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Common Fixed Points of Single-Valued and Multi-Valued Mappings in S-Metric Spaces

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Abstract. In this paper, the notion of limit property (-Tayyab kamran, 2004-) and common limit property (-Yicheng Liu & Jun Wu & Zhixiang Li, 2005-) for single-valued and multi-valued mappings on metric spaces are generalized to S-metric spaces. This idea is used to make some common fixed point theorems for single-valued and multi-valued mappings by using a generalization of coincidence point in S-metric spaces. We give an example of an S-metric which is not continuous.

Key words and Phrases: Coincidence point, Common fixed point, Hausdorff S-metric, Limit property.

1. INTRODUCTION

Metric spaces are very important in mathematics. Generalized metric spaces can be pointed out as b-metric, D-metric and fuzzy metric spaces. For more considerations, see [2, 13, 4, 15]. In 2012, another generalized metric space called S-metric space was introduced by Sedghi et al. [16]. In the setting of S-metric space see, for example [5, 9, 12, 14], and the references therein. For application of fixed points and common fixed points in different fields such as fractional calculus, existence theory in fractional boundary value problems, see [1, 3, 6, 7, 8, 11].
In this paper, some common fixed point theorems for single-valued and multi-valued mappings are proved in S-metric spaces by using a generalization of coincidence point for pairs (f, F), (f, F) and (g, G) in which the mappings f and g are single-valued and the mappings F and G are multi-valued mappings with values in S-metric space $(CB(X), S_H)$, where S_H is the Hausdorff S-metric.
In section 2, some preliminaries are recalled. In section 3, we state our main theorem. Section 4 is the conclusions.

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2. PRELIMINARIES

In this section some definitions, lemmas, theorems, and example are recalled.

Definition 2.1. [16] *For nonempty set X, $S : X^3 \longrightarrow [0, \infty)$ is called an S-metric on X if*

(1): $S(x, y, z) = 0$ *iff* $x = y = z$;
(2): $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$,

for all $x, y, z, a \in X$. (X, S) is called an S-metric space.

Example 2.2. **(1):** *Assume $\alpha \geq 0$ and $X = [\alpha, \infty)$. Define $S : X^3 \longrightarrow [0, \infty)$ by*

$$S(x,y,z) = \begin{cases} 0 & if \;\; x = y = z; \\ \max\{x, y, z\} - \alpha & otherwise. \end{cases}$$

The mapping S is an S-metric on X. We call it the max S-metric.

(2): *Let $X = [0, \infty)$. Define $S : X^3 \longrightarrow [0, \infty)$ by*

$$S(x,y,z) = \begin{cases} 0 & if \;\; x = y = z; \\ x + y + 2z & otherwise. \end{cases}$$

Then, S is an S-metric on X.

Definition 2.3. [16] *In S-metric space (X, S), assume that x is an element of X, and $r > 0$.*

(1): *An open ball $B_s(x, r)$ with center x and radius r is defined by $B_s(x, r) = \{y \in X : S(y, y, x) < r\}$.*
(2): *A sequence $\{y_n\}$ in X converges to y if $\lim_{n\to\infty} S(y_n, y_n, y) = 0$. In this case, we write $y_n \to y$ or $\lim_{n\to\infty} y_n = y$.*
(3): *A sequence $\{y_n\}$ in X is called a Cauchy sequence if $\lim_{n,m\to\infty} S(y_n, y_n, y_m) = 0$.*
(4): *(X, S) is called complete if every Cauchy sequence converges.*
(5): *A subset A of X is called bounded if there exists $\epsilon > 0$ such that for all $a, b \in A$, $S(a, a, b) < \epsilon$.*

In (X, S), we set $\tau = \{A \subseteq X : A$ is a union of open balls$\}$. τ is a topology and we set $CB(X) = \{A \subseteq X : A$ is nonempty closed and bounded$\}$.

Example 2.4. *Consider $X = [0, \infty)$ with the max S-metric. Then, for $a \in X$ and $r > 0$, we have:* $B_s(a, r) = \begin{cases} [0, r) & if \;\; a < r; \\ \{a\} & if \;\; a \geq r. \end{cases}$

Definition 2.5. *Let (X, S) be an S-metric space. We say S is continuous if $S(x_n, y_n, z_n) \to S(x, y, z)$, whenever $x_n \to x$, $y_n \to y$, $z_n \to z$.*

Example 2.6. *On $X = [0, \infty)$, define*

$$S(x,y,z) = \begin{cases} 1 & if \;\; (x, y, z) = (1, 2, 3); \\ |x - z| + |y - z| & otherwise. \end{cases}$$

S is a S-metric on X and it is not continuous. In fact, we have:

$$x_n = 1 + \frac{1}{n} \to 1,\ y_n = 2 + \frac{2}{n} \to 2,\ z_n = 3 + \frac{3}{n} \to 3.$$

But

$$3 = \lim_{n\to\infty} S(x_n, y_n, z_n) \neq S(1, 2, 3) = 1.$$

Definition 2.7. *Let (X, S) be an S-metric space. We define $S_H : CB(X)^3 \longrightarrow [0, \infty)$, by*

$$S_H(A, B, C) = H_s(A, C) + H_s(B, C),$$

where $H_s(A, B) = \max\{h_S(A, B), h_S(B, A)\}$,
$h_s(A, B) = \sup\{S(a, a, B) : a \in A\}$ and
$S(a, a, B) = \inf\{S(a, a, b) : b \in B\}$.
For more information see [14].

Theorem 2.8. [14] *S_H is an S-metric on $CB(X)$.*

We call S_H the Hausdorff S-metric on $CB(X)$ generated by S.

Remark 2.9. *In Example 2.2(1) let u be a nondecreasing continuous function on $X = [\alpha, \infty)$ and let $F(x) = [\alpha, u(x)]$. We have:*

$$H_s(Fx, Fy) = \begin{cases} u(y) - \alpha & if\ y \geq x; \\ u(x) - \alpha & if\ x > y. \end{cases}$$

Let (X, S) be an S-metric space. The set of all nonempty compact subsets of X is denoted by $K(X)$.

Theorem 2.10. [14] *Let (X, S) be a complete S-metric spaces.Then, $(K(X), S_H)$ is a complete S-metric space.*

The converse is also true. In fact, suppose that $\{x_n\}$ is a Cauchy sequence in (X, S). By Theorem 3.4 [14], we have $\lim_{n\to\infty} S_H(\{x_n\}, \{x_n\}, \{x_m\}) = 2\lim_{n\to\infty} S(x_n, x_n, x_m) \to 0$. That is, $\{\{x_n\}\}$ is a Cauchy sequence in $(K(X), S_H)$. So, by Lemma 3.9 [14], there exists $x \in X$ such that $\{x_n\} \to \{x\}$. That is, $x_n \to x$.

Definition 2.11. *Let (X, S) be an S-metric space.*

(1) *The mappings $f : X \longrightarrow X$ and $F : X \longrightarrow CB(X)$ are given. We say f and F have a coincidence point at $a \in X$ if $f(a) \in F(a)$, also, we say f and F have a common fixed point at $a \in X$ if $f(a) = a \in F(a)$.*

(2) *The mapping $F : X \longrightarrow CB(X)$ is given. We say the mapping $f : X \longrightarrow X$ is F-weakly commuting at $x \in X$ if $f(f(x)) \in F(f(x))$.*

Definition 2.12. *Let (X, S) be an S-metric space. The mappings $f, g : X \longrightarrow X$ and $F, G : X \longrightarrow CB(X)$ are given.*

(1) *We say the pair* (f, F) *satisfies the limit property if there exist a sequence* $\{x_n\}$ *in* X*, some* $t \in X$ *and* $A \in CB(X)$ *such that* $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$ (see [10]).

(2) *We say The pairs* (f, F) *and* (g, G) *satisfy the common limit property if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X*,* $t \in X$*, and* $A, B \in CB(X)$ *such that* $\lim_{n\to\infty} Fx_n = A$*,* $\lim_{n\to\infty} Gy_n = B$*,* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B$ (see [19]).

3. MAIN RESULT

In this section we state our mean theorem. Some examples and theorems follow up.

Theorem 3.1. *Let* f *be a self-mapping on an* S*-metric space* (X, S) *and let* F *be a multi-valued mapping from* X *into* $CB(X)$ *such that*

(1): *The pair* (f, F) *satisfies the limit property;*
(2): *For all two distinct elements* $x, y \in X$*,*

$$S_H(Fx, Fx, Fy) < \max\{S(fx, fx, fy), S(fx, fx, Fx) + S(fy, fy, Fy), S(fx, fx, Fy) + S(fy, fy, Fx)\}. \tag{1}$$

If fX *is a closed subset of* X*, then*

(a): f *and* F *have a coincidence point.*
(b): f *and* F *have a common fixed point provided that for each* $v \in C(f, F)$*, the mapping* f *is* F*-weakly commuting at* v *and* $ffv = fv$*, where* $C(f, F) = \{a \in X : fa \in Fa\}$*.*

Proof. By assumption, there exist a sequence $\{x_n\}$ in X, $t \in X$ and $A \in CB(X)$ such that $\lim_{n\to\infty} f(x_n) = t \in \lim_{n\to\infty} Fx_n = A$. Also there exists $a \in X$ such that $t = f(a)$. We put $x = x_n$ and $y = a$ in inequality (1) to obtain:

$$S_H(Fx_n, Fx_n, Fa) < \max\{S(fx_n, fx_n, fa), S(fx_n, fx_n, Fx_n) + S(fa, fa, Fa), S(fx_n, fx_n, Fa) + S(fa, fa, Fx_n)\}.$$

By Lemma 3.3 [14], It follows that

$$\lim_{n\to\infty} S_H(Fx_n, Fx_n, Fa) = S_H(A, A, Fa) \leqslant S(fa, fa, Fa).$$

By definition of S_H we have

$$2S(fa, fa, Fa) \leqslant S_H(A, A, Fa) \leqslant S(fa, fa, Fa).$$

That is, $S(fa, fa, Fa) = 0$. So, $f(a) \in F(a)$. This proves (a). To prove (b), by (a), there exist $t, a \in X$ such that $t = fa \in Fa$. Since $a \in C(f, F)$, So $ffa = fa$ and $ffa \in Ffa$. Hence, $ft = t \in Ft$. □

Example 3.2. *Consider $X = [1, \infty)$ with the max S-metric. Define $f : X \longrightarrow X$, $F : X \longrightarrow CB(X)$ as $f(x) = x^3$ and $F(x) = \left[1, \frac{x^2+1}{2x}\right]$ respectively. The pair (f, F) satisfies the limit property. In fact, we have*

$$\lim_{n\to\infty} f(1 + \frac{1}{n}) = 1 \in \lim_{n\to\infty} F(1 + \frac{1}{n}) = \{1\}.$$

For any two distinct elements $x, y \in X$, the inequality (1) holds. For example, in the case $x < y$, by Remark 2.9 we have

$$S_H(Fx, Fx, Fy) = 2H_S(Fx, Fy) = \frac{y^2 + 1}{y} - 2.$$

On the other hand, $S(fx, fx, fy) = S(Fx^3, Fx^3, Fy^3) = y^3 - 1$. So,

$$S_H(Fx, Fx, Fy) < \max\{S(fx, fx, fy), S(fx, fx, Fx) + S(fy, fy, Fy), S(fx, fx, Fy) + S(fy, fy, Fx)\}.$$

Hence, by Theorem 3.1, f and F have a coincidence point. That is, $f(1) \in F(1)$. Since $ff(1) = f(1)$ and $ff(1) \in F(1)$, f and F have common fixed point 1.

Theorem 3.3. *Let f be a self-mapping on a complete S-metric space (X, S) and let F be a multi-valued mapping from X into $K(X)$ and let $\lambda \in (0, \frac{2}{3})$ be a constant such that for all two distinct members $x, y \in X$:*

$$S_H(Fx, Fx, Fy) \leqslant \lambda \max\{S(fx, fx, fy), S(fx, fx, Fx), S(fy, fy, Fy), S(fx, fx, Fy) + S(fy, fy, Fx)\}. \quad (2)$$

If fX is a closed subset of X and $Fx \subseteq K(fX)$, then

- **(a):** *f and F have a coincidence point;*
- **(b):** *f and F have a common fixed point provided that for each $v \in C(f, F)$, f is F-weakly commuting at v and $ffv = fv$, where $C(f, F) = \{a \in X : fa \in Fa\}$.*

Proof. Since for each $x_0 \in X$, $\varnothing \neq Fx_0 \subseteq fX$, there exists $x_1 \in X$ such that $y_1 = fx_1 \in Fx_0$. So, by Lemma 3.11 [14], there exists $y_2 = fx_2 \in Fx_1$ such that

$$S(y_1, y_1, y_2) < \frac{1}{2} S_H(Fx_0, Fx_0, Fx_1) + \lambda.$$

We obtain a sequence $\{y_n\}$ such that $y_n = fx_n \in Fx_{n-1}$ and

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &< \frac{1}{2} S_H(Fx_{n-1}, Fx_{n-1}, Fx_n) + \lambda^n \\ &\leqslant \frac{\lambda}{2} \max\{S(fx_{n-1}, fx_{n-1}, fx_n), S(fx_{n-1}, fx_{n-1}, Fx_{n-1}), \\ &S(fx_n, fx_n, Fx_n), S(fx_{n-1}, fx_{n-1}, Fx_n) + S(fx_n, fx_n, Fx_{n-1})\} + \lambda^n. \end{aligned}$$

Set $a_n = S(y_n, y_n, y_{n+1})$. Since $fx_n \in Fx_{n-1}, S(fx_n, fx_n, Fx_{n-1}) = 0$. So,

$$a_n < \frac{\lambda}{2} \max\{a_{n-1}, S(fx_{n-1}, fx_{n-1}, Fx_{n-1}), S(fx_n, fx_n, Fx_n), \\ S(fx_{n-1}, fx_{n-1}, Fx_n)\} + \lambda^n.$$

We know

$$S(fx_{n-1}, fx_{n-1}, Fx_{n-1}) \leqslant S(fx_{n-1}, fx_{n-1}, fx_n) = a_{n-1}, S(fx_n, fx_n, Fx_n) \leqslant a_n, \\ S(fx_{n-1}, fx_{n-1}, Fx_n) \leqslant S(y_{n-1}, y_{n-1}, y_{n+1}) \leqslant 2S(y_{n-1}, y_{n-1}, y_n) \\ + S(y_{n+1}, y_{n+1}, y_n) = 2a_{n-1} + a_n.$$

So, $a_n < \frac{\lambda}{2}(2a_{n-1} + a_n) + \lambda^n$. That is, $a_n < \frac{\lambda}{1-\frac{\lambda}{2}} a_{n-1} + \frac{\lambda^n}{1-\frac{\lambda}{2}}$. By induction, we have

$$\begin{aligned} a_n &< \left(\frac{\lambda}{1-\frac{\lambda}{2}}\right)^n \left[a_0 + 1 + (1-\frac{\lambda}{2}) + (1-\frac{\lambda}{2})^2 + \cdots + \left(1-\frac{\lambda}{2}\right)^{n-1}\right] \\ &\leqslant \left(\frac{\lambda}{1-\frac{\lambda}{2}}\right)^n \left[a_0 + 1 + (n-1)(1-\frac{\lambda}{2})\right]. \\ \text{Set } b_n &= \left(\frac{\lambda}{1-\frac{\lambda}{2}}\right)^n \left[a_0 + 1 + (n-1)(1-\frac{\lambda}{2})\right]. \end{aligned}$$

Since $\lim_{n\to\infty} \dfrac{b_{n+1}}{b_n} = \dfrac{\lambda}{1-\frac{\lambda}{2}} < 1$, so, $\lim_{n\to\infty} a_n = 0$.

Now, we show that $\{y_n\}$ is a Cauchy sequence.

For all $m, n \in N, m \geqslant n$, by Lemma 3.1 [18]

$$S(y_n, y_n, y_m) \leq 2\sum_{i=n}^{m-2} a_i + a_{m-1} \\ \leq 2\sum_{i=n}^{\infty}(\frac{2\lambda}{2-\lambda})^i[a_0 + 1 + (i-1)(1-\frac{\lambda}{2})] + (\frac{2\lambda}{2-\lambda})^{m-1}[a_0 + 1 + (m-2)(1-\frac{\lambda}{2})].$$

Therefore, $\lim_{n,m\to\infty} S(y_n, y_n, y_m) = 0$. So, there exists $u \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = u$. Since fX is closed, there exists $a \in X$ such that $fa = u$. By putting $x = x_n, y = x_m$ in (2) :

$$S_H(Fx_n, Fx_n, Fx_m) \leqslant \lambda \max\{S(fx_n, fx_n, fx_m), S(fx_n, fx_n, Fx_n), \\ S(fx_m, fx_m, Fx_m), S(fx_n, fx_n, Fx_m) + S(fx_m, fx_m, Fx_n)\} \tag{3}$$

Also:

$$S(fx_n, fx_n, Fx_n) \leqslant S(y_n, y_n, y_{n+1}); \quad (4)$$
$$S(fx_m, fx_m, Fx_m) \leqslant S(y_m, y_m, y_{m+1}); \quad (5)$$
$$S(fx_n, fx_n, Fx_m) \leqslant S(y_n, y_n, y_{m+1}); \quad (6)$$
$$S(fx_m, fx_m, Fx_n) \leqslant S(y_m, y_m, y_{n+1}). \quad (7)$$

Relations $(4-7)$ imply $\lim_{m,n\to\infty} S_H(Fx_n, Fx_n, Fx_m) = 0$.
So, $\{Fx_n\}$ is a Cauchy sequence. Hence, by Theorem 2.10 there exists $A \in K(X)$ such that $\lim_{n\to\infty} Fx_n = A$. Since $S(y_n, y_n, A) \leqslant \frac{1}{2} S_H(Fx_{n-1}, Fx_{n-1}, A)$. So, $\lim_{n\to\infty} S(y_n, y_n, A) = 0$. By Lemma 3.4 [14], for every n, there exists $\alpha_n \in A$ such that $S(y_n, y_n, A) = S(y_n, y_n, \alpha_n)$. Hence, $\lim_{n\to\infty} S(y_n, y_n, \alpha_n) = 0$. Lemma 2.1 [17], implies $\lim_{n\to\infty} \alpha_n = u \in A$. So (f, F) satisfies the limit property. The rest of the proof is similar to Theorem 3.1. □

Example 3.4. *Consider $X = [0, 1]$ with the max S-metric. For $fx = x^3$ and $Fx = [0, \frac{x^3}{8}]$, the inequality (2) holds for all two distinct members $x, y \in X$. For example, in case $x < y$, by Remark 2.9, $H_S(Fx, Fy) = \frac{y^3}{8}$. Hence*

$$S_H(Fx, Fx, Fy) = 2H_S(Fx, Fy) = \frac{y^3}{4} = \frac{1}{4} S(fx, fx, fy) \leqslant \frac{1}{4} \max\{S(fx, fx, fy), S(fx, fx, Fx), S(fy, fy, Fy), S(fx, fx, Fy) + S(fy, fy, Fx)\}.$$

We have $fX = X$ and $FX \subseteq K(fX)$. So all conditions of Theorem 3.3 are satisfied. Hence, f and F have commn fixed point 0.

Theorem 3.5. *Let f, g be two self-mappings on an S-metric (X, S) and let F, G be two multi-valued mappings from X into $CB(X)$ such that*

(1): *The pairs (f, F) and (g, G) satisfy the common limit property;*
(2): *For all two distinct members $x, y \in X$:*

$$S_H(Fx, Fx, Gy) < \max\{S(fx, fx, gy), S(fx, fx, Fx) + S(gy, gy, Gy), S(fx, fx, Gy) + S(gy, gy, Fx)\}. \quad (8)$$

If fX, gX are closed subsets of X, then

(a): *f and F have coincidence point;*
(b): *g and G have coincidence point;*
(c): *f and F have common fixed point provided that for each $v \in C(f, F)$, f be an F-weakly commuting at v and $ffv = fv$;*
(d): *g and G have common fixed point provided that for each $v \in C(g, G)$, g be a G-weakly commuting at v and $ggv = gv$;*
(e): *If (c) and (d) hold, then f, g, F and G have common fixed point.*

Proof. By assumption, there exist sequences $\{x_n\}, \{y_n\}$ in X and $u \in X, A, B \in CB(X)$ such that $\lim_{n\to\infty} Fx_n = A$, $\lim_{n\to\infty} Gy_n = B$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n =$

$u \in A \cap B$. Assume that $v, w \in X$ such that $\lim_{n\to\infty} fx_n = fv$ and $\lim_{n\to\infty} gy_n = gw$. We have $fv = gw = u \in A \cap B$. To prove (a), we show, $u = fv \in Fv$. Put $x = v$ and $y = y_n$ in (8) and approach n to ∞, then
$S_H(Fv, Fv, B) \leqslant S(fv, fv, Fv)$. Since

$$u = fv \in B, 2S(fv, fv, Fv) \leqslant S_H(Fv, Fv, B),$$

so, $S(fv, fv, Fv) = 0$. Therefore, $u = fv \in Fv$. Similarly, put $x = x_n, y = w$ in (8) and we have $u = gw \in Gw$. Properties (c), (d), (e) are similar to Theorem 3.1(b). □

Example 3.6. *Consider $X = [0, \infty)$ with the max S-metric. For $fx = x^3, Fx = [0, \frac{x^3}{8}]$ and $gx = x^4, Gx = [0, \frac{x^4}{8}]$, the pairs (f, F) and (g, G) satisfy the common limit property, in fact*

$$\lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} g(\frac{1}{n}) = 0, \lim_{n\to\infty} F(\frac{1}{n}) = \lim_{n\to\infty} G(\frac{1}{n}) = \{0\}.$$

For all distinct members $x, y \in X$, the inquality (8) holds.
For example, in case $x < y$, first assume $x^3 < y^4$. Since, for $t \in Fx, S(t, t, Gy) = 0$, so, $h_S(Fx, Gy) = 0$. Also since, for $t \in Gy$, we have

$$S(t, t, Fx) = \begin{cases} 0 & if\ t \in Fx \\ t & if\ t \in Gy - Fx, \end{cases}$$

so, $h_S(Gy, Fx) = \sup\{S(t, t, Fx) : t \in Gy\} = \frac{y^4}{8}$.
Hence, $H_S(Fx, Gy) = \frac{y^4}{8}$ and $S_H(Fx, Fx, Gy) = \frac{y^4}{4}$.
On the other hand, we have $S(fx, fx, gy) = y^4$, therefore the inequality (8) holds.
Now, assume $y^4 < x^3$. It can be shown that $S_H(Fx, Fx, Gy) = \frac{x^3}{4}$.
On the other hand, we have $S(fx, fx, gy) = x^3$, so the inequality (8) holds. We have $ff0 = f0 = 0 \in Ff0$, and $gg0 = g0 = 0 \in Gg0$. So, all conditions of Theorem 3.5 are satisfied. Therefore, f, g, F and G have common fixed point. That is, $f0 = g0 = 0 \in F0 \cap G0 = \{0\}$.

Corollary 3.7. *If in Theorem 3.5 we set $F = G$, and $f = g$, Theorem 3.1 follows.*

Theorem 3.8. *Let f, g be two self-mappings on a complete S-metric space (X, S) and let F, G be two multi-valued mappings from X into $K(X)$ and let $\lambda \in (0, \frac{2}{3})$ be a constant such that for all two distinct members $x, y \in X$:*

$$\begin{aligned} S_H(Fx, Fx, Gy) \leqslant \lambda \max\{&S(fx, fx, gy), S(fx, fx, Fx), S(gy, gy, Gy), \\ &S(fx, fx, Gy) + S(gy, gy, Fx)\}. \end{aligned} \tag{9}$$

If fX, gX are closed subsets of X and $FX \subseteq K(gX), GX \subseteq K(fX)$, then

- **(a):** *f and F have coincidence point;*
- **(b):** *g and G have coincidence point;*
- **(c):** *f and F have common fixed point provided that for each $v \in C(f, F)$, f be an F-weakly commuting mapping at v and $ffv = fv$;*

(d): *g and G have common fixed point provided that for each $v \in C(g, G)$, g is an G-weakly commuting mapping at v and $ggv = gv$;*

(e): *If (c) and (d) hold, then f, g, F and G have common fixed point.*

Proof. For $x_0 \in X$, there exists $x_1 \in X$ such that $y_1 = gx_1 \in Fx_0$. So, by Lemma 3.11 [14], there exists $y_2 \in Gx_1$ such that

$$S(y_1, y_1, y_2) < \frac{1}{2} S_H(Fx_0, Fx_0, Gx_1) + \lambda.$$

There exists $x_2 \in X$ such that $y_2 = fx_2 \in Gx_1$. So, there exists $y_3 \in Fx_2$ such that

$$S(y_2, y_2, y_3) < \frac{1}{2} S_H(Gx_1, Gx_1, Fx_2) + \lambda^2.$$

We obtain a sequence $\{y_n\}$ such that for every $n \geqslant 1$,

$$y_{2n} = fx_{2n} \in Gx_{2n-1}, y_{2n+1} = gx_{2n+1} \in Fx_{2n}.$$

We have

$$S(y_{2n}, y_{2n}, y_{2n+1}) < \frac{1}{2} S_H(Gx_{2n-1}, Gx_{2n-1}, Fx_{2n}) + \lambda^{2n};$$

$$S(y_{2n-1}, y_{2n-1}, y_{2n}) < \frac{1}{2} S_H(Fx_{2n-2}, Fx_{2n-2}, Gx_{2n-1}) + \lambda^{2n-1}.$$

Set $a_n = S(y_n, y_n, y_{n+1})$. Similar to Theorem 3.3, it can be shown that

$$a_{2n} < \frac{\lambda}{2}(2a_{2n-1} + a_{2n}) + \lambda^{2n},\ a_{2n-1} < \frac{\lambda}{2}(2a_{2n-2} + a_{2n-1}) + \lambda^{2n-1}.$$

So, for every $n \in N$, we have $a_n < \frac{\lambda}{2}(2a_{n-1} + a_n) + \lambda^n$. Similar to Theorem 3.3, we have $\lim_{n\to\infty} a_n = 0$ and $\{y_n\}$ is a Cauchy sequence. So, there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$. Hence, $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = u$, and there exist $a, b \in X$ such that $fa = gb = u$. To show $\{Fx_{2n}\}$ is a Cauchy sequence, we have

$$\begin{aligned} S_H(Fx_{2n}, Fx_{2n}, Fx_{2m}) \leqslant & 2S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \\ & + S_H(Fx_{2m}, Fx_{2m}, Gx_{2n+1}). \end{aligned} \tag{10}$$

By (9) we have:

$$S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \leqslant \lambda \max\{S(fx_{2n}, fx_{2n}, gx_{2n+1}), S(fx_{2n}, fx_{2n}, Fx_{2n}),$$
$$S(gx_{2n+1}, gx_{2n+1}, Gx_{2n+1}), S(fx_{2n}, fx_{2n}, Gx_{2n+1}) + S(gx_{2n+1}, gx_{2n+1}, Fx_{2n})\}.$$

So, $\lim_{n\to\infty} S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) = 0$.

Similarly, we have $\lim_{n\to\infty} S_H(Fx_{2m}, Fx_{2m}, Gx_{2n+1}) = 0$. It follows from (10) that $\lim_{n,m\to\infty} S_H(Fx_{2n}, Fx_{2n}, Fx_{2m}) = 0$. So, by Theorem 2.10, there exists $A \in K(X)$ such that $\lim_{n\to\infty} Fx_{2n} = A$. Now, assume that the left side of inequality (9) is $S(fx, fx, gy)$. Then, we have

$$S_H(Fx, Fx, Gy) \leqslant \lambda\ S(fx, fx, gy). \tag{11}$$

Put $x = x_{2n}, y = x_{2n+1}$ in (11) to obtain

$$S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \leqslant \lambda\ S(fx_{2n}, fx_{2n}, gx_{2n+1}).$$

So,

$$\lim_{n\to\infty} S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) = 0.$$

Since $\lim_{n\to\infty} Fx_{2n} = A$, by Lemma 2.1 [17], $\lim_{n\to\infty} Gx_{2n+1} = A$. Assume that the left side of inequality (9) is $S(fx, fx, Gy) + S(gy, gy, Fx)$. Then, we have

$$S_H(Fx, Fx, Gy) \leqslant \lambda[S(fx, fx, Gy) + S(gy, gy, Fx)]. \tag{12}$$

Put $x = x_{2n}, y = x_{2n+1}$ in (12) to obtain

$$\begin{aligned} S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) \leqslant& \lambda[S(y_{2n}, y_{2n}, Gx_{2n+1}) + S(y_{2n+1}, y_{2n+1}, Fx_{2n})] \\ \leqslant& \lambda[S(y_{2n}, y_{2n}, y_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+1})]. \end{aligned}$$

So,

$$\lim_{n\to\infty} S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}) = 0.$$

Therefore, by Lemma 2.1 [17], $\lim_{n\to\infty} Gx_{2n+1} = A$. Similarly, if the left side of inequality (9) is $S(fx, fx, Fx)$ or $S(gy, gy, Gy)$, we have $\lim_{n\to\infty} Gx_{2n+1} = A$. On the other hand, we have:

$$S(y_{2n+1}, y_{2n+1}, A) \leqslant \frac{1}{2} S_H(Fx_{2n}, Fx_{2n}, A).$$

So, $\lim_{n\to\infty} S(y_{2n+1}, y_{2n+1}, A) = 0$. By Lemma 3.4 [14], for every n, there exists $\alpha_{2n+1} \in A$ such that,

$$S(y_{2n+1}, y_{2n+1}, A) = S(y_{2n+1}, y_{2n+1}, \alpha_{2n+1}).$$

So, $\lim_{n\to\infty} S(y_{2n+1}, y_{2n+1}, \alpha_{2n+1}) = 0$. Hence, by Lemma 2.1 [17], $\lim_{n\to\infty} \alpha_{2n+1} = u$. So $u \in A$. That is, $(f, F), (g, G)$ satisfy the common limit property. The rest of the proof is similar to Theorem 3.5. □

Example 3.9. *In Example 3.6, for all distinct members $x, y \in X$:*

$$\begin{aligned} S_H(Fx, Fx, Gy) =& \frac{1}{4} S(fx, fx, gy) \\ \leqslant& \frac{1}{4} \max\{S(fx, fx, gy), S(fx, fx, Fx), S(gy, gy, Gy), \\ & S(fx, fx, Gy) + S(gy, gy, Fx)\}. \end{aligned}$$

So, all conditions of Theorem 3.8 are satisfied. That is, f, g, F and G have common fixed point.

Corollary 3.10. *If in Theorem 3.8 we set $F = G$, and $f = g$, Theorem 3.3 follows.*

4. CONCLUSIONS

We generalized some theorems in fixed point theorem work. Theorem 3.1 is a generalization of Theorem 3.4 of Tayyab Kamran, 2004 [10]. Theorem 3.5 and Theorem 3.8 are generalizations of Theorem 2.3 and Theorem 2.8 of Yicheng Liu, Jun Wu, Zhixiang Li, 2005 [19], for single-valued and multi-valued mappings on S-metric and S_H-metric spaces respectively. We showed that not every S-metric is necessarily continuous.
The notion of compatible for single-valued and multi-valued mappings can be defined to investigate the existence of fixed points in S-metric spaces. Also, the existence of solution for certain nonlinear integral equations can be investigated in a future work.

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