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ON A CLASS OF REAL QUINTIC MOMENT PROBLEM

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Abstract. The present paper is centered on a class of real quintic moment problem. We state some conditions for the existence of a representative measure and we provide it explicitly. We also state some cases where no representative measure exists. Some numerical examples are presented to illustrate construction of the representative measure as well as to highlight the conflicts behind the irresolvability.

Key words and Phrases: truncated moment problem, recursively determinate moment matrix, representing measure, real quintic moment problem.

1. INTRODUCTION

Given a doubly indexed finite sequence of real numbers

$$\beta \equiv \beta^{(m)} = \{\beta_{ij}\}_{0\le i+j\le m} = \{\beta_{00}, \beta_{10}, \beta_{01}, \dots, \beta_{m0}, \dots, \beta_{0m}\},$$

with $\beta_{00} > 0$, the truncated moment problem (in short TRMP) associated to β means to find a Borel positive measure μ supported in $\mathbb{R}^2$ such that:

$$\beta_{ij} = \int x^i y^j d\mu \qquad (0 \le i+j \le m). \tag{1}$$

A sequence β satisfying (1) is called a sequence of truncated moment and the solution μ, the representative measure associated to the sequence β. There is an equivalent to the TRMP, that is the TCMP (truncated complex moment problem)[6], hence we use the term (TMP) problem of shortened moments.

The multidimensional truncated moment problem has been the subject of several studies, mainly by Curto, Fialkow and others as found for example in [3, 4,

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5, 6, 7, 8, 10, 11, 12, 13, 16, 18, 23, 24, 25]. In 1994 J. Stochel [23] showed that the truncated moment problem is more general than the full moment problem, i.e. a solution of the truncated moment problem implies a solution of the full one, which is widely studied as for instance see e.g [1, 2, 19, 22]. H. Richter pointed out in [20] that if a sequence of moments admits one or more representative measures, one of them must be of finite atomic type. So, if a real finite doubly indexed sequence $\beta^{(m)}$ has a representative measure μ, it can be of finitely atomic type. That is, we can write

$$\mu := \sum_{k=1}^{r} \rho_k \delta_{(x_k, y_k)},$$

where the positive numbers ρ_k and the couples $(x_k, y_k), 1 \leq k \leq r$, are called respectively weights and atoms of the measure μ which is said r-atomic, and we have

$$\beta_{ij} = \rho_1 x_1^i y_1^j + \cdots + \rho_r x_r^i y_r^j = \int x^i y^j d\mu, \quad 0 \leq i + j \leq m.$$

To solve the TMP for a sequence $\beta = \beta^{(m)}$ where $m = 2n$, Curto and Fialkow developed an approach based on positivity, on the flat extension theorem of the moment matrix $M(n)$ associated to the sequence β and on the core variety $\mathcal{V} \equiv \mathcal{V}(\beta)$ which contains the support of each representative measure of β, introduced in [17]. Dio and Shmdgen proved in [14] that if μ is a solution of (1), then the points of the core variety are exactly the atoms of the atomic measure μ. The problem has been completely solved for $n \in \{1, 2\}$ in [3, 6, 11]. For $n = 3$, it has been closely investigated and in particular the extreme case where the rank of the $M(3)$ matrix of moments associated to $\beta^{(6)}$ and the cardinal of the associated core variety are equal [8, 10, 12, 24, 25]. For $m = 3$, we can find a complete solution in [18] based on the commutativity conditions of the matrix associated to the sequence of cubic moment. In [13], R. Curto and S. Yoo presented an alternative solution for the non-singular cubic moment problem. For the resolution of the TMP associated to the $\beta^{(2n)}$, Curto and Fialkow introduced the notions of the recursively generated moment matrix and the moment matrix recursively determined.

In general, it is very difficult to prove existence results using the flat extension theorem. However, there are a number of exceptions and simple cases in the literature. Using a numerical algorithm, Fialkow [16, Algorithm 4.10] tests the existence or not of a positive flat extension for the class of moment matrix recursively determined which has finite core variety. In this paper, we study a class of real quintic moment problem ($m = 5$), using an approach based on matrix positivity and flat extension theorem.

Let $\beta \equiv \beta^{(5)} = \{\beta_{ij}\}_{0 \le i+j \le 5}$ be a doubly indexed finite sequence of real numbers, with $\beta_{00} > 0$, the following matrices $M(2)$ and $B(3)$ are associated to β.

$$M(2) := \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}; \; B(3) := \begin{pmatrix} \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\ \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\ \beta_{41} & \beta_{32} & \beta_{23} & \beta_{14} \\ \beta_{32} & \beta_{23} & \beta_{14} & \beta_{05} \end{pmatrix}. \tag{2}$$

If there exists a matrix W such that $B(3) = M(2)W$, which is equivalent to $\operatorname{Rang} B(3) \subseteq \operatorname{Rang} M(2)$ by applying Douglas factorization lemma [15]. Then from symmetry of $M(2)$, the matrix $W^T M(2) W$ is symmetric too and it takes the form:

$$W^T M(2) W = \begin{pmatrix} a & b & c & d \\ b & x & y & e \\ c & y & z & f \\ d & e & f & g \end{pmatrix}. \tag{3}$$

The relations between the entries x and c, y and d and z and e in $W^T M(2) W$ allow us to determine a positive extension $M(3)$ of $M(2)$ and $M(3)$ columns dependence relations, as well as the core variety $\mathcal{V}$ of $M(3)$ and the support of the minimum representative measure associated to the sequence β, when it exists. We focus on the cases $(x, y, z) = (c, d, e)$, $(x = c, y = d, z < e)$ and $(x < c)$. It is worth mentioning that the other cases related to $(x, y, z) \neq (c, d, e)$ represent an open problem that we plan to investigate in future work.
For the case $(x, y, z) = (c, d, e)$, we point out that β admits a unique finite representative measure $(\operatorname{rank} M(2))$-atomic. While for the cases $(x = c, y = d, z < e)$ and $(x < c)$, we establish necessary and sufficient conditions to have $M(3)$ positive semidefinite, recursively determined and verifying $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$. Furthermore, we also state sufficient conditions for the existence of a representative measure $(\operatorname{rank} M(2) + 1)$-atomic.

Since $M(2) \succeq 0$ and $\operatorname{Rang} B(3) \subset \operatorname{Rang} M(2)$ are two necessary conditions for the resolution of the quintic TRMP, our task is to determine sufficient conditions for the existence of an extension $M(3)$ of $M(2)$ which is positive semidefinite, recursively determined and verifying $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$. Hence, there are three $M(3)$ columns dependence relations, and we study the possible existence of a flat extension of $M(3)$. If this process fails, there will be no representative measure of β which is $(\operatorname{rank} M(2) + 1)$-atomic.

This article is organized as follows. In Section 2, we recall some useful tools that will be used for the resolution of the TMP. We also recall the notions of recursively determined matrix and recursively generated matrix according to Curto and Fialkow. Section 3 is devoted to solving a class of quintic TRMP and to present some numerical examples to illustrate our findings. The computations are done with Mathematica software.

2. PRELIMINARIES

In this section, we recall some results and notations that will be used in the sequel.
We denote by $\mathcal{M}_{(n,p)}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ the set of $n \times p$ matrices. For a sequence of moments $\beta = \beta^{(2n)} \equiv \{\beta_{ij}\}_{0 \leq i+j \leq 2n}$, we associate the matrix of moment $M(n)$, and if μ is a representative measure of β, then for every polynomial $p \equiv \sum_{l,k} a_{lk} x^l y^k \in \mathbb{R}[x,y]$, the space of polynomials with two indeterminates, we have

$$0 \leq \int |p(x,y)|^2 d\mu = \sum_{l,k,l',k'} a_{lk} a_{l'k'} \int x^{l+k'} y^{k+l'} d\mu = \sum_{l,k,l',k'} a_{lk} a_{l'k'} \beta_{l+k',k+l'}.$$

Hence, if β admits a representative measure then the matrix $M(n)$ is positive semidefinite. The matrix $M(n)$ admits a decomposition by blocks $M(n) = (B[i,j])_{0 \leq i,j \leq n}$

$$M(n) := \begin{pmatrix} B[0,0] & B[0,1] & \dots & B[0,n] \\ B[1,0] & M[1,1] & \dots & B[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ B[n,0] & B[n,1] & \dots & B[n,n] \end{pmatrix},$$

where

$$B[i,j] = \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \cdots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \cdots & \beta_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \cdots & \beta_{0,i+j} \end{pmatrix}, \quad 0 \leq i,j \leq n.$$

Thus, each block $B[i,j]$ has Hankel's property, i.e. it is constant on each cross diagonal. If we choose a labelling for the columns and rows of the moment matrix $M(n)$, considering the lexicographic order of degree $1, X, Y, X^2, XY, Y^2, \dots, X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n$. For example, the matrix $M(2)$ is written as follows

$$M(2) = \begin{array}{c} \\ 1 \\ \\ X \\ Y \\ \\ X^2 \\ XY \\ Y^2 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & & X & Y & & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} \beta_{00} & | & \beta_{10} & \beta_{01} & | & \beta_{20} & \beta_{11} & \beta_{02} \\ -- & - & -- & -- & - & -- & -- & -- \\ \beta_{10} & | & \beta_{20} & \beta_{11} & | & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & | & \beta_{11} & \beta_{02} & | & \beta_{21} & \beta_{12} & \beta_{03} \\ -- & - & -- & -- & - & -- & -- & -- \\ \beta_{20} & | & \beta_{30} & \beta_{21} & | & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & | & \beta_{21} & \beta_{12} & | & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & | & \beta_{12} & \beta_{03} & | & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix} \end{array}. \tag{4}$$

For a symmetric matrix A, we write $A \succeq 0$ if A is positive semidefinite and $A > 0$ if A is positive definite.

In the following theorem, Smul'jan [21] establishes a necessary and sufficient conditions which ensure positive extension and flatness of a positive matrix.

Theorem 2.1. *Let $A \in \mathcal{M}_{(n,n)}(\mathbb{C})$, $B \in \mathcal{M}_{(n,p)}(\mathbb{C})$, and $C \in \mathcal{M}_{(p,p)}(\mathbb{C})$ be matrices of complex numbers. We have,*

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0 \Longleftrightarrow \begin{cases} A \succeq 0 \\ B = AW \ (\text{ for some } W \in \mathcal{M}_{(n,p)}(\mathbb{C})). \\ C \succeq W^*AW \end{cases}$$

$\tilde{A}$ is called an extension of A. Moreover, $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A) \Longleftrightarrow C = W^*AW$ *for some W such that $AW = B$. If $A \succeq 0$ then every extension $\tilde{A}$ of A satisfying* $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A)$, *is said to be flat and it is necessary positive semidefinite.*

According to the Theorem 2.1, $M(n) \succeq 0$ admits a flat extension $M(n+1)$, which is necessary positive semidefinite, is equivalent to have the both next assertions,

(i) $B = M(n)W$ for some matrix W;
(ii) $C = W^T M(n)W$ is a Hankel matrix.

Let us note also, that for all matrices $\tilde{A}, B, W, C$ and A, defined in Theorem 2.1, with A symmetric we have,

$$\begin{pmatrix} I_n & 0 \\ -W^* & I_p \end{pmatrix} \tilde{A} \begin{pmatrix} I_n & -W \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - W^*AW \end{pmatrix}, \tag{5}$$

where I_n and I_p are the unit matrices with respective orders n and p.
From (5) we deduce that

$$\operatorname{rank}(\tilde{A}) = \operatorname{rank} A + \operatorname{rank}\left(C - W^*AW\right), \tag{6}$$

$\mathcal{P}_n$ will denote the space of polynomials with two indeterminate, and real coefficients with total degree is lower than or equal to n. We consider the Riesz functional $L_\beta : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ defined by

$$L_\beta\left(p = \sum_{0 \le i+j \le 2n} a_{ij} x^i y^j\right) = \sum_{0 \le i+j \le 2n} a_{ij} \beta_{ij.}$$

It is easy to see that if $\hat{p} = (a_{ij})$ and $\hat{q} = (b_{ij})$ are respectively the column vectors of the polynomial p and q in the basis of $\mathcal{P}_n$ made up of monomials in lexicographical order in degrees $1, x, y, x^2, xy, y^2, \cdots, x^n, \cdots, y^n$, then the action of the matrix $M(n)$ on the polynomials p and q is given by $\langle M_n \hat{p}, \hat{q}\rangle := L_\beta(pq)\,(p, q \in \mathcal{P}_n)$, and therefore the entry of the matrix $M(n)$ relative to the row X^kY^l and column $X^{k'}Y^{l'}$ is $\beta_{k'+k,l'+l} = \left\langle X^{k'}Y^{l'}, X^kY^l\right\rangle$.
The correspondence between $\mathcal{P}_n$ and $\mathcal{C}_{M(n)}$ the column space of the matrix $M(n)$ is given by $p(X,Y) = \sum\limits_{0 \le i+j \le 2n} a_{ij} X^i Y^j$ where $p = \sum\limits_{0 \le i+j \le 2n} a_{ij} x^i y^j$ so that $p(X,Y) = M(n)\hat{p}$ and $p(X,Y) \in \mathcal{C}_{M(n)}$. i.e $p(X,Y)$ is a linear combination of $M(n)$ columns.

By considering $\mathcal{Z}(p)$ the set of zeros of p, we define the core variety of $M(n)$ by $\mathcal{V} \equiv \mathcal{V}(M(n)) := \bigcap_{p \in \mathcal{P}_n} \mathcal{Z}(p)$.
The following two results will be useful for determining the representative measure.

Proposition 2.2. ([3, Proposition 3.1]). *Suppose that μ is a representative measure of β for $p \in \mathcal{P}_n$. We have,*

$$\operatorname{supp} \mu \subseteq \mathcal{Z}(p) \Longleftrightarrow P(X, Y) = \mathbf{0}.$$

So from the Corollary 3.7 in [3],we deduce that

$$\operatorname{supp} \mu \subseteq \mathcal{V}(M(n)) \quad \text{and} \quad \operatorname{rank} M(n) \leq \operatorname{card} \sup \mu \leq v := \operatorname{card} \mathcal{V}. \tag{7}$$

Theorem 2.3. ([3, Theorem 5.13]). *The truncated moment sequence $\beta^{(2n)}$ has a* rank $M(n)$*-atomic representative measure if and only if $M(n) \succeq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

If $M(n)$ admits positive semidefinite extension $M(n+1)$ such that $M(n+1)$ is flat or has a flat extension $M(n+2)$ then β admits a representative measure μ which is r-atomic where $r = \operatorname{rank} M(n+1)$. By virtue of the flat extension theorem 2.3, the core variety $\mathcal{V}$ of $M(n+1)$ consists of exactly r points.
Let us put $\mathcal{V} = \{(x_1, y_1), (x_2, y_2), \cdots, (x_r, y_r)\}$ and consider the Vandermonde matrix V given by

$$V = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{r-1} & x_r \\ y_1 & y_2 & y_3 & \dots & y_{r-1} & y_r \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_{r-1}^2 & x_r^2 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & \dots & x_{r-1} y_{r-1} & x_r y_r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n+1} & x_2^{n+1} & x_3^{n+1} & \dots & x_{r-1}^{n+1} & x_r^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{n+1} & y_2^{n+1} & y_3^{n+1} & \dots & y_{r-1}^{n+1} & y_r^{n+1} \end{pmatrix}. \tag{8}$$

If we denote by $\mathcal{B} = \{c_1, c_2, \cdots, c_r\}$ the basis of $\mathcal{C}_{M(n+1)}$, the column space of $M(n+1)$, and if $V_{|\mathcal{B}}$ is the compression of V at the columns of $\mathcal{B}$ then we can determine the densities ρ_s of atoms $\{(x_s, y_s)\}_{1 \leq s \leq r}$ by solving the following Vandermonde system

$$V_{|\mathcal{B}}.(\rho_1 \quad \rho_2 \quad \cdots \quad \rho_r)^T = (L(c_1) \quad L(c_2) \quad \cdots \quad L(c_r))^T. \tag{9}$$

Hence, the representative measure of β is $\mu = \sum_{s=1}^{r} \rho_s \delta_{(x_s, y_s)}$. Let $\beta = \beta^{(2n)} = (\beta_{ij})_{0 \leq i+j \leq 2n}$ with $\beta_{00} > 0$ be a doubly indexed finite sequence of real numbers, and let $M(n)$ be the matrix of moments associated to β. We denote by $\mathcal{C}_{M(n)}$ the column space of $M(n)$, that is to say,

$$\mathcal{C}_{M(n)} = \operatorname{span}\{1, X, Y, X^2, XY, Y^2, \dots, X^n, \dots, Y^n\}.$$

We express the $M(n)$ columns linear dependence by the following relations

$$p_1(X,Y) = \mathbf{0},\ p_2(X,Y) = \mathbf{0},\ \ldots,\ p_k(X,Y) = \mathbf{0},$$

for some polynomials $p_1, p_2, \ldots, p_k \in \mathcal{P}_n, k \in \mathbb{N}$ and $k \leq \frac{(n+2)(n+1)}{2}$.
From [16], $M(n)$ is recursively generated if the following property is verified

$$p, q, pq \in \mathcal{P}_n, p(X,Y) = 0 \Longrightarrow (pq)(X,Y) = 0, \tag{10}$$

and $M(n)$ is recursively determined [16] if it has the two following column dependence relations,

$$X^n = p(X,Y) = \sum_{0\leq i+j\leq n-1} a_{ij} X^i Y^j; \tag{11}$$

$$Y^n = q(X,Y) = \sum_{0\leq i+j\leq n, j\neq n} b_{ij} X^i Y^j. \tag{12}$$

or by similar relations when reversing the roles of p and q. By a column dependence relation we mean a linear dependence relation of the form $X^iY^j = r(X,Y)$, where $\deg r \leq i+j$ and each monomial term in r strictly precedes x^iy^j in the degree-lexicographic order. Such relation is said degree - reducing if $\deg r < i+j$. From [9, Theorem 2.1 and Corollary 2.2], if $M(n)$ is positive semidefinite and generated entirely by the above relations (11) and (12), then it admits a single extension $M(n+1)$, which is positive semidefinite and recursively generated given by,

$$M(n+1) = \begin{pmatrix} M(n) & B(n+1) \\ B(n+1)^T & C(n+1) \end{pmatrix}, \tag{13}$$

with $\mathrm{Rang}(B(n{+}1)) \subseteq \mathrm{Rang}(M(n))$, and we have in $M(n{+}1)$ the following columns dependence relations,

$$X^{n+i}Y^j = (x^iy^jp)(X,Y) \quad (0 \leq i+j \leq 1); \tag{14}$$

$$X^lY^{n+m} = (x^ly^mq)(X,Y) \quad (0 \leq l+m \leq 1). \tag{15}$$

Since the columns X^{n+1} and Y^{n+1} in $B(n+1)$ contain all the new moments of degree $2n+1$ and the old moments of degree $n+1, n+2, n+3, \ldots, 2n$, we will be interested by the new moments of degree $2n+1$ which are in the block $B[n,\ n+1]$ distributed on columns $X^{n+1}, X^nY, X^{n-1}Y^2, \ldots, Y^{n+1}$. We see that the relations (14), determine in a unique way the moments $\beta_{2n+1,0}, \beta_{2n,1}, \beta_{2n-1,2}, \ldots, \beta_{n+1,n}$ in the block $B[n, n+1]$, such that the moments belonging to the column X^{n+1} except $\beta_{2n+1,0}$, propagate in the most right columns up to the XY^n column, according to the Hankel structure. Using the relation (15), we determine the moments $\beta_{n,n+1}, \beta_{n-1,n+2}, \ldots, \beta_{0,n+1}$, which propagate in the most left columns up to the X^nY column, according to the Hankel structure. Hence, the construction of the block $B(n+1)$ is completed. Applying the same process on the columns of the block $B(n+1)^T$, we determine all the moments of the block $C(n+1)$. Thus the positive semidefinite extension $M(n+1)$ of $M(n)$ is well determined.

Now, if $M(n)$ is recursively determined and admits, in addition to relations (11) and (12), a third additional dependence relation between these columns as $X^uY^v = r(X,Y)$ where $\deg r \leq u+v = n$, $u \neq 0$ and $v \neq 0$, then to determine an extension which is recursively generated and positive semidefinite $M(n+1)$ of $M(n)$, we start by defining the block $B[n,n+1]$ with determining firstly the moments of degree $2n+1$ in column X^{n+1}, then in order those of columns $X^nY, X^{n-1}Y^2, \ldots, XY^n$ and Y^{n+1}. According to the principle of recursivity, if the system of dependence relations formed by the relations (11), (12) and the third additional relation leads to different and incompatible expressions for moments of degree $2n+1$ in $B[n,n+1]$, then by using [16, Corollary 4.6], $M(n)$ does not admit a positive semidefinite extension $M(n+1)$. So, the sequence β does not admit a representative measure. Otherwise, if the block $B[n,n+1]$ is well defined in a consistent manner and without conflict, we determine by similar method the block $C(n+1)$ to achieve the construction of $M(n+1)$. Finally, we test its flatness to ensure the existence of a representative measure for β.

3. MAIN RESULTS

Let $\beta = \beta^{(5)} \equiv \{\beta_{ij}\}_{0\le i+j\le 5}$ be a real doubly indexed finite sequence with $\beta_{00} > 0$. The real quintic moment problem, consists in determining the conditions of existence of a Borel positive measure μ supported on $\mathbb{R}^2$ such that $\beta_{ij} = \displaystyle\int x^i y^j \, d\mu, \quad 0 \le i+j \le 5$. From the initial data of β, distributed over two matrices $M(2)$ and $B(3)$ as in (2) with $M(2) \succeq 0$ and $\operatorname{Rang} B(3) \subseteq \operatorname{Rang} M(2)$, then to extend $M(2)$ to a positive semidefinite matrix $M(3)$ as described bellow in (17), we notice that $C(3)$, the block 4×4 in $M(3)$ at the bottom on the right, contains all the moments of degree six, that remain undefined. In the current section, we focus on determining these moments to ensure that $M(3)$ is a flat extension or admits a flat extension $M(4)$.
Put

$$C(3) = \begin{pmatrix} \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\ \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\ \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\ \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06} \end{pmatrix}, \tag{16}$$

in a way that $M(3)$ can be written in the form

$$M(3) = \begin{pmatrix} M(2) & B(3) \\ B(3)^T & C(3) \end{pmatrix}. \tag{17}$$

As $\operatorname{Rang} B(3) \subseteq \operatorname{Rang} M(2)$ then there exists a matrix W such that $M(2)W = B(3)$. We saw in Section 2 that $W^TM(2)W$ is symmetric and is written as:

$$W^TM(2)W = \begin{pmatrix} a & b & c & d \\ b & x & y & e \\ c & y & z & f \\ d & e & f & g \end{pmatrix}. \tag{18}$$

According to Smul'jan Theorem 2.1, the remaining condition which ensures the positivity of $M(3)$ is $C(3) - W^TM(2)W \succeq 0$.
Now, we are able to state our first main result.

Theorem 3.1. *Let β be a doubly indexed finite sequence of real numbers, with $M(2) \succeq 0$ and $\operatorname{Rang}(B) \subseteq \operatorname{Rang}(M(2))$. If $x = c, y = d$ and $z = e$ then β admits a finite* $(\operatorname{rank} M(2))$*-atomic representative measure.*

Proof. If $x = c$ and $y = d$ and $z = e$, then the matrix $W^T M(2) W$ has the Hankel's property. So, it suffices to take $C(3) = W^T M(2) W$, and the construction of $M(3)$ is achieved. According to (6), we deduce that $\operatorname{rank}(M(3)) = \operatorname{rank}(M(2))$. Then, $M(3)$ is a flat extension of $M(2)$. According to Theorem 2.3, $\beta^{(6)}$, and a fortiori $\beta = \beta^{(5)}$, admits a unique representative measure $\operatorname{rank}(M(2))$-atomic. □

To highlight the efficiency of Theorem 3.1 in solving quintic moment problem, we present the following numerical example.

Example 3.2. *Let $\beta = \beta^{(5)}$ be the quintic sequence, whose data are presented by the two next matrices $M(2)$ and $B(3)$,*

$$M(2) = \begin{pmatrix} 3 & \frac{7}{6} & \frac{5}{3} & \frac{17}{6} & \frac{1}{2} & \frac{16}{3} \\ \frac{7}{6} & \frac{17}{6} & \frac{1}{2} & \frac{25}{6} & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{3} & \frac{1}{2} & \frac{16}{3} & \frac{3}{2} & \frac{3}{2} & \frac{23}{3} \\ \frac{17}{6} & \frac{25}{6} & \frac{3}{2} & \frac{53}{6} & \frac{7}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{16}{3} & \frac{3}{2} & \frac{23}{3} & \frac{5}{2} & \frac{1}{2} & \frac{52}{3} \end{pmatrix} \text{ and } B(3) = \begin{pmatrix} \frac{25}{6} & \frac{3}{2} & \frac{3}{2} & \frac{23}{3} \\ \frac{53}{6} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{52}{3} \\ \frac{97}{6} & \frac{15}{2} & \frac{9}{2} & \frac{3}{2} \\ \frac{15}{2} & \frac{9}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{9}{2} & \frac{3}{2} & \frac{3}{2} & \frac{95}{3} \end{pmatrix}.$$

Calculations show that $M(2) \succeq 0$ and $\operatorname{rank} M(2) = 6$. W and $W^T M(2) W$ are given by

$$W = M(2)^{-1} B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{8}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 2 \\ \frac{3}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{6}{5} & \frac{7}{5} & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } W^T M(2) W = \begin{pmatrix} \frac{197}{6} & \frac{31}{2} & \frac{17}{2} & \frac{7}{2} \\ \frac{31}{2} & \frac{17}{2} & \frac{7}{2} & \frac{5}{2} \\ \frac{17}{2} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{196}{3} \end{pmatrix}.$$

Since $x = c = \frac{17}{2}$, $y = d = \frac{7}{2}$ *and* $z = e = \frac{5}{2}$, *Theorem 3.1 allows to deduce that* β *admits a finite representative measure with* 6 *atoms. The matrix* $M(3)$ *is given by*

$$M(3) = \begin{pmatrix} 3 & \frac{7}{6} & \frac{5}{3} & \frac{17}{6} & \frac{1}{2} & \frac{16}{3} & \frac{25}{6} & \frac{3}{2} & \frac{3}{2} & \frac{23}{3} \\ \frac{7}{6} & \frac{17}{6} & \frac{1}{2} & \frac{25}{6} & \frac{3}{2} & \frac{3}{2} & \frac{53}{6} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{5}{3} & \frac{1}{2} & \frac{16}{3} & \frac{3}{2} & \frac{3}{2} & \frac{23}{3} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{52}{3} \\ \frac{17}{6} & \frac{25}{6} & \frac{3}{2} & \frac{53}{6} & \frac{7}{2} & \frac{5}{2} & \frac{97}{6} & \frac{15}{2} & \frac{9}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{15}{2} & \frac{9}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{16}{3} & \frac{3}{2} & \frac{23}{3} & \frac{5}{2} & \frac{1}{2} & \frac{52}{3} & \frac{9}{2} & \frac{3}{2} & \frac{3}{2} & \frac{95}{3} \\ \frac{25}{6} & \frac{53}{6} & \frac{7}{2} & \frac{97}{6} & \frac{15}{2} & \frac{9}{2} & \frac{197}{6} & \frac{31}{2} & \frac{17}{2} & \frac{7}{2} \\ \frac{3}{2} & \frac{7}{2} & \frac{5}{2} & \frac{15}{2} & \frac{9}{2} & \frac{3}{2} & \frac{31}{2} & \frac{17}{2} & \frac{7}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{9}{2} & \frac{3}{2} & \frac{3}{2} & \frac{17}{2} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{23}{3} & \frac{1}{2} & \frac{52}{3} & \frac{3}{2} & \frac{3}{2} & \frac{95}{3} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{196}{3} \end{pmatrix}.$$

We check that $M(3) \succeq 0$ *and* $\operatorname{rank}(M(3)) = \operatorname{rank}(M(2)) = 6$ *and the* $M(3)$ *columns dependence relations are*

$$X^3 = \frac{8}{5}X + \frac{3}{5}X^2 + \frac{6}{5}XY, \quad Y^3 = -\frac{1}{5}X + 2Y - \frac{1}{5}X^2 + Y^2 - \frac{2}{5}XY,$$

$$XY^2 = \frac{2}{5}X + \frac{2}{5}X^2 - \frac{1}{5}XY \quad \textit{and} \quad Y^2Y = -\frac{1}{5}X + 2Y + \frac{1}{5}X^2 + \frac{7}{5}XY,$$

By computations, we get $\mathcal{V} = \{(-1, 0); (0, -1); (0, 0); (0, 2); (1, -1); (2, 1)\}$, *and the resolution of Vandermonde system* (9) *gives the densities of atoms, respectivly*

$$\rho_1 = \frac{1}{3}, \quad \rho_2 = \frac{1}{3}, \quad \rho_3 = \frac{1}{3}, \quad \rho_4 = 1, \quad \rho_5 = \frac{1}{2} \textit{ and } \rho_6 = \frac{1}{2}.$$

Then the representative measure of $\beta = \beta^{(5)}$ *is*

$$\mu = \frac{1}{3}\delta_{(-1,0)} + \frac{1}{3}\delta_{(0,-1)} + \frac{1}{3}\delta_{(0,0)} + \delta_{(0,2)} + \frac{1}{2}\delta_{(1,-1)} + \frac{1}{2}\delta_{(2,1)}.$$

Now, we focus on the case where $x \neq c$ or $y \neq d$ or $z \neq e$. Before stating our result, let us note that $M(3)$ is recursively determined, with a third additional relation, and that the matrix $W^T M(2) W$ is not Hankel. Then, for each (4×4) matrix $C(3)$ such that $M(3) \succeq 0$ with three dependence relations between its columns, it is necessary that $\operatorname{rank}(C(3) - W^T M(2) W) = 1$ and $C(3) - W^T M(2) W \succeq 0$. Under these conditions, we will define $M(3)$ and check if it admits a flat extension or not.

Theorem 3.3. *If* $M(2) \succeq 0$ *and* $\operatorname{Rang} B(3) \subseteq \operatorname{Rang} M(2)$ *then the extension* $M(3)$ *of* $M(2)$ *is semidefinite positive and recursively determined with* $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$ *if and only if one of the following conditions is satisfied*

- $x = c, y = d$ *and* $z < e$
- $x < c$.

Proof. Let $r = \operatorname{rank} M(2)$ and consider $\mathcal{B}$ the basis of $\mathcal{C}_{M(2)}$, the space of $M(2)$ columns. We note that the $(r \times r)$ matrix $M(2)_{|\mathcal{B}}$, the restriction of $M(2)$ at the basis $\mathcal{B}$ is invertible. Since $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$ then there exists in the matrix $M(3)$ a column in the block $B(3)$ which is linearly independent with the elements of the basis $\mathcal{B}$. Moreover, $M(3)$ is recursively determined if and only if the concerned column is X^2Y or XY^2 which imposes $\beta_{60} = a$, contrarily we will have

$$\det \begin{pmatrix} M(2)_{|\mathcal{B}} & (X^3)_{|\mathcal{B}} \\ \left((X^3)_{|\mathcal{B}}\right)^T & \beta_{60} \end{pmatrix} \neq 0, \tag{19}$$

and then, it is the column X^3 which will be linearly independent with the elements of the basis $\mathcal{B}$.
So, if $\operatorname{rank} M(3) = r+1$ (i.e. $\operatorname{rank}(C(3) - W^T M(2)W) = 1$), then, with $\beta_{60} = a$, we will have $\beta_{42} = c$. On the other hand, $M(3) \succeq 0$ implies that $C(3) - W^T M(2)W \succeq 0$.

Therefore, all entries of the main diagonal of $C(3) - W^T M(2)W$ are positive, hence $c - x \geq 0$. If we suppose that $x = c$ then necessarily $y = d$ and $\beta_{24} = e$. The positivity of $C(3) - W^T M(2)W$ requires $e - z \geq 0$. If $e \leq z$ then $x = c, y = d$ and $z = e$. Hence, $e > z$. Conversely, let us suppose that ($x = c, y = d$ and $z < e$) or ($x < c$), by simple algebraic techniques we construct explicitly $C(3)$, so that $M(3)$ is positive semidefinite and $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$. This construction will be done in the following five possible cases.

(i). If $x = c$, $y = d$ and $z < e$ then
- $\beta_{60} = a$, $\beta_{51} = b$, $\beta_{42} = c$, $\beta_{33} = d$, $\beta_{24} = e$, $\beta_{15} = f$ and $\beta_{06} = g$.

(ii). If $x < c$, $y = d$ and $z \neq e$ then
- $\beta_{60} = a$, $\beta_{51} = b$, $\beta_{42} = c$, $\beta_{33} = d$, $\beta_{24} = z$, $\beta_{15} = f$ and $\beta_{06} = g + \frac{(z-e)^2}{c-x}$.

(iii). If $x < c$, $d \neq y$ and $z = e$ then
- $\beta_{60} = a$, $\beta_{42} = c$, $\beta_{51} = b$, $\beta_{33} = d$, $\beta_{24} = \frac{(d-y)^2}{c-x} + z$, $\beta_{15} = \frac{(d-y)^3}{(c-x)^2} + f$ and $\beta_{06} = \frac{(d-y)^4}{(c-x)^3} + g$.

(iv). If $x < c$, $y \neq d$ and $z \neq e$ then
- $\beta_{60} = a$, $\beta_{42} = c$, $\beta_{51} = b$, $\beta_{33} = d$, $\beta_{24} = z + \lambda$, $\beta_{15} = f + \delta$ and $\beta_{06} = \alpha + g$, where $\lambda = \frac{(d-y)^2}{c-x} > 0$, $\delta = \frac{d-y}{(c-x)^2}((c-x)(z-e) + (d-y)^2)$, $\alpha = \frac{1}{(c-x)^3}((c-x)(z-e) + (d-y)^2)^2$.

(v). If $x < c$, $y = d$ and $z = e$ then
- $\beta_{60} = a$, $\beta_{51} = b$, $\beta_{42} = c$, $\beta_{33} = y$, $\beta_{24} = e$, $\beta_{15} = f$ and $\beta_{06} = g$.

Hence, $C(3)$ is well determined such that $C(3) - W^T M(2)W \succeq 0$ and $\operatorname{rank}(C(3) - W^T M(2)W) = 1$. That is, $M(3)$ is positive semidefinite, recursively determined and $\operatorname{rank} M(3) = \operatorname{rank} M(2) + 1$. $\square$

Under the conditions of the Theorem 3.3, we can always extend $M(2)$ to a positive semidefinite matrix $M(3)$, recursively determined with $\operatorname{rank} M(3) =$

$\operatorname{rank} M(2)+1$. Now, we seek to determine, when it is possible, a positive semi-definite extension $M(4)$ of $M(3)$ and we check its flatness. The two following propositions will be useful for further results.

Proposition 3.4. *If $x = c$, $y = d$ and $z < e$ then $M(3)$, the extension of $M(2)$, admits three linear columns dependence relations, which are degree-reducing as follows:*

$$X^3 = \sum_{0\le i+j\le 2} a_{ij}X^iY^j; \tag{20}$$

$$Y^3 = \sum_{0\le i+j\le 2} c_{ij}X^iY^j; \tag{21}$$

$$X^2Y = \sum_{0\le i+j\le 2} b_{ij}X^iY^j. \tag{22}$$

Proof. As $\beta_{60} = a$ and $\beta_{24} = e \neq z$ then the column, in $B(3)$ which is linearly independent with the elements of the basis $\mathcal{B}$ of the space $\mathcal{C}_{M(2)}$ is XY^2. Since $\operatorname{rank}(M(3)_{|\mathcal{B}\cup\{XY^2\}}) = \operatorname{rank} M(2)+1 = \operatorname{rank} M(3)$ then the columns X^3, X^2Y and Y^3, in the block $\begin{pmatrix} M(2) & B(3) \end{pmatrix}$, are linear combination of the elements of $\mathcal{B}\cup\{XY^2\}$. By the extension principle [5, Proposition 3.9], these columns are linearly dependent of the elements of $\mathcal{B}\cup\{XY^2\}$ in $M(3)$. So, we can write

$$\begin{aligned} X^3 &= \alpha_0 XY^2 + \sum_{0\le i+j\le 2} a_{ij}X^iY^j;\\ X^2Y &= \alpha_1 XY^2 + \sum_{0\le i+j\le 2} b_{ij}X^iY^j;\\ Y^3 &= \alpha_2 XY^2 + \sum_{0\le i+j\le 2} c_{ij}X^iY^j. \end{aligned} \tag{23}$$

with

$$\begin{aligned} \alpha_0 &= \frac{\det\begin{vmatrix} M(2)_{|\mathcal{B}} & X^3_{|\mathcal{B}} \\ (XY^2_{|\mathcal{B}})^T & \beta_{42}\end{vmatrix}}{\det\begin{vmatrix} M(2)_{|\mathcal{B}} & XY^2_{|\mathcal{B}(2)} \\ (XY^2_{|\mathcal{B}})^T & \beta_{24}\end{vmatrix}}\\ &= \frac{\det\begin{vmatrix} M(2)_{|\mathcal{B}} & X^3_{|\mathcal{B}} \\ (XY^2_{|\mathcal{B}})^T & c\end{vmatrix} + (\beta_{42}-c)\det\left|M(2)_{|\mathcal{B}}\right|}{\det\begin{vmatrix} M(2)_{|\mathcal{B}} & XY^2_{|\mathcal{B}} \\ (XY^2_{|\mathcal{B}})^T & \beta_{24}\end{vmatrix} + (\beta_{24}-z)\det\left|M(2)_{|\mathcal{B}}\right|}\\ &= \frac{(\beta_{42}-c)\det\left|M(2)_{|\mathcal{B}}\right|}{(\beta_{24}-z)\det\left|M(2)_{|\mathcal{B}}\right|}\\ &= \frac{\beta_{42}-c}{\beta_{24}-z} = 0 \qquad (\beta_{42} = c \text{ and } \beta_{24} = e). \end{aligned} \tag{24}$$

and by similar calculations as in (24), we find $\alpha_1 = \dfrac{\beta_{33} - y}{\beta_{24} - z} = 0$ and $\alpha_2 = \dfrac{\beta_{15} - f}{\beta_{24} - z} = 0$. Then, the relations (20), (21) and (22) hold. □

Proposition 3.5. *Under the same conditions of the Proposition 3.4 and if $M(4)$ is positive semidefinite, then $M(4)$ is flat.*

Proof. According to the Proposition 3.4, $M(3)$ admits three degree-reducing relations between its columns. This allows us to express the columns X^4, X^3Y, X^2Y^2, XY^3 and Y^4 in $M(4)$ as linear combination of columns of strictly lower degree. So, $M(4)$ is flat. □

Remark 3.6. *We have a similar results as in Propositions 3.4 and 3.5 if $x < c, y = d$ and $z = e$, except that the relation (22) is changed with*

$$XY^2 = \sum_{0 \le i+j \le 2} b_{ij} X^i Y^j, \tag{25}$$

since $\beta_{42} = c \neq x$.

As explained in the last paragraph of section 2, we build $M(4)$ following the next steps. Employing the relations (20), (21) and (22), we can construct the block $B[3,4]$, which contains the moments of degree 7. Since the column X^3Y can be obtained using the relation (20) or (22), and since recursivity requires consistency between these two expressions, we must ensure that the block $B[3,4]$ is well defined. Hence the block $B(4)$ is ended. With similar process on the block $B(4)^T$, we build the block $C(4)$. So, the construction of the matrix $M(4)$ is achieved. If there is any conflict during the construction of the block $B[3,4]$, then $M(3)$ does not admit any positive semidefinite extension $M(4)$. Hence, there is no representative measure for β. The Example 3.8 illustrates this case. On the other hand, if we succeed in constructing $M(4)$, positive semidefinite and since the relations (20), (21) and (22) are degree-reducing then $M(4)$ is necessary flat. Thus, the existence of a finite representative measure $(r+1)$-atomic for β. Examples 3.7 and 3.9 illustrate this last case.

Example 3.7. *Let $\beta = \beta^{(5)}$ be the quintic sequence, whose data are presented by the two matrices $M(2)$ and $B(3)$.*

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Simple calculations show that $M(2) \succeq 0$, $\operatorname{rank}(M(2)) = 5$ *and* $M(2)W = B(3)$ *with*

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad W^T M(2) W = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}.$$

We have $x = c = 1$, $y = d = 0$ *and* $z = 1 < e = 2$. *Computing the moments of degree 6 as pointed out in the proof of Theorem 3.3, we find* $C(3) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}$.
So the matrix $M(3)$, *the extension of* $M(2)$ *is given by*

$$M(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}.$$

Calculations lead to $M(3) \succeq 0$ *and* $\operatorname{rank}(M(3)) = 6$, *with the following columns dependence relations between* $M(3)$,

$$X^3 = X, \quad Y^3 = 2Y, \quad X^2Y = Y \quad \text{and} \quad X^2 = 1.$$

Thus, by functional calculation, we get,

$$X^4 = X^2, X^3Y = XY, X^2Y^2 = Y^2, XY^3 = 2XY \text{ and } Y^4 = 2Y^2. \tag{26}$$

Thus, the construction of recursively generated extension $M(4)$ of $M(3)$ without conflict is

$$M(4) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 8 \end{pmatrix}.$$

As the relations (26) are degree-reducing, then $M(4)$ is flat. Calculations show that the core variety of $\mathcal{V}$, consists of the atoms,

$$\omega_1 = (-1,0), \omega_2 = (-1,-\sqrt{2}), \omega_3 = (-1,\sqrt{2}), \omega_4 = (1,0), \omega_5 = (1,-\sqrt{2}) \text{ and } \omega_6 = (1,\sqrt{2}).$$

The resolution of the Vandermonde system (9) gives us the respective densities of the above atoms, as follows

$$\rho_1 = \frac{1}{4}, \rho_2 = \frac{1}{8}, \rho_3 = \frac{1}{8}, \rho_4 = \frac{1}{4}, \rho_5 = \frac{1}{8} \text{ and } \rho_6 = \frac{1}{8}.$$

Then, a representative measure of β is given by

$$\mu = \frac{1}{4}\delta_{(-1,0)} + \frac{1}{8}\delta_{(-1,-\sqrt{2})} + \frac{1}{8}\delta_{(-1,\sqrt{2})} + \frac{1}{4}\delta_{(1,0)} + \frac{1}{8}\delta_{(1,-\sqrt{2})} + \frac{1}{8}\delta_{(1,\sqrt{2})}.$$

Example 3.8. *Let $\beta = \beta^{(5)}$ be the sequence constituted by the data of degree 5 represented by the two matrices*

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 2 & 0 & 5 & 0 \\ 1 & 2 & 0 & 5 & 0 & 22 \end{pmatrix} \quad \text{and} \quad B(3) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 2 & 0 & 5 & 0 \\ 0 & 5 & 0 & 22 \\ -1 & -2 & 13 & 3 \\ -2 & 13 & 3 & \frac{894}{13} \\ 13 & 3 & \frac{894}{13} & \frac{336}{13} \end{pmatrix}.$$

we have $M(2) > 0$*. Then the matrix* $W = M(2)^{-1}B(3)$ *and* $W^TM(2)W$ *are given by*

$$W = \begin{pmatrix} 40 & 35 & \frac{381}{13} & \frac{501}{13} \\ -24 & -22 & -\frac{267}{13} & -\frac{360}{13} \\ 4 & -1 & -6 & -\frac{358}{13} \\ -53 & -46 & -\frac{521}{13} & -\frac{681}{13} \\ -2 & 3 & 3 & \frac{322}{13} \\ 13 & 11 & \frac{166}{13} & \frac{180}{13} \end{pmatrix}, \quad W^TM(2)W = \begin{pmatrix} 178 & 139 & 159 & \frac{1657}{13} \\ 139 & 159 & \frac{1657}{13} & \frac{4298}{13} \\ 159 & \frac{1657}{13} & \frac{54427}{169} & \frac{48015}{169} \\ \frac{1657}{13} & \frac{4298}{13} & \frac{48015}{169} & \frac{16877}{13} \end{pmatrix}.$$

Therefore, we have $x = c = 159$, $y = d = \frac{1657}{13}$ *and* $z = \frac{54427}{169} < e = \frac{4298}{13}$.
So,

$$C(3) = \begin{pmatrix} 178 & 139 & 159 & \frac{1657}{13} \\ 139 & 159 & \frac{1657}{13} & \frac{4298}{13} \\ 159 & \frac{1657}{13} & \frac{4298}{13} & \frac{48015}{169} \\ \frac{1657}{13} & \frac{4298}{13} & \frac{48015}{169} & \frac{16877}{13} \end{pmatrix}.$$

Then,

$$M(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 5 & 0 & 22 \\ 1 & 0 & 0 & 2 & 0 & 5 & -1 & -2 & 13 & 3 \\ 0 & 0 & 2 & 0 & 5 & 0 & -2 & 13 & 3 & \frac{894}{13} \\ 1 & 2 & 0 & 5 & 0 & 22 & 13 & 3 & \frac{894}{13} & \frac{336}{13} \\ 0 & 2 & 0 & -1 & -2 & 13 & 178 & 139 & 159 & \frac{1657}{13} \\ 0 & 0 & 5 & -2 & 13 & 3 & 139 & 159 & \frac{1657}{13} & \frac{4298}{13} \\ 2 & 5 & 0 & 13 & 3 & \frac{894}{13} & 159 & \frac{1657}{13} & \frac{4298}{13} & \frac{48015}{169} \\ 0 & 0 & 22 & 3 & \frac{894}{13} & \frac{336}{13} & \frac{1657}{13} & \frac{4298}{13} & \frac{48015}{169} & \frac{16877}{13} \end{pmatrix}.$$

Calculations show that $M(3) \succeq 0$, $\mathrm{rank}(M(3)) = 7$ *and the columns dependence relations are,*

$$\begin{aligned} X^3 &= p(X,Y) = 40 - 24X + 4Y - 53X^2 - 2XY + 13Y^2, \\ Y^3 &= q(X,Y) = \frac{501}{13} - \frac{360}{13}X - \frac{358}{13}Y - \frac{681}{13}X^2 + \frac{322}{13}XY + \frac{180}{13}Y^2, \\ X^2Y &= r(X,Y) = 35 - 22X - Y - 46X^2 + 3XY + 11Y^2. \end{aligned}$$

When determining the entries of the block $B[3,4]$ *of* $M(4)$ *we notice that* β_{43} *has two different values obtained by the two expressions of* X^3Y*. In fact, we get*

$$\langle yp(X,Y), XY^2 \rangle = -\frac{45762}{13} \neq \langle xr(X,Y), XY^2 \rangle = -\frac{44315}{13}.$$

Then $M(3)$ does not admit any positive semidefinite extension $M(4)$. Hence, $\beta^{(5)}$ does not admit any representative measure.

Example 3.9. *Let $\beta = \beta^{(5)}$ be represented by $M(2)$ and $B(3)$ as follows*

$$M(2) = \begin{pmatrix} 6 & 1 & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 3 \end{pmatrix} \quad \text{and} \quad B(3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & \frac{3}{2} & 1 & 1 \\ \frac{3}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

with $M(2) > 0$. Computing, we obtain,

$$W = M(2)^{-1}B(3) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 \\ -\frac{3}{4} & \frac{3}{4} & 0 & 0 \\ 2 & \frac{1}{4} & 1 & 0 \\ -\frac{3}{4} & \frac{1}{2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad W^T M(2) W = \begin{pmatrix} 4 & \frac{9}{8} & \frac{3}{2} & 1 \\ \frac{9}{8} & \frac{11}{8} & 1 & 1 \\ \frac{3}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}.$$

We have $x = \frac{11}{8} < c = \frac{3}{2}$, $y = d = 1$ and $z = e = 1$. As quoted in Theorem 3.3, the moments of degree 6 enable us to construct the following matrix $M(3)$,

$$M(3) = \begin{pmatrix} 6 & 1 & 1 & 3 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & \frac{3}{2} & 1 & 4 & \frac{9}{8} & \frac{3}{2} & 1 \\ 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & \frac{9}{8} & \frac{3}{2} & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

We have $M(3) \succeq 0$, $\mathrm{rank}(M(3)) = 7$ and the columns dependence relations are,

$$X^3 = -\frac{3}{4}X^2 + \frac{3}{4}X + 2XY - \frac{3}{4}Y^2 - \frac{1}{4}Y + \frac{1}{2}; \quad XY^2 = XY; \quad \text{and} \quad Y^3 = Y.$$

These relations allow to define a recursive extension $M(4)$ of $M(3)$ such that $\mathrm{rank}(M(4)) = \mathrm{rank}(M(3)) = 7$. Then, $\beta^{(5)}$ admits a representative measure μ which is 7-atomic. By calculations, we get $\mathcal{V} = \{(x_i, y_i\}_{i=1}^{i=7}$ where, $x_1 = 0$ and $y_1 = -1$, $x_2 \simeq -2,1425$ and $y_2 = 1$, $x_3 \simeq 0,19487$ and $y_3 = 1$, $x_4 \simeq 1,1976$ and $y_4 = 1$, $x_5 \simeq -0,59307$ and $y_5 = 0$, $x_6 \simeq 0,84307$ and $y_6 = 0$ and

$x_7 = -1$ *and* $y_6 = 0$. *Therefore,* $\mu = \sum_{i=1}^{i=7} \rho_i \delta_{(x_i,y_i)}$ *with* $\rho_1 \simeq 1$, $\rho_2 \simeq 0,00951481$, $\rho_3 \simeq 0,00951481$, $\rho_4 \simeq 0,630778$, $\rho_5 \simeq 0,90051$, $\rho_6 \simeq 1,4289$ *and* $\rho_7 \simeq 0,670594$.

To end the current paper, we will deal with the cases ($x < c, y = d$ and $z \neq e$), ($x < c, y \neq d$ and $z = e$) and ($x < c, y \neq d$ and $z \neq e$). In these cases, we have $\beta_{42} \neq x$. Then, the column X^2Y is linearly independent of the elements of the basis $\mathcal{B}$. Consequently, the columns X^3, XY^2 and Y^3, of $M(3)$, are in linearly dependent of elements of $\mathcal{B} \cup \{X^2Y\}$. So, we can write

$$\begin{aligned} X^3 &= \alpha_0 X^2Y + \sum_{0\leqslant i+j\leqslant 2} a_{ij} X^iY^j; \\ XY^2 &= \alpha_1 X^2Y + \sum_{0\leqslant i+j\leqslant 2} b_{ij} X^iY^j; \\ Y^3 &= \alpha_2 X^2Y + \sum_{0\leqslant i+j\leqslant 2} c_{ij} X^iY^j. \end{aligned} \tag{27}$$

With calculations as in (24), we find $\alpha_0 = \dfrac{\beta_{51} - b}{\beta_{42} - x} = 0$, $\alpha_1 = \dfrac{\beta_{33} - y}{\beta_{42} - x}$ and $\alpha_2 = \dfrac{\beta_{24} - e}{\beta_{42} - x}$.
Moreover, in the case ($x < c, y = d$ and $z \neq e$), we have $\alpha_1 = 0$ and $\alpha_2 \neq 0$. If ($x < c, y \neq d$ and $z = e$) then $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ and for the case ($x < c, y \neq d$ and $z \neq e$), we get $\alpha_1 \neq 0$ and α_2 is arbitrary chosen.

As done in the case ($x = c, y = d$ and $z < e$), we construct $M(4)$. If there is any conflict during the construction then β does not admit a representative measure. Otherwise, we check whether $M(4)$ is flat. If yes, by Theorem 2.3, β admits a representative measure ($\operatorname{rank} M(2) + 1$)-atomic. In the following two numerical examples, we present the case where the doubly sequence has representative measure.

Example 3.10. *Let* $\beta = \beta^{(5)}$ *be the quintic sequence with the following associated matrices* $M(2)$ *and* $B(3)$*,*

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\operatorname{rank} M(2) = 6$. *Since* $M(2) > 0$ *we get,*

$$W = M(2)^{-1}B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad W^T M(2) W = \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}.$$

We have $x = 1 < c = 2$, $y = d = 0$ *and* $z = 1 \neq e = 2$. *Hence, by calculating the moments of degree 6, given in the proof of Theorem 3.3, we get the following matrix* $M(3)$,

$$M(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \end{pmatrix}.$$

We have $\operatorname{rank} M(3) = 7$ *and the* $M(3)$ *columns dependence relations are*

$$X^3 = 2X, \quad Y^3 = -X^2Y + 3Y \quad \text{and} \quad XY^2 = X.$$

Then, we obtain $M(4)$,

$$M(4) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 5 \\ 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 8 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 14 \end{pmatrix}.$$

Calculations show that $M(4) \succeq 0$ *and* $\operatorname{rank} M(4) = 7$. *So* $M(4)$ *is flat, and the existence of a representative measure* μ *for* β *with 7 atoms.*

The core variety of $M(3)$ *is* $\mathcal{V} = \{(x_i, y_i)\}_{i=1}^{i=7}$ *where* $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (0, -\sqrt{3})$, $(x_3, y_3) = (0, \sqrt{3})$, $(x_4, y_4) = (-\sqrt{2}, -1)$, $(x_5, y_5) = (-\sqrt{2}, 1)$, $(x_6, y_6) = (\sqrt{2}, -1)$ *and* $(x_7, y_7) = (\sqrt{2}, 1)$. *Solving the Vandermonde system, we get the weights of* μ,

$\rho_1 = \frac{1}{3}$, $\rho_2 = \frac{1}{12}$, $\rho_3 = \frac{1}{12}$, $\rho_4 = \frac{1}{8}$, $\rho_5 = \frac{1}{8}$, $\rho_6 = \frac{1}{8}$ *and* $\rho_7 = \frac{1}{8}$.

Thus, the representative measure of $\beta^{(5)}$ *is*

$$\mu = \frac{1}{3}\delta_{(0,0)} + \frac{1}{12}\delta_{(0,-\sqrt{3})} + \frac{1}{12}\delta_{(0,\sqrt{3})} + \frac{1}{8}\delta_{(-\sqrt{2},-1)} + \frac{1}{8}\delta_{(-\sqrt{2},1)} + \frac{1}{8}\delta_{(\sqrt{2},-1)} + \frac{1}{8}\delta_{(\sqrt{2},1)}.$$

Example 3.11. *Consider $\beta = \beta^{(5)}$ a real doubly indexed sequence whose data are,*

$$M(2) = \begin{pmatrix} 7 & 0 & 0 & 4 & 2 & 4 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 \\ 4 & 0 & 0 & 2 & 2 & 4 \end{pmatrix} \quad \text{and} \quad B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As $\operatorname{rank} M(2) = 6$ *and* $M(2) > 0$ *then we obtain,*

$$W = M(2)^{-1}B(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad W^T M(2) W = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & \frac{4}{3} & \frac{4}{3} & 2 \\ 2 & \frac{4}{3} & \frac{4}{3} & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}.$$

We have $x = \frac{4}{3} < c = 2, y = \frac{4}{3} \neq d = 2$ and $z = \frac{4}{3} \neq e = 2$. Calculations of the moments of degree 6 *given in the proof of Theorem 3.3, we determine $C(3)$ and then $M(3)$ the extension of $M(2)$ such that,*

$$M(3) = \begin{pmatrix} 7 & 0 & 0 & 4 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 & 4 & 2 & 2 & 2 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 4 \\ 4 & 0 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 2 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 & 4 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 4 \end{pmatrix},$$

with $\operatorname{rank}(M(3)) = 7$ *and the $M(3)$ columns dependence relations are*

$$X^3 = X,\ Y^3 = Y, \quad \text{and} \quad XY^2 = X^2Y.$$

From these last relations, we define the moments of degree 7 and 8, without conflict, and we deduce the construction of the extension $M(4)$ given below

$$M(4) = \begin{pmatrix} 7 & 0 & 0 & 4 & 2 & 4 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 4 \\ 0 & 4 & 2 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 4 & 0 & 0 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 \\ 0 & 4 & 2 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 4 & 0 & 0 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}.$$

We check that $\operatorname{rank}(M(4)) = \operatorname{rank}(M(3)) = 7$. *So, $M(4)$ is flat and $\beta^{(5)}$ is a sequence of moments with a representative measure 7-atomic. The core variety of $M(3)$ is* $\mathcal{V} = \{(0,0);(1,0);(-1,0);(1,1);(0,-1);(1,1);(-1,-1)\}$.
Solving the Vandermonde system, we get the weights of the atoms $\rho_i = 1$ for $1 \leq i \leq 7$. Finally, the measure is

$$\mu = \delta_{(0,0)} + \delta_{(1,0)} + \delta_{(-1,0)} + \delta_{(1,1)} + \delta_{(0,-1)} + \delta_{(1,1)} + \delta_{(-1,-1)}.$$

The following example treats the case ($x < c, d \neq y$ and $z = e$), where there is no representative measure.

Example 3.12. *Let $\beta = \beta^{(5)}$ be the quintic sequence, which data are represented in the two following matrices,*

$$M(2) = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 & 4 \end{pmatrix}, \quad \textit{and} \quad B(3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 \\ 1 & 2 & 1 & 4 \\ 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 2 \end{pmatrix}.$$

Calculations show that $\operatorname{rank} M(2) = 6$ *and* $M(2) > 0$. *So we have,*

$$W = M(2)^{-1}B(3) = \begin{pmatrix} -\frac{7}{5} & 0 & -1 & -\frac{13}{10} \\ \frac{13}{5} & 0 & 1 & -\frac{4}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{11}{5} \\ 1 & 0 & 0 & 0 \\ -\frac{2}{5} & 0 & 0 & \frac{6}{5} \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \textit{and} \quad W^T M(2) W = \begin{pmatrix} \frac{56}{5} & 1 & 4 & -\frac{3}{5} \\ 1 & 2 & 1 & 4 \\ 4 & 1 & 4 & 2 \\ -\frac{3}{5} & 4 & 2 & \frac{113}{10} \end{pmatrix}$$

We have $x = 2 < c = 4, y = 1 \neq d = -\frac{3}{5}$ *and* $z = e = 4$ *and by calculations of the moments of degree 6 given in the proof or Theorem 3.3, we determine* $C(3)$ *then* $M(3)$ *with,*

$$M(3) = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 4 & 1 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 & 4 & 1 & 1 & 3 & 2 \\ 1 & 4 & 1 & 3 & 1 & 1 & \frac{56}{5} & 1 & 4 & -\frac{3}{5} \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 4 & -\frac{3}{5} & \frac{132}{25} \\ 1 & 2 & 1 & 1 & 1 & 3 & 4 & -\frac{3}{5} & \frac{132}{25} & \frac{122}{125} \\ 1 & 1 & 4 & 1 & 3 & 2 & -\frac{3}{5} & \frac{132}{25} & \frac{122}{125} & \frac{15149}{1250} \end{pmatrix}.$$

We have $\operatorname{rank}(M(3)) = 7$ *and the* $M(3)$ *columns dependence relations are,*

$$\begin{aligned} X^3 &= p(X,Y) = -\frac{7}{5} + \frac{13}{5}X - \frac{2}{5}Y + X^2 - \frac{2}{5}XY, \\ XY^2 &= r(X,Y) = -1 + X + \frac{4}{5}Y + Y^2 - \frac{4}{5}X^2Y, \\ Y^3 &= q(X,Y) = -\frac{13}{10} - \frac{4}{5}X + \frac{39}{25}Y + \frac{6}{5}XY + \frac{1}{2}Y^2 + \frac{16}{25}X^2Y. \end{aligned}$$

When determining the inputs of the block $B[3,4]$ *of* $M(4)$*, we notice that* β_{34} *has two distinct values, obtained by the two following expressions of* XY^3*,*

$$\left\langle xq(X,Y), X^2Y \right\rangle = \frac{3931}{625} \neq \left\langle yr(X,Y), X^2Y \right\rangle = \frac{4531}{625}.$$

Therefore, $M(3)$ *does not admit any positive semidefinite extension* $M(4)$*. Hence,* $\beta^{(5)}$ *has no representing measure.*

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