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ON CONGRUENT DOMINATION NUMBER OF DISJOINT AND ONE POINT UNION OF GRAPHS

S.K. Vaidya[1], H.D. Vadhel[2]

[1,2]**Department of Mathematics, Saurashtra University, Rajkot, Gujarat, India.**

[1]**samirkvaidya@yahoo.co.in**, [2]**harshad.vadhel18@gmail.com**

Abstract. A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set of G if $\sum\limits_{v\in V(G)} d(v) \equiv 0 \left(\bmod \sum\limits_{v\in D} d(v) \right)$ The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of G which is denoted by $\gamma_{cd}(G)$. We establish the bounds on congruent domination number in terms of order of disjoint union of graphs as well as one point union of graphs.

Key words and Phrases: Dominating Set, Domination Number, Congruent Dominating Set, Congruent Domination Number.

1. INTRODUCTION

All the graphs considered here are finite, connected and undirected with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ and $E(G)$ are respectively called order and degree of G. A set $D \subseteq V(G)$ is called a dominating set of G if every vertex $v \in V(G)-D$ is adjacent to at least one vertex in D. A dominating set D is said to be a minimal dominating set if no proper set $D' \subset D$ is a dominating set of G. The minimum cardinality of a minimal dominating set in G is called the domination number $\gamma(G)$ of a graph G.

The study of dominating sets and its related concepts are the major areas of research within graph theory. Many domination models have been introduced in the recent past. Global domination[10], total domination[3], independent domination[1, 9], equitable domination[11] are worth to mention. An expository discussion on domination and related concepts can be found in [4, 5, 6, 7, 8]. For standard

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terminology in graph theory, we refer to West [14] while the terms related to number theory are used in the sense of Burton [2].

The following new concept is recently introduced and explored by Vaidya and Vadhel [12, 13].

A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set of G if

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\bmod \sum_{v \in D} d(v) \right).$$

A congruent dominating set $D \subseteq V(G)$ is said to be a minimal congruent dominating if no proper subset D' of D is a congruent dominating set. The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of a graph G which is denoted by $\gamma_{cd}(G)$.

In this paper, we have investigated the bounds on congruent domination number in terms of order of disjoint union of graphs and one point union of graphs. The present work is a combination of two major research areas of mathematics namely graph theory and number theory.

Definition 1.1. *Let G_1 and G_2 be two graphs with vertex sets $V(G_1)$, $V(G_2)$ and edge sets $E(G_1)$, $E(G_2)$ then the disjoint union of G_1 and G_2 is denoted as $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.*

Definition 1.2. *A graph G in which a vertex can be distinguished from other vertices is called a rooted graph and the distinguished vertex is called the root of G. If G is a rooted graph then the graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called the one point union of n copies of G.*

2. MAIN RESULTS

We prove following two trivial but important Lemmas.

Lemma 2.1. *Let A, B, X and Y be positive integers such that*

(1) $\frac{A}{X} = \frac{B}{Y} = k\,; k \in \mathbb{N}$ or

(2) $A \equiv 0 \pmod{X}$ and $B \equiv 0 \pmod{Y}$ when A and B are even and $X = Y$ are odd.

Then $A + B \equiv 0(\bmod\ (X+Y))$.

Proof.

(1) Let $\frac{A}{X} = \frac{B}{Y} = k$, for some $k \in \mathbb{N}$. Then $A = kX$ and $B = kY$. Now, $kX \equiv 0 \pmod{X}$ and $kY \equiv 0 \pmod{Y}$. Then, $kX + kY = k(X+Y)$ and so $k(X+Y) \equiv 0\ (\bmod (X+Y))$. Hence, $A + B \equiv 0(\bmod\ (X+Y))$.

(2) Let $A \equiv 0 \pmod X$ and $B \equiv 0 \pmod X$ as $X = Y$. Then, $A + B \equiv 0(\text{mod } X)$. Note that $A + B$ is even as A and B both are even. Similarly $X = Y$ are odd. Hence, $A + B \geqslant 2X$ and so $A + B \equiv 0(\text{mod } 2X)$. Thus, $A + B \equiv 0(\text{mod } (X + Y))$. □

The following Lemma 2.2 is the generalization of Lemma 2.1, where unlike in Lemma 2.1 the condition (2) is not required.

Lemma 2.2. *Let A_i and X_i be any positive integers such that $\dfrac{A_i}{X_i} = k$, for each $i = 1, 2, ..., n$, then for at least one i,*

$$\sum_{i=1}^{n} A_i \equiv 0 \left(\text{mod } \sum_{i=1}^{n} X_i\right) \; if \sum_{j=1, j\neq i}^{n} A_j \equiv 0 \left(\text{mod } \sum_{j=1, j\neq i}^{n} X_j\right)$$

Proof.
Let $\dfrac{A_i}{X_i} = k$, $\forall\ i = 1, 2, ..., n$ then $A_i \equiv 0 \pmod{X_i}$ and so $(A_i + A_j) \equiv 0$ (mod $(X_i + X_j)$), $\forall i, j = 1, 2, ..., n$ by Lemma 2.1.

Now let

$$\sum_{j=1, j\neq i}^{n} A_j \equiv 0 \left(\text{mod } \sum_{j=1, j\neq i}^{n} X_j\right), \; \forall i = 1, 2, ..., n.$$

Denote

$$\sum_{j=1, j\neq i}^{n} A_j = A \text{ and } \sum_{j=1, j\neq i}^{n} X_j = X, \text{ for each } i = 1, 2, ..., n.$$

Then $A \equiv 0 \pmod X$ and we have $A_i \equiv 0 \pmod{X_i}$, $\forall i = 1, 2, ..., n$. Now $\dfrac{A}{X} = k$ as for each i, $\dfrac{A_i}{X_i} = k$. Therefore, by Lemma 2.1, we have, $(A + A_i) \equiv 0$ (mod $(X + X_i)$). i.e.,

$$\sum_{i=1}^{n} A_i \equiv 0 \left(\text{mod } \sum_{i=1}^{n} X_i\right).$$

□

In the previously established Lemma 2.1 and Lemma 2.2, we take into account that $A = \sum_{v \in V(G)} d(v)$ and $A_i = \sum_{v \in V(G_i)} d(v)$, for each $i = 1, 2, ..., n$ are the degree sum of vertices of a graph G and G_i, respectively while $X = \sum_{v \in D} d(v)$ and $X_i = \sum_{v \in D_i} d(v)$ are the degree sum of vertices of the dominating set D in G and degree sum of vertices of the the dominating set D_i in G_i. The two lemma mentioned before are helpful in proving the next two theorems.

Theorem 2.3. *Let G be union of n disjoint graphs $G_1, G_2, ..., G_n$ and for any positive integer A_i and X_i if $A_i \equiv 0 \pmod{X_i}$ such that $\dfrac{A_i}{X_i} = k,\ \forall i = 1, 2, ..., n$. Then for at least one i,*

$$\gamma_{cd}(G) \leqslant \sum_{i=1}^{n} \gamma_{cd}(G_i)\ if \sum_{j=1, j\neq i}^{n} A_j \equiv 0 \left(\bmod \sum_{j=1, j\neq i}^{n} X_j\right),$$

where $A_i = \sum_{v\in V(G_i)} d(v)$ and $X_i = \sum_{v\in D_i} d(v),\ \forall i = 1, 2, ..., n$.

Proof.
Let D_i be a congruent dominating set of graph G_i such that $\gamma_{cd}(G_i) = |D_i|,\ \forall i = 1, 2, ..., n$ with

$$\sum_{v\in V(G_i)} d(v) = A_i \text{ and } \sum_{v\in D_i} d(v) = X_i,\ \forall i = 1, 2, ..., n.$$

Let

$$\sum_{j=1, j\neq i}^{n} \left(\sum_{v\in D_j} d(v)\right) = \sum_{j=1, j\neq i}^{n} X_j = X, \text{ for each } i = 1, 2, ..., n.$$

Let G be a graph with

$$\sum_{j=1, j\neq i}^{n} \left(\sum_{v\in V(G_j)} d(v)\right) = \sum_{j=1, j\neq i}^{n} A_j = A.$$

Since D_i is a congruent dominating set of graph G_i, for each i, we have,

$$\sum_{v\in V(G_i)} d(v) \equiv 0 \left(\bmod \sum_{v\in D_i} d(v)\right), \text{ for each } i.$$

Then by Lemma, 2.2, we have,

$$\sum_{v\in V(G)} d(v) \equiv 0 \left(\bmod \sum_{i=1}^{n} \left(\sum_{v\in D_i} d(v)\right)\right), \text{ for each } i.$$

Then, $D = D_1 \cup D_2 \cup .... \cup D_n$ is congruent dominating set of graph G. Hence,

$$\gamma_{cd}(G) \leqslant \sum_{i=1}^{n} \gamma_{cd}(G_i).$$

□

Remark 2.4. *The bounds obtained in Theorem 2.3 are sharp when*

$$\gamma(G_i) = \gamma_{cd}(G_i) \text{ for } 1 \leqslant i \leqslant n.$$

i.e. for each i, $\gamma(G_i) = \gamma_{cd}(G_i)$, then

$$\gamma_{cd}(G) = \sum_{i=1}^{n} \gamma_{cd}(G_i).$$

Corollary 2.5. *Let G be an union of two disjoint graphs G_1 and G_2 such that $\frac{A}{X} = \frac{B}{Y}$ or each A and B are even and $X = Y$ are odd, where*

$$A = \sum_{v \in V(G_1)} d(v),\ B = \sum_{v \in V(G_2)} d(v),\ X = \sum_{v \in D_1} d(v) \text{ and } Y = \sum_{v \in D_2} d(v)$$

while D_1 and D_2 are the congruent dominating sets of graphs G_1 and G_2, respectively such that $|D_1| = \gamma_{cd}(G_1)$ and $|D_2| = \gamma_{cd}(G_2)$, then

$$\gamma_{cd}(G) \leqslant \gamma_{cd}(G_1) + \gamma_{cd}(G_2).$$

Proof.

Let G be a graph with vertex set $V(G)$ and let each G_i be the graph with vertex set $V(G_i)$ and congruent dominating set D_i such that $|D_i| = \gamma_{cd}(G_i)$, for each $i = 1, 2$. Let

$$A = \sum_{v \in V(G_1)} d(v),\ B = \sum_{v \in V(G_2)} d(v),\ X = \sum_{v \in D_1(G_1)} d(v) \text{ and } Y = \sum_{v \in D_2} d(v)$$

Then

$$A \equiv 0(\text{mod} X) \text{ and } B \equiv 0(\text{mod} Y)$$

Here,

$$A + B = \sum_{v \in V(G_1) \cup V(G_2)} d(v) = \sum_{v \in V(G)} d(v)$$

Now if $\frac{A}{X} = \frac{B}{Y}$ or $A \equiv 0(\bmod\ X)$ and $B \equiv 0(\bmod\ Y)$ such that each A and B are even and $X = Y$ are odd, then from Lemma 2.1, we have,

$A + B \equiv 0(\text{mod } (X + Y))$

That is,

$$\sum_{v \in V(G_1)} d(v) + \sum_{v \in V(G_2)} d(v) \equiv 0 \left(\text{mod} \left(\sum_{v \in D_1} d(v) + \sum_{v \in D_2} d(v)\right)\right)$$

Thus,

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\text{mod} \left(\sum_{v \in D_1} d(v) + \sum_{v \in D_2} d(v)\right)\right)$$

Which implies that $D_1 \cup D_2$ is a congruent dominating set of graph G. Hence,

$$\gamma_{cd}(G) \leqslant \gamma_{cd}(G_1) + \gamma_{cd}(G_2).$$

□

Corollary 2.6. *Let G be an union of n disjoint copies of graph H, then $\gamma_{cd}(G) \leqslant n \cdot \gamma_{cd}(H)$.*

Proof.
Let D_i be a congruent dominating set of graph $H_i = H$ such that $\gamma_{cd}(H_i) = |D_i|$, $\forall i = 1, 2, ..., n$ with

$$\sum_{v \in V(H_i)} d(v) = X_i \text{ and } \sum_{v \in V(D_i)} d(v) = A_i, \ \forall i = 1, 2, ..., n.$$

Let

$$\sum_{i=1}^{n} \left(\sum_{v \in V(D_i)} d(v) \right) = \sum_{i=1}^{n} A_i = A, \ \forall i = 1, 2, ..., n.$$

Let G be a graph with

$$\sum_{v \in V(G)} d(v) = \sum_{i=1}^{n} \left(\sum_{v \in V(H_i)} d(v) \right) = \sum_{i=1}^{n} X_i = X.$$

Since D_i is a congruent dominating set of graph $H_i = H$, we have

$$\sum_{v \in V(H_i)} d(v) \equiv 0 \left(\text{mod} \sum_{v \in V(D_i)} d(v) \right), \text{ for each } i.$$

i.e., $X_i \equiv 0(\text{mod } A_i)$ i.e., $nX_i \equiv 0(\text{mod } nA_i)$ i.e., $X \equiv 0(\text{mod } A)$ i.e.,

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\text{mod} \sum_{i=1}^{n} \left(\sum_{v \in V(D_i)} d(v) \right) \right), \text{ for each } i.$$

Thus, $D = D_1 \cup D_2 \cup .... \cup D_n$ is congruent dominating set of graph G. Hence, $\gamma_{cd}(G) \leqslant n \cdot \gamma_{cd}(H)$. □

Theorem 2.7. *Let G be the one point union of two graphs G_1 and G_2 such that $\frac{A}{X} = \frac{B}{Y} = k$, $k \in \mathbb{N}$ or each A and B are even and $X = Y$ are odd, where*

$$A = \sum_{v \in V(G_1)} d(v), \ B = \sum_{v \in V(G_2)} d(v), \ X = \sum_{v \in D_1} d(v) \text{ and } Y = \sum_{v \in D_2} d(v)$$

with D_1 and D_2 are the congruent dominating sets of graphs G_1 and G_2, respectively such that $|D_1| = \gamma_{cd}(G_1)$ and $|D_2| = \gamma_{cd}(G_2)$, then $\gamma_{cd}(G) \leqslant \gamma_{cd}(G_1) + \gamma_{cd}(G_2)$.

Proof.
Let G be a graph with vertex set $V(G)$ and let each G_i be the graph with vertex set $V(G_i)$ and congruent dominating set D_i such that $|D_i| = \gamma_{cd}(G_i)$, for each $i = 1, 2$. Let

$$A = \sum_{v \in V(G_1)} d(v), \ B = \sum_{v \in V(G_2)} d(v), \ X = \sum_{v \in D_1(G_1)} d(v) \text{ and } Y = \sum_{v \in D_2} d(v)$$

Then

$$A \equiv 0(\mathrm{mod} X) \text{ and } B \equiv 0(\mathrm{mod} Y)$$

Since, G is the one point union of two graph G_1 and G_2, we have,

$$A + B = \sum_{v \in V(G_1) \cup V(G_2)} d(v) = \sum_{v \in V(G)} d(v)$$

Now if $\dfrac{A}{X} = \dfrac{B}{Y}$ or each A and B are even and $X = Y$ are odd, then from Lemma 2.1,

$$A + B \equiv 0(\mathrm{mod}\ (X + Y))$$

i.e.,

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\mathrm{mod} \left(\sum_{v \in D_1} d(v) + \sum_{v \in D_2} d(v) \right) \right)$$

This implies that $D_1 \cup D_2$ is a congruent dominating set of graph G. Hence,

$$\gamma_{cd}(G) \leqslant \gamma_{cd}(G_1) + \gamma_{cd}(G_2).$$

□

3. CONCLUDING REMARKS

The congruences are the equivalence relations compatible with algebraic structures while domination parameters are defined in the context of graph structures. The present work is an attempt to unify two important concepts namely, congruence relation and domination numbers. The new domination parameter is termed as congruent domination number[12, 13]. Here, we have contributed some more results on this newly introduced concept.

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