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K-CONTINUOUS FUNCTIONS AND RIGHT B_1 COMPOSITORS

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Abstract. A function $g : \mathbb{R} \to \mathbb{R}$ from the real line to itself is called a right B_1 compositor if for any Baire class one function $f : \mathbb{R} \to \mathbb{R}$, $f \circ g : \mathbb{R} \to \mathbb{R}$ is Baire class one. In this study, we first apply Jayne-Rogers Theorem [2] to prove that every right B_1 compositor is $\mathcal{D}$-continuous where $\mathcal{D}$ is the class of all positive functions on $\mathbb{R}$ and thus give a positive answer to a problem posed by D. Zhao. This result then characterizes the right B_1 compositor as a class of naturally defined functions. Furthermore, we also improved some of the results in [4]. Lastly, a counterexample was constructed to a claim in [4] that every function with a finite number of discontinuity points is left B_1 compositor.

Key words: Baire class one, right B_1 compositor, $\mathcal{D}$-continuous, k-continuous.

Abstrak. Sebuah fungsi $g : \mathbb{R} \to \mathbb{R}$ dari bilangan riil ke bilangan riil disebut kompositor B_1 kanan jika untuk setiap fungsi Baire kelas satu $f : \mathbb{R} \to \mathbb{R}$, $f \circ g : \mathbb{R} \to \mathbb{R}$ adalah Baire kelas satu. Pada artikel ini, pertama-tama kami menerapkan Teorema Jayne-Rogers [2] untuk membuktikan bahwa setiap kompositor B_1 kanan adalah kontinu-$\mathcal{D}$ dengan $\mathcal{D}$ adalah kelas dari semua fungsi positif pada $\mathbb{R}$ sehingga memberikan jawaban positif untuk sebuah masalah yang diajukan oleh D. Zhao. Hasil ini kemudian mengkarakterisasi kompositor B_1 kanan sebagai sebuah kelas dari fungsi yang terdefinisi secara alami. Lebih jauh, kami juga memperbaiki beberapa hasil di [4]. Terakhir, sebuah contoh penyangkal dikonstruksi untuk sebuah klaim di [4] yaitu setiap fungsi kontinu dengan sejumlah hingga titik-titik diskontinu adalah kompositor B_1 kiri.

Kata kunci: Baire kelas satu, kompositor B_1 kanan, kontinu-$\mathcal{D}$, kontinu-k.

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1. **Introduction**

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be a right B_1 compositor if $f \circ g$ is Baire class one whenever f is Baire class one while a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{D}$-continuous if for every positive function $\epsilon(\cdot) \in \mathcal{D}$ there exists a function $\delta : \mathbb{R} \to \mathbb{R}^+$ such that for any $x,\, y \in \mathbb{R}$

$$|x-y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x)-f(y)| < \epsilon(f(x)) \wedge \epsilon(f(y)),$$

where $\mathcal{D}$ is a class of positive functions on $\mathbb{R}$. It turned out that right B_1 compositors have a similar characterization as that of the characterization of Baire class one functions discovered by P.Y. Lee, W.K. Tang and D. Zhao recently[1].

Furthermore, Zhao [4] studied a subclass of right B_1 compositors by considering $\mathcal{D}$ to be the class of all positive real valued functions defined on $\mathbb{R}$ and called them k-continuous. He asked whether every right B_1 compositor is k-continuous. We answered this query in the affirmative. In the last part of the paper, we also provide a counterexample to the claim in [4, page 550] that every function with a finite number of discontinuity points is left B_1 compositor.

2. **Equivalence of Right B_1 Compositors and k-Continuous Functions**

We shall denote $\min\{a, b\}$ by $a \wedge b$ for any two real numbers a and b. The following two definitions are given in [4].

Definition 2.1. *A function $g : \mathbb{R} \to \mathbb{R}$ is called a right B_1 compositor if $f \circ g$ is Baire class one whenever $f : \mathbb{R} \to \mathbb{R}$ is Baire class one. Left B_1 compositor is defined similarly.*

Definition 2.2. *Let A be a subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is $\mathcal{D}$-continuous if for any $\epsilon(\cdot) \in \mathcal{D}$ there is a positive real valued function $\delta : A \to \mathbb{R}^+$ such that for any $x, y \in A$*

$$|x-y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x)-f(y)| < \epsilon(f(x)) \wedge \epsilon(f(y)).$$

A function $f : \mathbb{R} \to \mathbb{R}$ is k-continuous if f is $\mathcal{D}$-continuous where $\mathcal{D}$ is the class of all positive real valued functions on $\mathbb{R}$.

It is an easy exercise to prove that every continuous function is k-continuous.

Proposition 2.3. *Let $\mathcal{D}$ be a class of positive real valued functions on $\mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $\mathbb{R} = \bigcup_{i=1}^{+\infty} F_i$ where each F_i is an F_σ set and that $g|_{F_i}$ is $\mathcal{D}$-continuous for each i. Then $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{D}$-continuous.*

Proof: By Lemma 1 of [4], we may assume that the countable collection of F_σ sets $\{F_i\}_{i=1}^{+\infty}$ are pairwise disjoint. By [4, Lemma 2] , there is a positive function $\delta_0(\cdot)$ on $\mathbb{R}$ such that if $x \in F_n$ and $y \in F_m$, $m \neq n$ then

$$|x-y| \geq \delta_0(x) \wedge \delta_0(y).$$

Let $\epsilon(\cdot)$ be any positive real valued function on $\mathbb{R}$ that belongs to $\mathcal{D}$. Since $g|_{F_i}$ is $\mathcal{D}$-continuous for each i then for every i there is a positive function $\delta_i(\cdot)$ on F_i such that for x, y in F_i

$$|x-y| < \delta_i(x) \wedge \delta_i(y) \Longrightarrow |g(x)-g(y)| < \epsilon(g(x)) \wedge \epsilon(g(y)).$$

Put

$$\delta(x) = \delta_0(x) \wedge \delta_i(x), \quad x \in F_i.$$

Suppose $|x-y| < \delta(x) \wedge \delta(y)$. Then there is some n such that $x, y \in F_n$. Since $g|_{F_n}$ is $\mathcal{D}$-continuous then $|g(x)-g(y)| < \epsilon(g(x)) \wedge \epsilon(g(y))$. Therefore, g is $\mathcal{D}$-continuous on $\mathbb{R}$. □

Corollary 2.4. *Suppose that* $\mathbb{R} = \bigcup_{i=1}^{+\infty} F_i$ *where each* F_i *is an* F_σ *set and that* $g|_{F_i}$ *is continuous. Then* $g : \mathbb{R} \to \mathbb{R}$ *is* k*-continuous.*

A function $f : X \to Y$ from a metric space X into a metric space Y is said to be a Δ_2^0-function if $f^{-1}(S)$ is F_σ in X for every F_σ set S in Y. Furthermore, $f : X \to Y$ is said to be piecewise continuous if X can be expressed as the union of an increasing sequence $X_0, X_1, \ldots$ of closed sets in X such that $f|_{X_n}$ is continuous for every $n \in \mathbb{N}$. Theorem 2.5 [2, Theorem 2.1] is used in the proof of Theorem 2.6. Here we use the fact that $\mathbb{R}$ is a complete metric space with the usual metric in $\mathbb{R}$.

Theorem 2.5. *([2]) Let* X *and* Y *be metric spaces such that the metric of* X *is complete, and let* $f : X \to Y$ *be of Baire class 1. If* f *is a* Δ_2^0*-function then it is piecewise continuous.*

The following is the main result of the paper which gives a positive answer to the question posed in [4].

Theorem 2.6. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is a right* B_1 *compositor if and only if* f *is* k*-continuous.*

Proof: The sufficiency has been proved in [4]. Now we prove the necessity. Assume that f is a right B_1 compositor. Then by [4, Theorem 1], $f^{-1}(F)$ is F_σ for every F_σ set F in $\mathbb{R}$. By Theorem 2.5, there exists an increasing sequence of closed sets $\{E_i\}_{i=1}^{\infty}$ such that $\mathbb{R} = \bigcup_{i=1}^{\infty} E_i$ and $f|_{E_i}$ is continuous for each i. We apply Corollary 2.4 to conclude the proof. □

By combining [4, Theorem 1] and the above theorem we have the following:

Theorem 2.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function from the real line to itself. Then the following statements are equivalent:*
(1) For every closed set $A \subseteq \mathbb{R}$, $f^{-1}(A)$ *is* F_σ.
(2) For every F_σ *set* $A \subseteq \mathbb{R}$, $f^{-1}(A)$ *is* F_σ.

(3) For every positive Baire class one function $\epsilon(\cdot)$ *on* $\mathbb{R}$ *there is a positive function* $\delta : \mathbb{R} \to \mathbb{R}^+$ *such that for any x,y in* $\mathbb{R}$

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon(f(x)) \wedge \epsilon(f(y)).$$

(4) For every positive function $\epsilon(\cdot)$ *on* $\mathbb{R}$ *there is a positive function* $\delta : \mathbb{R} \to \mathbb{R}^+$ *such that for any x,y in* $\mathbb{R}$

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon(f(x)) \wedge \epsilon(f(y)).$$

(5) f *is a right* B_1 *compositor.*

3. Further Results on k-Continuous Functions

In [4] the following were stated without proof:

(1) If the range $g(\mathbb{R})$ is a finite set then g is k-continuous if and only if it is Baire class one.
(2) If f and g have different values at only a finite number of points then f is k-continuous if and only if g is k-continuous.
(3) If the set of discontinuity of f is discrete then f is k-continuous.

We sharpen the first two observations by considering a discrete set instead of a finite set. It is shown here that a discrete set cannot be replaced by a countable set. In this sense the improvement is optimal. We use the improvement of (1) to give a characterization of Baire class one functions in terms of k-continuous functions. We also include a proof of (3) and give some remarks on it.

Note that $f : \mathbb{R} \to \mathbb{R}$ is Baire class one if $f^{-1}(a, b)$ is F_σ for any $a < b$.

Theorem 3.1. *If the range* $g(\mathbb{R})$ *is a discrete set then* g *is* k*-continuous if and only if it is Baire class one.*

Proof: Suppose $g(\mathbb{R})$ is discrete. We only need to verify the sufficiency.
Assume that g is Baire class one and let $g(\mathbb{R}) = \{r_1, r_2, \ldots\}$. Since $g(\mathbb{R})$ is discrete then for every i we can find an open interval U_{r_i} of r_i such that $U_{r_i} \cap g(\mathbb{R}) = \{r_i\}$. It follows that for each i, $F_i = g^{-1}\{r_i\} = g^{-1}(U_{r_i})$ is an F_σ set. Furthermore, $\mathbb{R} = \bigcup_{i=1}^{+\infty} F_i$, $F_i \cap F_j = \phi$ for $i \neq j$ and $g|_{F_i}$ is continuous for each i. By Corollary 2.4, g is k-continuous. □

Remark 1: Theorem 3.1 is no longer true if the range of g is a non-discrete countable set. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be the set of all rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if} \quad x = r_n; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that f is Baire class one and the range of f is the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Furthermore, the range of f is countable but not discrete since 0 is a limit point of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. One can show that f is not k-continuous.

Observe that χ_A is Baire class one if and only if A is of type $G_\delta \cap F_\sigma$. So we have

Corollary 3.2. *The characteristic function χ_A of a set A is k-continuous if and only if A is of type $G_\delta \cap F_\sigma$.*

The next lemma is used in the proof of Theorem 3.4.

Lemma 3.3. *The sum(product) of two k-continuous functions is k-continuous.*

Proof: We will do the proof for the sum. The proof for the product is similar. Let f and g be k-continuous functions. By Theorem 2.7 and Theorem 2.5 f and g are piecewise continuous. By definition there exist increasing sequences of closed sets $\{E_i\}$ and $\{F_i\}$ for f and g, respectively such that $\mathbb{R} = \bigcup_{i=1}^{+\infty} E_i = \bigcup_{i=1}^{+\infty} F_i$ and $f|_{E_i}$ and $g|_{F_i}$ are continuous for each i. By Lemma 1 of [4], we can find disjoint sequences of F_σ sets $\{H_i\}$ and $\{J_i\}$ such that $\mathbb{R} = \bigcup_{i=1}^{+\infty} H_i = \bigcup_{i=1}^{+\infty} J_i$, $J_i \cap J_j = \emptyset$, $H_i \cap H_j = \emptyset$ for $i \neq j$ and $H_i \subseteq E_i$ and $J_i \subseteq F_i$ for all i. Now, $\mathbb{R} = \bigcup\{H_i \cap J_k : i, k \in \mathbb{N}\}$, different $H_i \cap J_k$ are disjoint and $(f+g)|_{H_i \cap J_k}$ is continuous for any i, k. Applying Corollary 2.4, we conclude that $f+g$ is k-continuous. □

Theorem 3.4. *If f and g have different values at only a discrete set of points then f is k-continuous if and only if g is k-continuous.*

Proof: Suppose $A = \{x : f(x) \neq g(x)\}$ is discrete and let f be a k-continuous function. Let $h(x) = g(x) - f(x)$. First, we need to show that h is k-continuous. Let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$. Since A is discrete then A is of type $G_\delta \cap F_\sigma$. It will also follow that $G = \mathbb{R} - A$ is of type F_σ. Furthermore, $h|_{\{x_i\}}$ is continuous for each i and $h|_G$ is continuous as well. Since $\mathbb{R} = G \cup \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \cup \cdots$ then by Corollary 2.4, h is k-continuous. Since the sum of two k-continuous functions is again a k-continuous function and $g(x) = f(x) + h(x)$ then g is k-continuous. The other direction can be proved similarly. □

Remark 2: Again, Theorem 3.4 can no longer be improved in the sense that f and g cannot have different values on a non-discrete countable set. Consider the Cantor set K. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1, & x \in K; \\ 0, & \text{otherwise} \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is a two-sided limit point of } K; \\ 0, & \text{otherwise.} \end{cases}$$

Since K is closed then K is of type $G_\delta \cap F_\sigma$. By Corollary 3.2, f is k-continuous. On the other hand, we can show that $g|_K$ has no points of continuity on K and hence g is not Baire class one. Observe further that f and g agree except on a countable set. Clearly, this set is not discrete. All these show that we cannot replace the condition of discreteness in Theorem 3.4 by countability.

Theorem 3.5. *Every function g with a discrete set of discontinuities on $\mathbb{R}$ is k-continuous.*

Proof: Let $D_g = \{r_1, r_2, \cdots, r_n, \cdots\}$ be the set of points of discontinuity of g. Since D_g is of type $G_\delta \cap F_\sigma$ then $\mathbb{R} - D_g$ is F_σ. Now, $g|_{\{r_i\}}$ is continuous for each i, $g|_{\mathbb{R}-D_g}$ is continuous on $\mathbb{R} - D_g$ and $\mathbb{R} = (\mathbb{R} - D_g) \cup D_g$. By Corollary 2.4, g is k-continuous on $\mathbb{R}$. □

Remark 3: The above theorem is false if the set of discontinuities of g is a non-discrete countable set. Here's a counterexample. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be the set of all rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{if} \quad x = r_n; \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that f is discontinuous on the set of rational numbers $\mathbb{Q}$ but continuous on the set of irrational numbers. Hence, f is Baire class one.
Consider another function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \begin{cases} 1 & \text{if} \quad x = \frac{1}{n}, \quad n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Again, g is Baire class one. However,

$$(g \circ f)(x) = \begin{cases} 1 & \text{if} \quad x = r_n; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $g \circ f$ is the well-known Dirichlet function which is not a Baire class one function. Hence, all these show that f with countable discontinuity $\mathbb{Q}$ fails to be a right B_1 compositor and hence not a k-continuous function.

Remark 4: The converse of Theorem 3.5 is not true. Let K be the Cantor set. Consider the characteristic function χ_K. We have already shown above that χ_K is k-continuous. One can show that χ_K is discontinuous on K and continuous on $\mathbb{R} - K$. Therefore, we have exhibited a k-continuous function which has a non-discrete set of discontinuity.

It was shown in [4, Example 2] that the class of k-continuous functions is not closed under uniform limit. However, in that example, the sequence of k-continuous functions converges to a Baire class one function. Theorem 3.6 characterizes Baire class one functions in terms of k-continuous functions.

Theorem 3.6. *Let $g : [0,1] \to \mathbb{R}$. Then g is Baire class one if and only if there exists a sequence of k-continuous functions on $[0,1]$ that converges uniformly to g.*

Proof: The sufficiency follows from the facts that the uniform limit of a sequence of Baire class one functions is Baire class one and that every k-continuous function is Baire class one. Now we prove the necessity. Suppose g is Baire class one. By [3, Proposition 2], for each n there is a Baire class one g_n such that $|g - g_n| < \frac{1}{n}$ and $g_n([0,1])$ is discrete. Obviously $\{g_n\}$ converges uniformly to g. Also each g_n is k-continuous by Theorem 3.1. □

It is also natural to ask how a left B_1 compositor is characterized. This is another open problem. It was stated in [4, page 550] that a function with a finite set of discontinuity points is a left B_1 compositor. The following counterexample shows that this is not true.

Example: Let $q_1, q_2, \cdots, q_n, \cdots$ be an enumeration of the set $\mathbb{Q}$ of rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} \frac{1}{k}, & x = q_k; \\ 0, & \text{otherwise.} \end{cases}$$

and let

$$g(x) = \begin{cases} x+1, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

The function f is well-known to be continuous at the set of irrational numbers and discontinuous at the set of rational numbers. Hence, f is a Baire class one function. We will show that $g \circ f$ is not Baire class one so that g is not a left B_1 compositor. Now,

$$(g \circ f)(x) = \begin{cases} 1 + \frac{1}{k}, & x = q_k; \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to show that $g \circ f$ is discontinuous everywhere and so it is not Baire class one. Therefore g, which has a single discontinuity at $x = 0$, is not a left B_1 compositor. □

The counterexample above suggests that a left B_1 compositor must have a very nice property. It is conjectured that the only left B_1 compositors are the continuous functions.

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