-L(3,2,1). Let f be a k_1-L(3,2,1) labeling on a graph G. $G' \subset G$, $V(G') \subset V(G)$, $E(G') \subset E(G)$ and f labeling k_1-L(3,2,1) on graph G. Thus, for every vertices $u, v \in V(G') \subset V(G)$, $|f(u) - f(v)| \geq 3$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 1$, $|f(u) - f(v)| \geq 2$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 2$ and $|f(u) - f(v)| \geq 1$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u, v) = 3$. This means that there is a k_1-L(3,2,1) on graph G'. $\lambda_{3,2,1}(G') = k_2$ and $k_2 > k_1$, meaning that there is a number smaller than k_2 (named k_1) so that graph G' has labeling k_1-L(3,2,1). This contradicts the minimality of $\lambda_{3,2,1}(G')$. Therefore $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$. □

Corollary 1.4. [1] *For any graph G with $\Delta(G) = \Delta > 0$ then $\lambda_{3,2,1}(G) \geq 2\Delta + 1$. If $\lambda_{3,2,1}(G) = 2\Delta+1$ and f is any labeling 2Δ+1-L(3,2,1), then for every $v \in V(G)$ where $d(v) = \Delta$ such that $f(v) \in \{0, 2\Delta + 1\}$.*

Corollary 1.5. [4] *Let G be a graph with $\Delta \geq 1$. If there is $v_1, v_2 \in V(G)$ with $d(v_1, v_2) = 2$ and $d(v_1) = d(v_2) = \Delta$, then $\lambda_{3,2,1}(G) \geq 2\Delta + 2$.*

2. MAIN RESULTS

Firecracker graph $F_{m,n}$ is a graph obtained form m graphs S_n by connecting a leaf from each S_n through a path P_m.[3]

Theorem 2.1. *Let m and n be positive integers with $m \geq 3$ and $n \geq 2$. If $F_{m,n}$ is a firecracker graph, then*

$$\lambda_{3,2,1}(F_{m,n}) = \begin{cases} 2\Delta + 1, & \text{for } n \geq 5 \text{ or} \\ & n = 2 \text{ and } 4 \geq m \text{ or} \\ & n = 4 \text{ and } 4 \geq m; \\ 2\Delta + 2, & \text{for } n = 3 \text{ or} \\ & n = 2 \text{ and } m \geq 5 \text{ or} \\ & n = 4 \text{ and } m \geq 5. \end{cases}$$

Proof. Let $V(F_{m,n}) = \{v_1, \ldots, v_m\} \cup \{v_0^1, \ldots, v_{n-1}^1\} \cup \cdots \cup \{v_0^m, \ldots, v_{n-1}^m\}$ and $E(F_{m,n}) = \{v_i v_{i+1} |\ i \in [1, m-1]\} \cup \{v_i v_0^i |\ i \in [1, m]\} \cup \{v_0^i v_j^i |\ i \in [1, m], j \in [1, n-1]\}$. Determine the lower bound of $\lambda_{3,2,1}(F_{m,n})$ divided into two cases. For illustration, see figure 1.

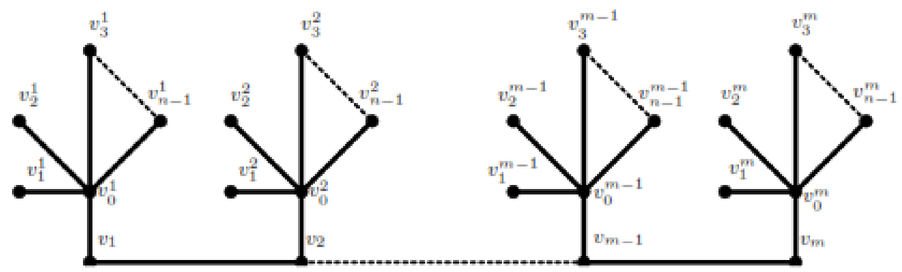

FIGURE 1. Firecracker graph $F_{m,n}$

- Case I : $n \geq 5$ or $n = 2$ and $4 \geq m$ or $n = 4$ and $4 \geq m$.
  In this case, the distance of each vertex of degree Δ is not equal to two. Based on corollary 1.4, we get $\lambda_{3,2,1}(F_{m,n}) \geq 2\Delta + 1$.
- Case II : $n = 3$ and $m \geq 3$ or $n = 2$ and $m \geq 5$ or $n = 4$ and $m \geq 5$.
  It is divided into three subcases to determine the lower bound for this case.
  (1) Subcase $n = 3$ and $m \geq 3$
  Note that, in this category $d(v_i) = \Delta = d(v_0^{i+1})$ for each $i \in [2, m-1]$ and $d(v_i, v_0^{i+1}) = 2$. Based on corollary 1.5, we get $\lambda_{3,2,1}(F_{m,n}) \geq 2\Delta + 2$ for $n = 3$.
  (2) Subcase $n = 2$ and $m \geq 5$.

In this case, $d(v_i) = \Delta = d(v_{i+1})$ for every $i \in [2, m-2]$ and $d(v_i, v_{i+1}) = 2$. Based on corollary 1.5, we get $\lambda_{3,2,1}(F_{m,n}) \geq 2\Delta + 2$ for $m \geq 5$ and $n = 2$.

(3) Subcase $n = 4$ and $m \geq 5$
By contradiction, suppose that $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$. In this case $2\Delta + 1 = 9$, so based on corollary 1.4, then $f(v) \in \{0, 9\}$ with $v \in V(F_{m,n})$. Since the vertex v_0^i is Δ, then $f(v_0^i) \in \{0, 9\}$ for $i \in [1, m]$. Let $f(v_0^i) = 0$, for odd i, and $f(v_0^i) = 9$ for even i. Consequently, $f(v_i) \in \{3, 5, 7, 9\}$ $\forall i$ is odd and $f(v_i) \in \{0, 2, 4, 6\}$ $\forall i$ is even. Note that $f(v_i) \neq 9$ $\forall i$ is odd and $f(v_i) \neq 0$ $\forall i$ is even. Therefore, $f(v_i) \in \{3, 5, 7\}$ $\forall i$ is odd and $f(v_i) \in \{2, 4, 6\}$ $\forall i$ is even.
Consider the following subcases :

(a) If $f(v_1) = 3$, then $f(v_2) = 6$. But there is no label for $f(v_3)$ that satisfies $|f(v_3) - f(v_2)| \geq 3$, because $f(v_3) \in \{5, 7\}$. Contradiction.

(b) If $f(v_1) = 5$ then $f(v_2) = 2$, $f(v_3) = 7$, and $f(v_4) = 4$. However $f(v_5)$ with condition $|f(v_5) - f(v_4)| \geq 3$, $|f(v_5) - f(v_3)| \geq 2$ and $|f(v_5) - f(v_2)| \geq 1$. Contradiction.

(c) If $f(v_1) = 7$ then
   - For $f(v_2) = 4$, the label $f(v_3)$ does not satisfy.
   - For $f(v_2) = 2$ and $f(v_3) = 5$, $f(v_4)$ there is no label that satisfies. .

   Contradiction.

Consequently , $\lambda_{3,2,1}(F_{m,n}) \geq 2\Delta + 2$ for $m \geq 5$ and $n = 4$.

Furthermore, the determination of the upper bound of $\lambda_{3,2,1}(F_{m,n})$ is divided into to cases.

(1) The case $n \geq 5$ or $n = 2$ and $4 \geq m$ or $n = 4$ and $4 \geq m$.
This subcase is further subdivided into three subcases to demonstrate the case's upper bound.

(a) Subcase $n \geq 5$.
Claim: If $n \geq 5$ then $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$.
To proof that $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$ for $n \geq 5$, define the labeling function L(3,2,1) on $F_{m,n}$ as follows :

$$f(v_i) = \begin{cases} 2, & if \quad i \equiv 1 \; mod \; 4;\; i \in [1, m]; \\ 9, & if \quad i \equiv 2 \; mod \; 4;\; i \in [1, m]; \\ 4, & if \quad i \equiv 3 \; mod \; 4;\; i \in [1, m]; \\ 7, & if \quad i \equiv 0 \; mod \; 4;\; i \in [1, m]. \end{cases}$$

$$f(v_0^i) = \begin{cases} 2n + 1, & if \quad i \equiv 1 \; mod \; 2;\; i \in [1, m]; \\ 0, & if \quad i \equiv 0 \; mod \; 2;\; i \in [1, m]. \end{cases}$$

for $i \equiv 1 \; mod \; 4$,

$$f(v_j^i) = \begin{cases} 0, & if \quad j = 1; \\ 2j, & if \quad 2 \leq j \leq n-1. \end{cases}$$

for $i \equiv 2 \; mod \; 4$,

$$f(v_j^i) = \begin{cases} 2j+1, & if \quad 1 \leq j \leq 3; \\ 2j+3, & if \quad 4 \leq j \leq n-1. \end{cases}$$

for $i \equiv 3 \; mod \; 4$,

$$f(v_j^i) = \begin{cases} 2j-2, & if \quad 1 \leq j \leq 2; \\ 2j, & if \quad 3 \leq j \leq n-1. \end{cases}$$

for $i \equiv 0 \; mod \; 4$,

$$f(v_j^i) = \begin{cases} 2j+1, & if \quad 1 \leq j \leq 2; \\ 2j+3, & if \quad 4 \leq j \leq n-1. \end{cases}$$

Moreover, it is demonstrated that f is $L(3,2,1)$-labeling.

(i) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u,v) = 1$. These are all possible vertices $u, v \in V(F_{m,n})$.

- The vertex $u = v_i$ and $v = v_{i+1}$ for an $i \in [1, m]$.
  - For $i \equiv 0 \; mod \; 4$ applies $|f(v_{i+1}) - f(v_i)| = |9-2| \geq 3$.
  - For $i \equiv 1 \; mod \; 4$ applies $|f(v_{i+1}) - f(v_i)| = |4-9| \geq 3$.
  - For $i \equiv 2 \; mod \; 4$ applies $|f(v_{i+1}) - f(v_i)| = |7-4| \geq 3$.
  - For $i \equiv 3 \; mod \; 4$ applies $|f(v_{i+1}) - f(v_i)| = |2-7| \geq 3$.
- The vertex $u = v_i$ and $v = v_0^i$ for an $i \in [1, m]$
  - If $i \equiv 0 \; mod \; 4$ then $|f(v_0^i) - f(v_i)| = |2n+1-2| \geq 3$.
  - If $i \equiv 1 \; mod \; 4$ then $|f(v_0^i) - f(v_i)| = |9-0| \geq 3$.
  - If $i \equiv 2 \; mod \; 4$ then $|f(v_0^i) - f(v_i)| = |2n+1-4| \geq 3$.
  - If $i \equiv 3 \; mod \; 4$ then $|f(v_0^i) - f(v_i)| = |7-0| \geq 3$.
- The vertex $u = v_j^i$ and $v = v_0^i$ for an $i \in [1, m]$.
  - For $i \equiv 1 \; mod \; 4$, we get $|f(v_j^i) - f(v_0^i)| = |0-2n+1| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2n+1-2j| \geq 3$.
  - For $i \equiv 2 \; mod \; 4$, apply $|f(v_j^i) - f(v_0^i)| = |2j+1-0| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2j+3-0| \geq 3$.
  - For $i \equiv 3 \; mod \; 4$, apply $|f(v_j^i) - f(v_0^i)| = |(2j-2)-(2n-1)| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2j-(2n+1)| \geq 3$.
  - For $i \equiv 0 \; mod \; 4$, we get $|f(v_j^i) - f(v_0^i)| = |2j+1-0| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2j+3-0| \geq 3$.

(ii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u,v) = 2$. The following is given all the possible vertices $u, v \in V(F_{m,n})$.

- The vertex $u = v_i$ and $v = v_{i+2}$ for an $i \in [1, m]$.
  - if $i \equiv 1 \; mod \; 4$ then $i+2 \equiv 3 \; mod \; 4$. Consequently $|f(v_{i+2}) - f(v_i)| = |4-2| \geq 2$.

– For $i \equiv 2$ mod 4, then we have $|f(v_{i+2})-f(v_i)| = |7-9| \geq 2$.

- The vertex $u = v_i$ and $v = v_0^{i+1}$ for an $i \in [1, m]$.
  – For $i \equiv 1$ mod 4 then $i + 1 \equiv 0$ mod 2. That means we get $|f(v_{i+1}) - f(v_i)| = |0 - 2| \geq 2$.
  – For $i \equiv 2$ mod 4,we have $|f(v_{i+1}) - f(v_i)| = |(2n + 1) - 9| \geq 2$.
  – If $i \equiv 3$ mod 4 such that $|f(v_{i+1}) - f(v_i)| = |0 - 4| \geq 2$.
  – IF $i \equiv 0$ mod 4 then we get $|f(v_{i+1}) - f(v_i)| = |(2n + 1) - 7| \geq 2$.
- The vertex $u = v_i$ and $v = v_j^i$ for an $i \in [1, m]$.
  – For $i \equiv 1$ mod 4, then $|f(v_j^i) - f(v_i)| = |0 - 2| \geq 2$ and $|f(v_j^i) - f(v_i)| = |0 - 2j| \geq 2$.
  – For $i \equiv 2$ mod 4, then $|f(v_j^i) - f(v_i)| = |2j + 1 - 9| \geq 2$ and $|f(v_j^i) - f(v_i)| = |2j + 3 - 9| \geq 2$.
  – For $i \equiv 3$ mod 4, then $|f(v_j^i) - f(v_i)| = |2j - 2 - 4| \geq 2$ and $|f(v_j^i) - f(v_i)| = |2j - 4| \geq 2$.
  – For $i \equiv 0$ mod 4, then $|f(v_j^i) - f(v_i)| = |2j + 1 - 7| \geq 2$ and $|f(v_j^i) - f(v_i)| = |2j + 3 - 7| \geq 2$.
- The vertex $u = v_j^i$ and $v = v_j^{i+1}$ for an $i \in [1, m]$.
  – For $i \equiv 1$ mod 4 then $|f(v_j^i) - f(v_j^{i+1})| = |2j - 0| \geq 2$.
  – For $i \equiv 2$ mod 4 then $|f(v_j^i) - f(v_j^{i+1})| = |2j + 3 - (2j + 1)| \geq 2$.
  – For $i \equiv 3$ mod 4 then $|f(v_j^i) - f(v_j^{i+1})| = |2j - (2j - 2)| \geq 2$.
  – For $i \equiv 0$ mod 4 then $|f(v_j^i) - f(v_j^{i+1})| = |2j + 3 - (2j + 1)| \geq 2$.

(iii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 3$. The following are possible of all vertices $u, v \in V(F_{m,n})$.

- Vertex $u = v_i$ and $v = v_{i+3}$ for an $i \in [1, m]$.
  – If $i \equiv 1$ mod 4 then $i + 3 \equiv 0$ mod 4. Thus we get $|f(v_{i+3}) - f(v_i)| = |7 - 2| \geq 1$.
  – If $i \equiv 2$ mod 4 then $|f(v_{i+3}) - f(v_i)| = |9 - 2| \geq 1$.
  – If $i \equiv 3$ mod 4 then $|f(v_{i+3}) - f(v_i)| = |9 - 4| \geq 1$.
  – If $i \equiv 0$ mod 4 then $i + 3 \equiv 3$ mod 4. Hence $|f(v_{i+3}) - f(v_i)| = |7 - 4| \geq 1$.
- Vertex $u = v_i$ and $v = v_0^{i+2}$ for an $i \in [1, m]$.
  – If $i \equiv 1$ mod 4 then $i + 2 \equiv 0$ mod 2. Thus we get $|f(v_0^{i+2}) - f(v_i)| = |0 - 2| \geq 1$.
  – For $i \equiv 2$ mod 4 then $|f(v_0^{i+2}) - f(v_i)| = |2n + 1 - 9| \geq 1$.
  – For $i \equiv 3$ mod 4, then $|f(v_0^{i+2}) - f(v_i)| = |0 - 4| \geq 1$.
  – If $i \equiv 0$ mod 4 then $i+2 \equiv 1$ mod 2. Such that $|f(v_0^{i+2}) - f(v_i)| = |2n + 1 - 7| \geq 1$.

- Vertex $u = v_i$ and $v = v_j^{i+1}$ for an $i \in [1, m]$.
  - If $i \equiv 1 \; mod \; 4$ then $i + 1 \equiv 2 \; mod \; 4$. Thus implies $|f(v_j^{i+1}) - f(v_i)| = |2j + 1 - 2| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |2j + 3 - 2| \geq 1$.
  - If $i \equiv 2 \; mod \; 4$ then, we get $|f(v_j^{i+1}) - f(v_i)| = |2j - 2 - 9| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |2j - 9| \geq 1$.
  - If $i \equiv 3 \; mod \; 4$, then $|f(v_j^{i+1}) - f(v_i)| = |2j + 1 - 4| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |2j + 3 - 4| \geq 1$.
  - If $i \equiv 0 \; mod \; 4$ then $i + 1 \equiv 1 \; mod \; 4$. thus $|f(v_j^{i+1}) - f(v_i)| = |0 - 2| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |2j - 2| \geq 1$.

Therefore, then f is labeling $2\Delta + 1$ - L(3,2,1) on $F_{m,n}$, with $n \geq 5$.

(b) Subcase $n = 2$ and $4 \geq m$.
Claim: If $n = 2$ and $4 \geq m$ then $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$.
For prove that $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$ for $n = 2$, define the labeling function L(3,2,1) on graph $F_{m,n}$ as follows:

$$f(v_i) = \begin{cases} 4, & if \quad i \equiv 1 \; mod \; 4; \; i \in [1, 4]; \\ 7, & if \quad i \equiv 2 \; mod \; 4; \; i \in [1, 4]; \\ 0, & if \quad i \equiv 3 \; mod \; 4; \; i \in [1, 4]; \\ 5, & if \quad i \equiv 0 \; mod \; 4; \; i \in [1, 4]. \end{cases}$$

$$f(v_0^i) = \begin{cases} n - 1, & if \quad i \equiv 1 \; mod \; 4; \; i \in [1, 4]; \\ n, & if \quad i \equiv 2 \; mod \; 4; \; i \in [1, 4]; \\ n + 1, & if \quad i \equiv 3 \; mod \; 4; \; i \in [1, 4]; \\ n, & if \quad i \equiv 0 \; mod \; 4; \; i \in [1, 4]. \end{cases}$$

With

$$f(v_j^i) = \begin{cases} 6, & if \quad i \equiv 1 \; mod \; 4; \; i \in [1, 4]; \\ 5, & if \quad i \equiv 2 \; mod \; 4; \; i \in [1, 4]; \\ 6, & if \quad i \equiv 3 \; mod \; 4; \; i \in [1, 4]; \\ 7, & if \quad i \equiv 0 \; mod \; 4; \; i \in [1, 4]. \end{cases}$$

Further, it will be shown that f labeling $L(3, 2, 1)$.

(i) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 1$. These are the all possible of vertices $u, v \in V(F_{m,n})$.
- Vertex $u = v_i$ and $v = v_{i+1}$ for an $i \in [1, m]$.
  - If $i \equiv 1 \; mod \; 4$ then $i + 1 \equiv 2 \; mod \; 4$. Thus $|f(v_{i+1}) - f(v_i)| = |7 - 4| \geq 3$.
  - For $i \equiv 2 \; mod \; 4$ then $|f(v_{i+1}) - f(v_i)| = |0 - 7| \geq 3$.
  - If $i \equiv 3 \; mod \; 4$ then $i + 1 \equiv 0 \; mod \; 4$. Thus $|f(v_{i+1}) - f(v_i)| = |5 - 0| \geq 3$.
- Vertex $u = v_i$ and $v = v_0^i$ for an $i \in [1, m]$.
  - For $i \equiv 1 \; mod \; 4$ then $|f(v_0^i) - f(v_i)| = |n - 1 - 4| = |n - 5| \geq 3$.

- If $i \equiv 2\ mod\ 4$, then $|f(v_0^i) - f(v_i)| = |n - 7| \geq 3$.
- If $i \equiv 3\ mod\ 4$ then we get $|f(v_0^i) - f(v_i)| = |n+1-0| \geq 3$.
- For $i \equiv 0\ mod\ 4$, then $|f(v_0^i) - f(v_i)| = |n - 5| \geq 3$.

• Vertex $u = v_j^i$ and $v = v_0^i$ for an $i \in [1, m]$.
- For $i \equiv 1\ mod\ 4$, we get $|f(v_j^i) - f(v_0^i)| = |n - 1 - 6| \geq 3$.
- For $i \equiv 2\ mod\ 4$ then $|f(v_j^i) - f(v_0^i)| = |n - 5| \geq 3$.
- If $i \equiv 3\ mod\ 4$, then $|f(v_j^i) - f(v_0^i)| = |(n + 1) - 6| \geq 3$.
- If $i \equiv 0\ mod\ 4$ then $i \equiv 0\ mod\ 4$. Hence $|f(v_j^i) - f(v_0^i)| = |n - 7| \geq 3$.

(ii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 2$. These are the all possible of vertices $u, v \in V(F_{m,n})$.

• Vertex $u = v_i$ and $v = v_{i+2}$ for and $i \in [1, m]$.
- If $i \equiv 1\ mod\ 4$ then $i + 2 \equiv 3\ mod\ 4$. Thus $|f(v_{i+2}) - f(v_i)| = |0 - 4| \geq 2$.
- For $i \equiv 2\ mod\ 4$ then $|f(v_{i+2}) - f(v_i)| = |5 - 7| \geq 2$.

• Vertex $u = v_i$ and $v = v_0^{i+1}$ for an $i \in [1, m]$.
- For $i \equiv 1\ mod\ 4$ then $|f(v_0^{i+1}) - f(v_i)| = |n - 4| \geq 2$.
- If $i \equiv 2\ mod\ 4$ then $i + 1 \equiv 3\ mod\ 4$. Thus $|f(v_0^{i+1}) - f(v_i)| = |(n + 1) - 7| \geq 2$.
- For $i \equiv 3\ mod\ 4$, then, we get $|f(v_0^{i+1}) - f(v_i)| = |n-5| \geq 2$.

• Vertex $u = v_i$ and $v = v_j^i$ for and $i \in [1, m]$.
- For $i \equiv 1\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |6 - 4| \geq 2$.
- If $i \equiv 2\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |5 - 7| \geq 2$.
- For $i \equiv 3\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |6 - 0| \geq 2$.
- For $i \equiv 0\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |7 - 5| \geq 2$.

(iii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 3$. These are the all possible of vertices $u, v \in V(F_{m,n})$.

• If $u = v_i$ and $v = v_{i+3}$ for an $i \in [1, m]$ with $i \equiv 1\ mod\ 4$, then $|f(v_{i+3}) - f(v_i)| = |5 - 4| \geq 1$.

• Vertex $u = v_i$ and $v = v_0^{i+2}$ for an $i \in [1, m]$.
- If $i \equiv 1\ mod\ 4$ then $|f(v_0^{i+2}) - f(v_i)| = |n + 1 - 4| \geq 1$.
- If $i \equiv 2\ mod\ 4$ then $|f(v_0^{i+2}) - f(v_i)| = |n - 7| \geq 1$.
- If $i \equiv 0\ mod\ 4$, then $|f(v_0^{i+2}) - f(v_i)| = |n - 5| \geq 1$.
- For $i \equiv 3\ mod\ 4$, then $|f(v_0^{i+2}) - f(v_i)| = |n - 1 - 0| \geq 1$.

• Vertex $u = v_i$ and $v = v_j^{i+1}$ for an $i \in [1, m]$.
- If $i \equiv 1\ mod\ 4$ then $i + 1 \equiv 2\ mod\ 4$. Thus $|f(v_j^{i+1}) - f(v_i)| = |5 - 4| \geq 1$
- For $i \equiv 2\ mod\ 4$ then $|f(v_j^{i+1}) - f(v_i)| = |6 - 7| \geq 1$.
- For $i \equiv 3\ mod\ 4$, then $|f(v_j^{i+1}) - f(v_i)| = |7 - 0| \geq 1$.

Therefore, f is labeling $2\Delta + 1$ - L(3,2,1) on graph $F_{m,n}$ with $n = 2$ and $4 \geq m$.

(c) Subcase $n = 4$ and $4 \geq m$.

Claim: if $n = 4$ and $4 \geq m$ then $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$.
For prove that $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 1$ for $n = 4$ and $4 \geq m$, define the labeling function $L(3,2,1)$ on graph $F_{m,n}$ as follows:

$$f(v_i) = \begin{cases} 5, & if \quad i \equiv 1 \ mod\ 4;\ i \in [1, m]; \\ 2, & if \quad i \equiv 2 \ mod\ 4;\ i \in [1, m]; \\ 7, & if \quad i \equiv 3 \ mod\ 4;\ i \in [1, m]; \\ 4, & if \quad i \equiv 0 \ mod\ 4;\ i \in [1, m]. \end{cases}$$

$$f(v_0^i) = \begin{cases} 0, & if \quad i \equiv 1 \ mod\ 2;\ i \in [1, m]; \\ 2n+1, & if \quad i \equiv 0 \ mod\ 2;\ i \in [1, m]. \end{cases}$$

for $i \equiv 1 \ mod\ 4$,

$$f(v_j^i) = \begin{cases} 2j+1, & if \quad j = 1; \\ 2j+3, & if \quad 2 \leq j \leq 3. \end{cases}$$

for $i \equiv 2 \ mod\ 4$,

$$f(v_j^i) = \begin{cases} 0, & if \quad j = 1; \\ 2j, & if \quad 2 \leq j \leq 3. \end{cases}$$

for $i \equiv 3 \ mod\ 4$,

$$f(v_j^i) = \begin{cases} 2j+1, & if \quad 1 \leq j \leq 2; \\ 2j+3, & if \quad j = 3. \end{cases}$$

for $i \equiv 0 \ mod\ 4$,

$$f(v_j^i) = \begin{cases} 2j-2, & if \quad 1 \leq j \leq 2; \\ 2j, & if \quad j = 3. \end{cases}$$

It will also be demonstrated that f labels $L(3,2,1)$.

(i) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u,v) = 1$. These are the all possible of vertices $u, v \in V(F_{m,n})$.
- Vertex $u = v_i$ and $v = v_{i+1}$ for an $i \in [1, m]$.
  - For $i \equiv 0 \ mod\ 4$ then $|f(v_{i+1}) - f(v_i)| = |5 - 2| \geq 3$.
  - If $i \equiv 1 \ mod\ 4$ then $|f(v_{i+1}) - f(v_i)| = |2 - 7| \geq 3$.
  - If $i \equiv 2 \ mod\ 4$, then $|f(v_{i+1}) - f(v_i)| = |7 - 4| \geq 3$.
- Vertex $u = v_i$ and $v = v_0^i$ for an $i \in [1, m]$.
  - For $i \equiv 0 \ mod\ 4$ then $|f(v_0^i) - f(v_i)| = |5 - 0| \geq 3$.
  - If $i \equiv 1 \ mod\ 4$, then $|f(v_0^i) - f(v_i)| = |2n + 1 - 2| \geq 3$.
  - If $i \equiv 2 \ mod\ 4$, then $|f(v_0^i) - f(v_i)| = |0 - 7| \geq 3$.
  - For $i \equiv 3 \ mod\ 4$ then $|f(v_0^i) - f(v_i)| = |2n + 1 - 4| \geq 3$.
- Vertex $u = v_j^i$ and $v = v_0^i$ for an $i \in [1, m]$.
  - If $i \equiv 1 \ mod\ 4$ then $|f(v_j^i) - f(v_0^i)| = |2j + 1 - 0| \geq 3$, and $|f(v_j^i) - f(v_0^i)| = |2j + 3 - 0| \geq 3$.

 - If $i \equiv 2\ mod\ 4$ then $|f(v_j^i) - f(v_0^i)| = |2n + 1 - 0| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2n + 1 - 2j| \geq 3$.
 - For $i \equiv 3\ mod\ 4$, then $|f(v_j^i) - f(v_0^i)| = |(2j + 1) - 0| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2j + 3 - 0| = |2j + 3| \geq 3$.
 - If $i \equiv 0\ mod\ 4$ then $|f(v_j^i) - f(v_0^i)| = |2j - 2 - (2n+1)| \geq 3$ and $|f(v_j^i) - f(v_0^i)| = |2j - (2n + 1)| \geq 3$.

(ii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 2$. These are the all possible of vertices Vertex $u, v \in V(F_{m,n})$.

- Vertex $u = v_i$ and $v = v_{i+2}$ for an $i \in [1, m]$.
  - If $i \equiv 1\ mod\ 4$, then $i+2 \equiv 3\ mod\ 4$. thus then $|f(v_{i+2}) - f(v_i)| = |5 - 7| \geq 2$.
  - For $i \equiv 2\ mod\ 4$ then $|f(v_{i+2}) - f(v_i)| = |2 - 4| \geq 2$.
- Vertex $u = v_i$ and $v = v_0^{i+1}$ for an $i \in [1, m]$.
  - For $i \equiv 1\ mod\ 4$ then $|f(v_0^{i+1}) - f(v_i)| = |5 - (2n+1)| \geq 2$.
  - If $i \equiv 2\ mod\ 4$, then $|f(v_0^{i+1}) - f(v_i)| = |2 - 0| \geq 2$.
  - If $i \equiv 3\ mod\ 4$, then $|f(v_0^{i+1}) - f(v_i)| = |7 - (2n+1)| \geq 2$.
- Vertex $u = v_i$ and $v = v_j^i$ for an $i \in [1, m]$.
  - If $i \equiv 1\ mod\ 4$ then then $|f(v_j^i) - f(v_i)| = |5 - (2j + 1)| \geq 2$, and $|f(v_j^i) - f(v_i)| = |5 - (2j + 3)| \geq 2$.
  - If $i \equiv 2\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |2 - 0| \geq 2$ and $|f(v_j^i) - f(v_i)| = |2 - (2j)| \geq 2$.
  - For $i \equiv 3\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |7 - (2j + 1)| \geq 2$ and $|f(v_j^i) - f(v_i)| = |7 - (2j + 3)| \geq 2$.
  - For $i \equiv 0\ mod\ 4$ then $|f(v_j^i) - f(v_i)| = |4 - (2j - 2)| \geq 2$ and $|f(v_j^i) - f(v_i)| = |4 - (2j)| \geq 2$
- Vertex $u = v_j^i$ and $v = v_j^{i+1}$ for an $i \in [1, m]$.
  - For $i \equiv 1\ mod\ 4$ then $|f(v_j^i) - f(v_j^{i+1})| = |2j + 1 - (2j + 3)| \geq 2$.
  - If $i \equiv 2\ mod\ 4$ then $|f(v_j^i) - f(v_j^{i+1})| = |2j - 0| \geq 2$.
  - For $i \equiv 3\ mod\ 4$ then $|f(v_j^i) - f(v_j^{i+1})| = |2j + 1 - (2j + 3)| = |4j + 4| \geq 2$.
  - If $i \equiv 0\ mod\ 4$ then $|f(v_j^i) - f(v_j^{i+1})| = |(2j - 2) - (2j)| \geq 2$.

(iii) Take two arbitrary vertices $u, v \in V(F_{m,n})$ with $d(u, v) = 3$. These are the all possible of vertices $u, v \in V(F_{m,n})$.

- If $u = v_i$ then $v = v_{i+3}$ for an $i \in [1, m]$ with $i \equiv 1\ mod\ 4$, then $|f(v_{i+3}) - f(v_i)| = |5 - 4| \geq 1$.
- Vertex $u = v_i$ and $v = v_0^{i+2}$ for an $i \in [1, m]$.
  - If $i \equiv 1\ mod\ 4$ then $i+2 \equiv 0\ mod\ 2$. thus then $|f(v_0^{i+2}) - f(v_i)| = |5 - 0| \geq 1$.
  - If $i \equiv 2\ mod\ 4$ then $|f(v_0^{i+2}) - f(v_i)| = |2n + 1 - 2| \geq 1$.
  - For $i \equiv 3\ mod\ 4$, then $|f(v_0^{i+2}) - f(v_i)| = |7 - 0| \geq 1$.

- If $i \equiv 0 \ mod \ 4$, then $|f(v_0^{i+2}) - f(v_i)| = |2n + 1 - 4| = |2n - 3| \geq 1$.

- Vertex $u = v_i$ and $v = v_j^{i+1}$ for an $i \in [1, m]$.
  - If $i \equiv 1 \ mod \ 4$ then $|f(v_j^{i+1}) - f(v_i)| = |5 - (0)| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |5 - (2j)| \geq 1$.
  - For $i \equiv 2 \ mod \ 4$ then $|f(v_j^{i+1}) - f(v_i)| = |2-(2j+1)| \geq 1$ and $|f(v_j^{i+1}) - f(v_i)| = |2 - (2j + 3)| \geq 1$.

Therefore, f is labeling $2\Delta + 1$ - $L(3,2,1)$ on $F_{m,n}$, with $n = 4$ and $4 \geq m$.

(2) Case $n = 3$, atau $n = 2$ and $m \geq 5$, atau $n = 4$ and $m \geq 5$.

To prove the upper bound of this case, claim that $\lambda_{3,2,1}(F_{m,n}) = 2\Delta + 2$. Further labeling $\lambda_{3,2,1}$-$L(3,2,1)$ for subcase $n = 3$ shown in figure 2, and subcase $n = 2$ with $m \geq 5$ is implementation in figure 3 moreover for subcase $n = 4$ and $m \geq 5$ in figure 4.

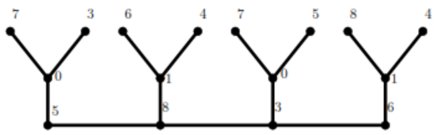

FIGURE 2. $\lambda_{3,2,1}(F_{4,3}) = 8$

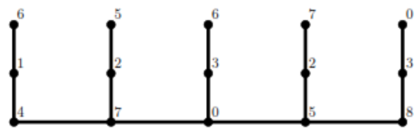

FIGURE 3. $\lambda_{3,2,1}(F_{5,2}) = 8$

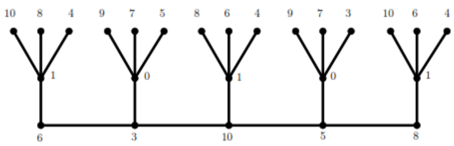

FIGURE 4. $\lambda_{3,2,1}(F_{5,4}) = 10$

□

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