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REGULAR OPEN SETS ON THE INTUITIONISTIC FUZZY TOPOLOGICAL SPACES IN ŠOSTAK'S SENCE

T. Menahadevi[1], P. Maragatha Meenakshi[2] and Neelamegarajan Rajesh[3]

[1]**Department of Mathematics, Vivekanandha College of Arts and Science for Women, Tiruchengode-637 205, Tamilnadu, India.**
[2]**Department of Mathematics, Periyar E.V.R College, Tiruchirappalli 624024, (affliated to Bharathidasan University), Tamilnadu, India.**
[3]**Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamilnadu, India.**

Abstract. The aim of this paper is to introduce and study the concepts of intuitionistic r-fuzzy regular open sets and their related notions in intuitionistic fuzzy topological spaces.

Key words and Phrases: Smooth fuzzy topological spaces, intuitionistic r-fuzzy regular open set, intuitionistic r-fuzzy regular closed set.

1. INTRODUCTION

After the introduction of fuzzy sets by Zadeh [17], there have been a number of generalizations of this fundamental concept. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [2, 4, 5]. The idea of intuitionistic fuzzy sets suggested by Atanassov [3] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [9, 10, 11]. Using the notion of intuitionistic fuzzy sets, Coker [7] introduced the notion of intuitionistic

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smooth fuzzy topological spaces. Samanta and Mondal [14, 15] introduced the definitions of the intuitionistic smooth fuzzy topological space in Sŏstak sense. The aim of this paper is to introduce and study the concepts of intuitionistic r-fuzzy regular open sets and their related notions in intuitionistic fuzzy topological spaces.

2. PRELIMINARIES

Throughout this paper, let X be a non-empty set, I the unit interval $[0,1]$, and $I_0 = (0,1]$. The family of all intuitionistic fuzzy sets on X is denoted by I^X. By $\overline{0}$ and $\overline{1}$, we denote the smallest and the greatest intuitionistic fuzzy sets on X. For an intuitionistic fuzzy set $\lambda \in I^X, \overline{1} - \lambda$ denotes its complement.

Definition 2.1. *Let X be a nonempty set and I the closed interval $[0,1]$. An intuitionistic fuzzy set A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, where the function $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every fuzzy set A on a nonempty set X is an intuitionistic fuzzy set of the following form $A = \{\langle x, \mu_A(x), \overline{1} - \mu_A(x)\rangle : x \in X\}$.*

Definition 2.2. *If $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$ are intuitionistic fuzzy sets of X. Then*

(1) $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$;
(2) $\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$
(3) $A \wedge B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$.
(4) $A \vee B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$.

We will use the notation $A = \langle x, \mu_A, \gamma_A\rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$. A constant fuzzy set α taking value $\alpha \in [0,1]$ will be denoted by $\underline{\alpha}$. The intuitionistic fuzzy sets $\overline{0}$ and $\overline{1}$ are defined by $\overline{0} = \{\langle x, \underline{0}, \underline{1}\rangle : x \in X\}$ and $\overline{1} = \{\langle x, \underline{1}, \underline{0}\rangle : x \in X\}$. Let f be a function from an ordinary set X into an ordinary set Y. If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ is an intuitionistic fuzzy set in Y, then the inverse image of B under f is intuitionistic fuzzy set defined by $f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$. The image of intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ under f is an intuitionistic fuzzy set defined by $f(A) = \{\langle y, f(\mu_A)(y), f(\gamma_A)(y)\rangle : y \in Y\}$ where

$$f(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(\mu_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x), & f^{-1}(y) \neq 0, \\ 1 & \text{otherwise} \end{cases}$$

for each $y \in Y$.

Definition 2.3. [14, 15] *An intuitionistic gradation of openness on X is an ordered pair $(\tau, \tau^\star)$ of functions from I^X to I such that*

(1) $\tau(\lambda) + \tau^\star(\lambda) \leq 1$ for all $\lambda \in I^X$,
(2) $\tau(0) = \tau(1) = 1$, $\tau^\star(0) = \tau^\star(1) = 0$
(3) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.
(4) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^\star(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Gamma$.

The triplet $(X, \tau, \tau^\star)$ is called an intuitionistic smooth fuzzy topological space.

Definition 2.4. *An intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X, \tau, \tau^\star)$ is called an intuitionistic r-fuzzy open if $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq 1 - r$ for each $r \in I_0$, λ is called an intuitionistic r-fuzzy closed if and only if $\overline{1} - \lambda$ is an intuitionistic r-fuzzy open set. We set $\tau_r = \{\lambda \in I^X : \tau(\lambda) \geq r, \tau^\star(\lambda) \leq \overline{1} - r\}$.*

Theorem 2.5. [14, 15] *Let $(X, \tau, \tau^\star)$ be an intuitionistic smooth fuzzy topological space. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operators $\mathrm{Cl}, \mathrm{Int} : I^X \times I_0 \to I^X$ as follows*

$$\mathrm{Cl}(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\underline{1} - \mu) \geq r, \tau^\star(\underline{1} - \mu) \leq \overline{1} - r\},$$

$$\mathrm{Int}(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq \overline{1} - r\}.$$

Definition 2.6. [8] *Let X be a nonempty set and $c \in X$ a fixed element in X. If $a \in (0, 1]$ and $b \in [0, 1)$ are two fixed real numbers such that $a + b \leq 1$, then the intuitionistic fuzzy set $c(a, b) = \langle x, c_a, \overline{1} - c_{1-b}\rangle$ is called an intuitionistic fuzzy point in X, where a denotes the degree of membership of $c(a, b)$, b the degree of nonmembership of $c(a, b)$, and $c \in X$ the support of $c(a, b)$.*

Definition 2.7. [8] *Let $c(a, b)$ be an intuitionistic fuzzy point in X and $A = \langle x, \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set in X. Suppose that $a, b \in (0, 1)$. $c(a, b)$ is said to be properly contained in A ($c(a, b) \in U$, for short) if and only if $a < \mu_A(c)$ and $b > \gamma_A(c)$.*

Definition 2.8. [8]

(1) An intuitionistic fuzzy point $c(a,b)$ in X is said to be quasi-coincident with the intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A \rangle$ denoted by $c(a,b)\ q\ A$, if and only if $a < \mu_A(c)$ or $b > \gamma_A(c)$.

(2) Let $A = \langle x, \mu_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \gamma_B \rangle$ are two intuitionistic fuzzy sets in X. Then A and B are said to be quasi-coincident, denoted by $A\ q\ B$ if and only if there exists an element $x \in X$ such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$.

The expression not quasi-coincident will be abbreviated as $\overline{q}$.

Proposition 2.9. [8] *Let U and V be two intuitionistic fuzzy sets and $c(a,b)$ an intuitionistic fuzzy point in X. Then*

(1) $U\ \overline{q}\ \overline{1} - V$ if and only if $U \leq V$,
(2) $U\ q\ V$ if and only if $U \not\leq \overline{1} - V$,
(3) $c(a,b) \leq U$ if and only if $c(a,b)\ \overline{q}\ \overline{1} - U$,
(4) $c(a,b)\ q\ U$ if and only if $c(a,b) \not\leq \overline{1} - U$.

Definition 2.10. [8] *Let $f : X \to Y$ be a function and $c(a,b)$ an intuitionistic fuzzy point in X. Then the image of $c(a,b)$ under f, denoted by $f(c(a,b))$, is defined by $f(c(a,b)) = \langle y, f(c)_a, \overline{1} - f(c)_{1-b} \rangle$.*

Proposition 2.11. [13] *Let $f : X \to Y$ be a function and $c(a,b)$ an intuitionistic fuzzy point in X.*

(1) If $f(c(a,b))\ q\ V$ holds for an intuitionistic fuzzy set V of Y, then we have $c(a,b)\ q\ f^{-1}(V)$.
(2) If $c(a,b)\ q\ U$ holds for an intuitionistic fuzzy set U of X, then we have $f(c(a,b))\ q\ f(U)$.

Definition 2.12. *Let (X, τ) be an intuitionistic fuzzy topological space on X and $c(a,b)$ an intuitionistic fuzzy point in X. An intuitionistic fuzzy set A is called q-neighbourhood of $c(a,b)$, denoted by $\mathcal{N}_q(c(a,b))$, if there exists an intuitionistic fuzzy open set U in X such that $c(a,b)\ q\ U$ and $U \leq A$.*

Definition 2.13. [1, 12] *An intuitionistic fuzzy set λ of an intuitionistic fuzzy topological space (X, τ) is said to be an:*

(1) intuitionistic r-fuzzy strongly semiopen if $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$,
(2) intuitionistic r-fuzzy semiopen if $\lambda \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$,
(3) intuitionistic r-fuzzy preopen if $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$,
(4) intuitionistic r-fuzzy semi-preopen if $\lambda \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r)$.

The complement of an intuitionistic r-fuzzy strongly semiopen (resp. semiopen, preopen, semi-preopen) set is called an intuitionistic r-fuzzy strongly semiclosed (resp. semiclosed, preclosed, semi-preclosed) set.

3. INTUTIONISTIC SMOOTH FUZZY REGULAR OPEN SETS

Definition 3.1. *An intuitionistic fuzzy subset λ of an intuitionistic smooth fuzzy topological space $(X, \tau, \tau^\star)$ is said to be intuitionistic r-fuzzy regular open set if* $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) = \lambda$. *We call an intuitionistic fuzzy subset λ of X is intuitionistic r-fuzzy regular closed if its complement is intuitionistic r-fuzzy regular open.*

Theorem 3.2. *Let $\lambda \in I^X$, then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3),

(1) λ *is intuitionistic r-fuzzy clopen (= intuitionistic r-fuzzy open and intuitionistic r-fuzzy closed).*
(2) $\lambda = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$.
(3) $\overline{1} - \lambda$ *is intuitionistic r-fuzzy regular open.*

Proof. (1) $\Rightarrow$ (2): This is obvious.
(2) $\Rightarrow$ (3): By (2), $\overline{1} - \lambda = \overline{1} - \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) = \mathrm{Int}(\overline{1} - \mathrm{Int}(\lambda, r), r) = \mathrm{Int}(\mathrm{Cl}(\overline{1} - \lambda, r), r)$, and hence $\overline{1} - \lambda$ is intuitionistic fuzzy r-regular open set. □

Lemma 3.3. *For an intuitionistic smooth fuzzy topological space $(X, \tau, \tau^\star)$, we have*

(1) $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$.
(2) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$.

Proof. (1). Since $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$ holds for any intuitionistic fuzzy set, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r) \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$. On the other hand, since $\mathrm{Int}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$ implies that $\mathrm{Int}(\lambda, r) \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$ and $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r)$. Then $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$.
(2). Similar to (1). □

Lemma 3.4. *Let λ and μ be intuitionistic fuzzy subsets of an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$. Then the following properties hold:*

(1) $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ *is intuitionistic r-fuzzy regular open.*
(2) If λ and μ are intuitionistic r-fuzzy regular open, then $\lambda \wedge \mu$ is also intuitionistic r-fuzzy regular open.
(3) If λ and μ are intuitionistic r-fuzzy regular closed, then $\lambda \vee \mu$ is also intuitionistic r-fuzzy regular closed.

Proof. (1). Follows from Lemma 3.3.
(2). Let λ and μ be intuitionistic r-fuzzy regular open sets of$X$. Then we have $\lambda \wedge \mu = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \wedge \mathrm{Int}(\mathrm{Cl}(\mu, r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r) \wedge \mathrm{Cl}(\mu, r), r) \geq \mathrm{Int}(\mathrm{Cl}(\lambda \wedge \mu, r), r) \geq \mathrm{Int}(\lambda \wedge \mu, r) = \lambda \wedge \mu$. Then $\lambda \wedge \mu = \mathrm{Int}(\mathrm{Cl}(\lambda \wedge \mu, r), r)$. Hence $\lambda \wedge \mu$ is

intuitionistic r-fuzzy regular open.
(3). Let λ and μ be any two intuitionistic r-fuzzy regular closed sets in X. Then λ and μ are intuitionistic r-fuzzy closed sets and hence $\overline{1}-\lambda$ and $\overline{1}-\mu$ are intuitionistic r-fuzzy open sets. Then $\overline{1}-(\lambda \vee \mu)$ is an intuitionistic r-fuzzy open set. Thus $\lambda \vee \mu$ is an intuitionistic r-fuzzy closed set. Since $\text{Int}(\lambda \wedge \mu, r) \leq \lambda \wedge \mu$, $\text{Cl}(\text{Int}(\lambda \wedge \mu, r), r) \geq \text{Cl}(\lambda \wedge \mu, r) = \lambda \wedge \mu$. Now, $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$ imply $\text{Cl}(\text{Int}(\lambda \wedge \mu, r), r) \leq \text{Cl}(\text{Int}(\lambda, r), r) = \lambda$ and $\text{Cl}(\text{Int}(\lambda \wedge \mu, r), r) \leq \text{Cl}(\text{Int}(\mu, r), r) = \mu$. Thus $\text{Cl}(\text{Int}(\lambda \wedge \mu, r), r) \leq \lambda \wedge \mu$. Therefore $\text{Cl}(\text{Int}(\lambda \wedge \mu, r), r) = \lambda \wedge \mu$ and hence $\lambda \wedge \mu$ is an intuitionistic r-fuzzy regular closed set. □

Remark 3.5. *The following example shows that the union of any two intuitionistic r-fuzzy regular open sets need not be intuitionistic r-fuzzy regular open.*

Example 3.6. *Let $X = [0, 1]$. The intuitionistic fuzzy subsets $\lambda_1, \lambda_2 \in I^X$ are defined as follows:*

$$\lambda_1 = \{\langle x, \mu_{\lambda_1}(x), \gamma_{\lambda_1}(x)\rangle : x \in X\},$$
$$\lambda_2 = \{\langle x, \mu_{\lambda_2}(x), \gamma_{\lambda_2}(x)\rangle : x \in X\},$$
$$\mu_{\lambda_1}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$
$$\gamma_{\lambda_1}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2x+2 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$
$$\mu_{\lambda_2}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4} \\ -4x+2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$
$$\gamma_{\lambda_2}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4} \\ 4x-1 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We define $\tau, \tau^\star : I^X \to I$ is defined as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \{\lambda_1, \lambda_2, \lambda_1 \vee \lambda_2\} \\ 0 & \text{otherwise} \end{cases}$$
$$\tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda \in \{\lambda_1, \lambda_2, \lambda_1 \vee \lambda_2\} \\ 1 & \text{otherwise.} \end{cases}$$

Since $\text{Int}(\text{Cl}(\lambda_1, \frac{1}{2}), \frac{1}{2}) = \text{Int}(\overline{1} - \lambda_2, \frac{1}{2}) = \lambda_1$ *and* $\text{Int}(\text{Cl}(\lambda_2, \frac{1}{2}), \frac{1}{2}) = \text{Int}(\overline{1} - \lambda_1, \frac{1}{2}) = \lambda_2$, *$\lambda_1$ and λ_2 are intuitionistic $\frac{1}{2}$-fuzzy regular open sets. Also* $\text{Int}(\text{Cl}(\lambda_1 \vee \lambda_2, \frac{1}{2}), \frac{1}{2}) = \overline{1}$. *Hence $\lambda_1 \vee \lambda_2$ is not intuitionistic $\frac{1}{2}$-fuzzy regular open.*

Theorem 3.7. *The following statements are true:*

(1) An intuitionistic r-fuzzy open set λ is intuitionistic r-fuzzy regular open if, and only if $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \lambda$ *holds.*
(2) For every intuitionistic r-fuzzy closed set λ, $\mathrm{Int}(\lambda, r)$ *is intuitionistic r-fuzzy regular open.*
(3) For every intuitionistic r-fuzzy open set λ, $\mathrm{Cl}(\lambda, r)$ *is intuitionistic r-fuzzy regular closed.*

Proof. (1). It suffices to prove that every intuitionistic r-fuzzy open set λ satisfying $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \lambda$ is intuitionistic r-fuzzy regular open. Since $\lambda \leq \mathrm{Cl}(\lambda, r)$, $\lambda = \mathrm{Int}(\lambda, r) \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Hence $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$.
(2). If λ is an intuitionistic r-fuzzy closed set, then $\mathrm{Int}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \mathrm{Cl}(\lambda, r) = \lambda$. Hence $\mathrm{Int}(\lambda, r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) \leq \mathrm{Int}(\lambda, r)$. So, $\mathrm{Int}(\lambda, r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$ holds. That is, $\mathrm{Int}(\lambda, r)$ is intuitionistic r-fuzzy regular open.
(3). Let λ be an intuitionistic r-fuzzy open set of X. Then $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \mathrm{Cl}(\lambda, r)$ implies that $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) \leq \mathrm{Cl}(\mathrm{Cl}(\lambda, r), r) = \mathrm{Cl}(\lambda, r)$. Since λ is intuitionistic r-fuzzy open, $\lambda = \mathrm{Int}(\lambda, r)$. Also since $\lambda \leq \mathrm{Cl}(\lambda, r)$, $\lambda = \mathrm{Int}(\lambda, r) \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Thus $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r)$. Hence $\mathrm{Cl}(\lambda, r)$ is an intuitionistic r-fuzzy regular closed set. □

Theorem 3.8. *Let λ be any intuitionistic fuzzy subset of an intuitionistic smooth fuzzy topological space $(X, \tau, \tau^\star)$. Then*

(1) λ is intuitionistic r-fuzzy clopen if, and only if it is intuitionistic r-fuzzy regular open and intuitionistic r-fuzzy regular closed.
(2) λ is intuitionistic r-fuzzy regular open if, and only if it is intuitionistic r-fuzzy preopen and intuitionistic fuzzy r-semiclosed.
(3) λ is intuitionistic r-fuzzy regular closed if, and only if it is intuitionistic r-fuzzy semiopen and intuitionistic r-fuzzy preclosed.

Proof. Follows from their respective definitions. □

Theorem 3.9. *Let λ be any intuitionistic fuzzy subset of an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$. Then the following statements are equivalent:*

(1) λ is intuitionistic r-fuzzy regular open.
(2) λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy semiclosed.
(3) λ is intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy semiclosed.
(4) λ is intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy semiclosed.
(5) λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy semi-preclosed.
(6) λ is intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy semi-preclosed.

Proof. (1) ⇒ (2): Let λ be an intuitionistic r-fuzzy regular open set. But every intuitionistic r-fuzzy regular open set is intuitionistic r-fuzzy open and every intuitionistic r-fuzzy regular open set is intuitionistic r-fuzzy semiclosed. Then λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy semiclosed.
(2) ⇒ (3): Let λ be an intuitionistic r-fuzzy open and intuitionistic r-fuzzy semiclosed set. Also every intuitionistic r-fuzzy open set is intuitionistic r-fuzzy strongly semiopen. Then λ is intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy semiclosed.
(3) ⇒ (4): Let λ be an intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy semiclosed set. Also every intuitionistic r-fuzzy strongly semiopen set is intuitionistic r-fuzzy preopen. Then λ is intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy semiclosed.
(4) ⇒ (5): Let λ be an intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy semiclosed set. Then $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ and $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \lambda$. Then $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Hence λ is intuitionistic r-fuzzy regular open set and hence it is intuitionistic r-fuzzy open. Since every intuitionistic r-fuzzy semiclosed set is intuitionistic r-fuzzy semi-preclosed. Then λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy semi-preclosed.
(5) ⇒ (6): It is obvious since every intuitionistic r-fuzzy open set is intuitionistic r-fuzzy strongly semiopen.
(6) ⇒ (1): Let λ be an intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy semi-preclosed set, $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$ and $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) \leq \lambda$. It follows that $\mathrm{Int}(\lambda, r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) = \lambda$ and $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) = \lambda$. Hence λ is an intuitionistic r-fuzzy regular open set. □

Corollary 3.10. *Let λ be any intuitionistic fuzzy subset of an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$. Then the following statements are equivalent:*

(1) *λ is intuitionistic r-fuzzy regular closed.*
(2) *λ is intuitionistic r-fuzzy closed and intuitionistic r-fuzzy semiopen.*
(3) *λ is intuitionistic r-fuzzy strongly semiclosed and intuitionistic r-fuzzy semiopen.*
(4) *λ is intuitionistic r-fuzzy preclosed and intuitionistic r-fuzzy semiopen.*
(5) *λ is intuitionistic r-fuzzy closed and intuitionistic r-fuzzy semi-preopen.*
(6) *λ is intuitionistic r-fuzzy strongly semiclosed and intuitionistic r-fuzzy semi-preopen.*

Proof. Similar to Theorem 3.9. □

Theorem 3.11. *Let λ be any intuitionistic fuzzy subset of an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$. Then the following statements are equivalent:*

(1) *λ is intuitionistic r-fuzzy clopen.*
(2) *λ is intuitionistic r-fuzzy regular open and intuitionistic r-fuzzy regular closed.*
(3) *λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy strongly semiclosed.*

(4) λ *is intuitionistic* r*-fuzzy open and intuitionistic* r*-fuzzy preclosed.*
(5) λ *is intuitionistic* r*-fuzzy strongly semiopen and intuitionistic* r*-fuzzy preclosed.*
(6) λ *is intuitionistic* r*-fuzzy strongly semiopen and intuitionistic* r*-fuzzy closed.*
(7) λ *is intuitionistic* r*-fuzzy preopen and intuitionistic* r*-fuzzy closed.*
(8) λ *is intuitionistic* r*-fuzzy strongly semiopen and intuitionistic* r*-fuzzy preopen.*

Proof. (1) $\Rightarrow$ (2): Let λ be any intuitionistic r-fuzzy clopen set of X. Then $\lambda = \mathrm{Cl}(\mathrm{Int}(\lambda))$ and $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda))$. Hence λ is intuitionistic r-fuzzy regular open and intuitionistic r-fuzzy regular closed.
(2) $\Rightarrow$ (3): Let λ be any intuitionistic r-fuzzy regular open and intuitionistic r-fuzzy regular closed set. Since every intuitionistic r-fuzzy regular open set is intuitionistic r-fuzzy open and every intuitionistic r-fuzzy regular closed set is intuitionistic r-fuzzy strongly semiclosed, λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy strongly semiclosed.
(3) $\Rightarrow$ (4): Let λ be any intuitionistic r-fuzzy open and intuitionistic r-fuzzy strongly semiclosed set. Since every intuitionistic r-fuzzy strongly semiclosed set is intuitionistic r-fuzzy preclosed, λ is intuitionistic r-fuzzy open and intuitionistic r-fuzzy preclosed.
(4) $\Rightarrow$ (5): Let λ be any intuitionistic r-fuzzy open and intuitionistic r-fuzzy preclosed set. Since every intuitionistic r-fuzzy open set is intuitionistic r-fuzzy strongly semiopen, λ is intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy preclosed.
(5) $\Rightarrow$ (6): Let λ be an intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy preclosed set. Then $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$ and $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) \leq \lambda$. Then $\lambda = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r)$ and hence $\mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r)$. Then $\mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$. Since $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \lambda$, then $\mathrm{Cl}(\lambda, r) \leq \lambda$. But in general $\lambda \leq \mathrm{Cl}(\lambda, r)$. Then $\mathrm{Cl}(\lambda, r) = \lambda$. Hence λ is intuitionistic r-fuzzy closed.
(6) $\Rightarrow$ (7): Let λ be any intuitionistic r-fuzzy strongly semiopen and intuitionistic r-fuzzy closed. Since every intuitionistic r-fuzzy strongly semiopen set is intuitionistic r-fuzzy preopen, λ is intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy closed.
(7) $\Rightarrow$ (8): Let λ be any intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy closed. Then $\lambda = \mathrm{Cl}(\lambda)$ and $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda))$. Hence $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda)) = \mathrm{Int}(\lambda)$; hence λ is intuitionistic r-fuzzy open. Since every intuitionistic r-fuzzy open set is intuitionistic r-fuzzy strongly semiopen, λ is intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy strongly semiopen.
(8) $\Rightarrow$ (1): Let λ be an intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy strongly semiopen set. Then $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) \leq \lambda$. Hence $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) \leq \lambda$ and hence $\mathrm{Cl}(\lambda, r) \leq \lambda$. Hence $\mathrm{Cl}(\lambda, r) = \lambda$; λ is intuitionistic r-fuzzy closed. Since $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) \leq \lambda$, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r), r) \leq \mathrm{Int}(\lambda, r)$. Then $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r) \leq \mathrm{Int}(\lambda, r)$ and hence $\lambda \leq \mathrm{Int}(\lambda, r)$. But in general $\mathrm{Int}(\lambda, r) \leq \lambda$. Then $\mathrm{Int}(\lambda, r) = \lambda$. Hence λ is intuitionistic r-fuzzy open. Therefore, λ is intuitionistic r-fuzzy clopen. □

Definition 3.12. *An intuitionistic fuzzy topological space* $(X, \tau, \tau^\star)$ *is said to be intuitionistic* r*-fuzzy extremally disconnected if* $\mathrm{Cl}(\lambda, r) \in \tau_r$ *for every* $\lambda \in \tau_r$*.*

Theorem 3.13. *For an intuitionistic fuzzy topological space* $(X, \tau, \tau^\star)$*, the following properties are equivalent:*

(1) X *is intuitionistic* r*-fuzzy extremally disconnected.*
(2) Every intuitionistic r*-fuzzy regular open subset of* X *is intuitionistic* r*-fuzzy closed.*
(3) Every intuitionistic r*-fuzzy regular closed subset of* X *is intuitionistic* r*-fuzzy open.*

Proof. (1) $\Rightarrow$ (2): Let X be an intuitionistic r-fuzzy extremally disconnected space. Let λ be an intuitionistic r-fuzzy regular open set of X. Then $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Since λ is an intuitionistic r-fuzzy open set, then $\mathrm{Cl}(\lambda, r) \in \tau_r$. Thus, $\lambda = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r) = \mathrm{Cl}(\lambda, r)$; hence λ is intuitionistic r-fuzzy closed.
(2) $\Rightarrow$ (3): Suppose that every intuitionistic r-fuzzy regular open subset of X is intuitionistic r-fuzzy closed in X. Let $\lambda \in \tau_r$. Since $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ is intuitionistic r-fuzzy regular open, then it is intuitionistic r-fuzzy closed in X. Then $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ since $A \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Thus, $\mathrm{Cl}(\lambda, r) \in \tau_r$; hence X is intuitionistic r-fuzzy extremally disconnected.
(2) $\Leftrightarrow$ (3): Obvious. □

Theorem 3.14. *An intuitionistic fuzzy topological space* $(X, \tau, \tau^\star)$ *is intuitionistic* r*-fuzzy extremally disconnected if and only if intuitionistic* r*-fuzzy regular open sets coincide with intuitionistic* r*-fuzzy regular closed sets.*

Proof. Suppose λ is an intuitionistic r-fuzzy regular open subset of X. Since intuitionistic r-fuzzy regular open sets are intuitionistic r-fuzzy open, by (1), $\lambda = \mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$ and so λ is intuitionistic r-fuzzy regular closed. If λ is intuitionistic r-fuzzy regular closed, $\lambda = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) = \mathrm{Int}(\lambda, r)$ so λ is intuitionistic r-fuzzy open. It follows that $\lambda = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Hence λ is intuitionistic r-fuzzy regular open. Conversely, let λ be an intuitionistic r-fuzzy open subset of X. Then we have $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ is intuitionistic r-fuzzy regular open and so it is intuitionistic r-fuzzy regular closed. Hence $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ which implies that $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Then $\mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ and so $\mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$. Hence $\mathrm{Cl}(\lambda, r)$ is intuitionistic r-fuzzy open. Hence X is intuitionistic r-fuzzy extremally disconnected. □

Theorem 3.15. *An intuitionistic fuzzy topological space* $(X, \tau, \tau^\star)$ *is intuitionistic* r*-fuzzy extremally disconnected if and only if every intuitionistic* r*-fuzzy regular closed sets is intuitionistic* r*-fuzzy preopen.*

Proof. If λ is intuitionistic r-fuzzy open, then $\mathrm{Cl}(\mathrm{Int}(\lambda, r), r)$ is intuitionistic r-fuzzy regular closed and so it is intuitionistic r-fuzzy preopen. Therefore, $\mathrm{Cl}(\lambda, r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \leq \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r), r) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Thus, $\mathrm{Cl}(\lambda, r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ which implies that $\mathrm{Cl}(\lambda, r)$ is intuitionistic r-fuzzy open. Hence X is intuitionistic r-fuzzy extremally disconnected. The converse follows from from the fact that every intuitionistic r-fuzzy regular open sets is intuitionistic r-fuzzy preopen set. □

Theorem 3.16. *If $(X, \tau, \tau^\star)$ is an intuitionistic r-fuzzy extremally disconnected space, then the following properties hold:*

(1) $\lambda \wedge \mu$ is intuitionistic r-fuzzy regular closed for all intuitionistic r-fuzzy regular closed subsets of λ and μ of X.
(2) $\lambda \wedge \mu$ is intuitionistic r-fuzzy regular open for all intuitionistic r-fuzzy regular open subsets of λ and μ of X.

Proof. (1). Let λ and μ be intuitionistic r-fuzzy regular closed subsets of X. Since λ and μ are intuitionistic r-fuzzy closed, $\mathrm{Int}(\lambda, r)$ and $\mathrm{Int}(\mu, r)$ are intuitionistic r-fuzzy closed. Hence $\lambda \wedge \mu = \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \wedge \mathrm{Cl}(\mathrm{Int}(\mu, r), r) = \mathrm{Int}(\lambda, r) \wedge \mathrm{Int}(\mu, r) = \mathrm{Int}(\lambda \wedge \mu) \leq \mathrm{Cl}(\mathrm{Int}(\lambda \wedge \mu, r), r)$. On the other hand, $\mathrm{Cl}(\mathrm{Int}(\lambda \wedge \mu, r), r) = \mathrm{Cl}(\mathrm{Int}(\lambda, r) \wedge \mathrm{Int}(\mu, r), r) \leq \mathrm{Cl}(\mathrm{Int}(\lambda, r), r) \wedge \mathrm{Cl}(\mathrm{Int}(\mu, r), r) = \lambda \wedge \mu$. Thus, $\lambda \wedge \mu$ is intuitionistic r-fuzzy regular closed.
(2). It follows from (1). □

Theorem 3.17. *For an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$, the following properties are equivalent:*

(1) X is intuitionistic r-fuzzy extremally disconnected.
(2) $\mathrm{Cl}(\lambda, r) \in \tau_r$ for every intuitionistic r-fuzzy semiopen set λ of X.
(3) $\mathrm{Cl}(\lambda, r) \in \tau_r$ for every intuitionistic r-fuzzy preopen set λ of X.
(4) $\mathrm{Cl}(\lambda, r) \in \tau_r$ for every intuitionistic r-fuzzy regular open set λ of X.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): Let λ be a intuitionistic r-fuzzy semiopen (intuitionistic r-fuzzy preopen) set. Then λ is intuitionistic r-fuzzy semi-preopen; hence $\mathrm{Cl}(\lambda, r) \in \tau_r$.
(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4): Let λ be an intuitionistic r-fuzzy regular open set of X. Then λ is an intuitionistic r-fuzzy semiopen set of X and λ is an intuitionistic r-fuzzy preopen set of X and hence $\mathrm{Cl}(\lambda, r) \in \tau_r$.
(4) $\Rightarrow$ (1): Suppose that the intuitionistic r-fuzzy closure of every intuitionistic r-fuzzy open subset of X is intuitionistic r-fuzzy open. Let $\lambda \in I^X$ be an intuitionistic r-fuzzy open set. Then $\mathrm{Int}(\mathrm{Cl}(\lambda, r))$ is a intuitionistic r-fuzzy open set. Then $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r)$ is intuitionistic r-fuzzy open. We have $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r), r) = \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Thus, $\mathrm{Cl}(\lambda, r) \in \tau_r$; hence X is intuitionistic r-fuzzy extremally disconnected. □

Proposition 3.18. *Let λ be an intuitionistic fuzzy of an intuitionistic fuzzy topological space $(X, \tau, \tau^\star)$. Then*

(1) The intuitionistic r-fuzzy closure of a intuitionistic r-fuzzy open set is intuitionistic r-fuzzy regular closed.

(2) The intuitionistic r-fuzzy interior of a intuitionistic r-fuzzy closed set is intuitionistic r-fuzzy regular open.

Proof. (1). Let λ be an intuitionistic r-fuzzy open set in X. Then we have $\text{Int}(\text{Cl}(\lambda, r), r) \leq \text{Cl}(\lambda, r)$ implies that $\text{Cl}(\text{Int}(\text{Cl}(\lambda, r), r), r) \leq \text{Cl}(\text{Cl}(\lambda, r), r) = \text{Cl}(\lambda, r)$. Since λ is intuitionistic r-fuzzy open, $\lambda = \text{Int}(\lambda, r)$. Also since $\lambda \leq \text{Cl}(\lambda, r)$, $\lambda = \text{Int}(\lambda, r) \leq \text{Int}(\text{Cl}(\lambda, r), r)$. Thus $\text{Cl}(\lambda, r) \leq \text{Cl}(\text{Int}(\text{Cl}(\lambda, r), r), r)$. Hence $\text{Cl}(\lambda, r) = \text{Cl}(\text{Int}(\text{Cl}(\lambda, r), r), r)$ and hence $\text{Cl}(\lambda, r)$ is an intuitionistic r-fuzzy regular closed set.
(2). Let λ be an intuitionistic r-fuzzy closed set in X. Then $\text{Cl}(\text{Int}(\lambda, r), r) \geq \text{Int}(\lambda, r)$ implies that $\text{Int}(\text{Cl}(\text{Int}(\lambda, r), r), r) \geq \text{Int}(\text{Int}(\lambda, r), r) = \text{Int}(\lambda, r)$. Since λ is an intuitionistic r-fuzzy closed set, $\lambda = \text{Cl}(\lambda, r)$. Also since $\lambda \geq \text{Int}(\lambda, r)$, $\lambda = \text{Cl}(\lambda, r) \geq \text{Cl}(\text{Int}(\lambda, r), r)$. Thus $\text{Int}(\lambda, r) \leq \text{Int}(\text{Cl}(\text{Int}(\lambda, r), r), r)$. Hence $\text{Int}(\lambda, r) = \text{Int}(\text{Cl}(\text{Int}(\lambda, r), r), r)$ and hence $\text{Int}(\lambda, r)$ is an intuitionistic r-fuzzy regular open set. □

4. INTUTIONISTIC SMOOTH FUZZY ALMOST CONTINUOUS FUNCTIONS

Definition 4.1. *A function $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ is called an intuitionistic r-fuzzy almost continuous if $f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set of X for each intuitionistic r-fuzzy regular open set μ of Y.*

Remark 4.2. *Every intuitionistic r-fuzzy continuous is intuitionistic r-fuzzy almost continuous function. The following example shows that the converse statement may not be true.*

Example 4.3. *Let $X = \{a, b\}$. Define the intuitionistic fuzzy subset λ as $\lambda = \langle x, (\frac{a}{0.4}, \frac{b}{0.5})\rangle$. Let $\tau, \tau^\star, \sigma, \sigma^\star : I^X \to I$ defined as follows:*

$$\tau(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \tau^\star(\mu) = \begin{cases} 0 & \text{if } \mu = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } \mu = \lambda \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \sigma^\star(\mu) = \begin{cases} 0 & \text{if } \mu = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } \mu = \lambda \\ 1 & \text{otherwise.} \end{cases}$$

Let $r = \frac{1}{2}$. *Then the identity function* $f : (X, \tau, \tau^\star) \to (X, \sigma, \sigma^\star)$ *is an intuitionistic* $\frac{1}{2}$*-fuzzy almost continuous but not intuitionistic* $\frac{1}{2}$*-fuzzy continuous.*

Theorem 4.4. *For a function* $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, *the following statements are equivalent:*

(1) f *is an intuitionistic* r*-fuzzy almost continuous function;*
(2) $f^{-1}(\mu)$ *is an intuitionistic* r*-fuzzy closed set of* X *for each intuitionistic* r*-fuzzy regular closed set* μ *of* Y*;*
(3) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r) \leq f^{-1}(\mu)$ *for each intuitionistic* r*-fuzzy closed set* μ *of* Y*;*
(4) $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu, r), r)), r)$ *for each intuitionistic* r*-fuzzy open set* μ *of* Y*;*
(5) $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)), r)$ *for each intuitionistic* r*-fuzzy strongly semiopen set* μ *of* Y*;*
(6) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mu, r), r), r)), r) \leq f^{-1}(\mu)$ *for each intuitionistic* r*-fuzzy strongly semiclosed set* μ *of* Y*;*
(7) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r) \leq f^{-1}(\mu)$ *for each intuitionistic* r*-fuzzy preclosed set* μ *of* Y*;*
(8) $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu, r), r)), r)$ *for each intuitionistic* r*-fuzzy preopen set* μ *of* Y*;*
(9) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mu, r), r), r)), r) \leq f^{-1}(\mathrm{Cl}(\mu, r))$ *for each intuitionistic fuzzy set* μ *of* Y*;*
(10) $f^{-1}(\mathrm{Int}(\mu, r)) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)), r)$ *for each intuitionistic fuzzy set* μ *of* Y.

Proof. (1) $\Rightarrow$ (2): Let μ be any intuitionistic r-fuzzy regular closed set of Y. Then $\overline{1} - \mu$ is an intuitionistic r-fuzzy open set of Y. By assumption, $f^{-1}(\overline{1} - \mu)$ is an intuitionistic r-fuzzy open set of X. From $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$ it follows that $f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set of X.
(2) $\Rightarrow$ (3): Let μ be any intuitionistic r-fuzzy closed set of Y. Then $\mathrm{Cl}(\mathrm{Int}(\mu, r), r) \leq \mu$, so $f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)) \leq f^{-1}(\mu)$. Since $\mathrm{Cl}(\mathrm{Int}(\mu, r), r)$ is an intuitionistic r-fuzzy regular closed set, by (2), $f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set. Hence $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r) \leq f^{-1}(\mu)$.
(3) $\Rightarrow$ (4): It can be proved by using the complement.
(4) $\Rightarrow$ (5): Let μ be any intuitionistic r-fuzzy strongly semiopen set of Y. Then $f^{-1}(\mu) \leq f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)$. Since $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)$ is intuitionistic r-fuzzy open set, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)), r)$.
(5) $\Rightarrow$ (6): It can be proved by using the complement.
(6) $\Rightarrow$ (7): Let μ be any intuitionistic r-fuzzy preclosed set of Y. Then $\mu \geq \mathrm{Cl}(\mathrm{Int}(\mu, r), r)$, so $f^{-1}(\mu) \geq f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r))$. Since $\mathrm{Cl}(\mathrm{Int}(\mu, r), r)$ is an intuitionistic r-fuzzy strongly semiclosed set, by (6), $f^{-1}(\mu) \geq f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)) \geq \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r), r), r)), r) \geq \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)), r)$.
(7) $\Rightarrow$ (8): It can be proved by using the complement.
(8) $\Rightarrow$ (9): Let μ be any intuitionistic fuzzy set of Y. Then $\overline{1} - \mathrm{Int}(\mu, r)$ is

an intuitionistic r-fuzzy preopen set, so $f^{-1}(\overline{1} - \text{Int}(\mu, r)) \leq f^{-1}(\text{Int}(\text{Cl}(\text{Int}(\overline{1} - \mu, r), r), r)) \leq f^{-1}(\text{Cl}(\mu, r))$.
(9) $\Rightarrow$ (10): It can be proved by using the complement.
(10) $\Rightarrow$ (1): Let μ be an intuitionistic r-fuzzy regular open set of Y. Then $f^{-1}(\mu) = f^{-1}(\text{Int}(\mu, r)) \leq \text{Int}(f^{-1}(\text{Int}(\text{Cl}(\text{Int}(\mu, r), r), r)), r) = \text{Int}(f^{-1}(\mu), r)$, so $f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set of X: Hence f is intuitionistic r-fuzzy almost continuous. □

Theorem 4.5. *For a function $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, the following statements are equivalent:*

(1) f *is an intuitionistic r-fuzzy almost continuous function.*
(2) $\text{Cl}(f^{-1}(\mu), r) \leq f^{-1}(\text{Cl}(\mu, r))$ *for each intuitionistic r-fuzzy semiopen set μ of Y.*
(3) $f^{-1}(\text{Int}(\mu, r)) \leq \text{Int}(f^{-1}(\mu), r)$ *for each intuitionistic r-fuzzy semiclosed set μ of Y.*
(4) $f^{-1}(\text{Int}(\mu, r)) \leq \text{Int}(f^{-1}(\mu), r)$ *for each intuitionistic r-fuzzy semipreclosed set μ of Y.*
(5) $\text{Cl}(f^{-1}(\mu), r) \leq f^{-1}(\text{Cl}(\mu, r))$ *for each intuitionistic r-fuzzy semipreopen set μ of Y.*

Proof. (1) $\Rightarrow$ (2): Let μ be any intuitionistic r-fuzzy semiopen set of Y. Then $\mu \leq \text{Cl}(\text{Int}(\mu, r), r)$, so $f^{-1}(\mu) \leq f^{-1}(\text{Cl}(\text{Int}(\mu, r), r))$. Since $\text{Cl}(\text{Int}(\mu, r), r)$ is an intuitionistic r-fuzzy regular closed set, by assumption $f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set. Hence $\text{Cl}(f^{-1}(\text{Cl}(\text{Int}(\mu, r), r)), r) \leq f^{-1}(\text{Cl}(\mu, r))$.
(2) $\Rightarrow$ (3): It can be proved by using the complement.
(3) $\Rightarrow$ (4): Let μ be any intuitionistic r-fuzzy semipreclosed open set of Y. From $\mu \leq \text{Int}(\text{Cl}(\text{Int}(\mu, r), r), r)$ it follows that $\text{Int}(\mu, r) \geq \text{Int}(\text{Cl}(\text{Int}(\mu, r), r), r)$, so $\text{Int}(\mu, r)$ is an intuitionistic r-fuzzy semiclosed set. By assumption, $f^{-1}(\text{Int}(\mu, r)) \leq \text{Int}(f^{-1}(\text{Int}(\mu, r)), r) \leq \text{Int}(f^{-1}(\mu), r)$.
(4) $\Rightarrow$ (5): It can be proved by using the complement.
(5) $\Rightarrow$ (1): Let μ be any intuitionistic r-fuzzy regular closed set of Y. Then μ is an intuitionistic r-fuzzy semipreopen set. By assumption, $f^{-1}(\text{Cl}(\mu, r)) \leq \text{Cl}(f^{-1}(\mu), r)$, so $f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set. Hence f is an intuitionistic r-fuzzy almost continuous. □

Corollary 4.6. *Let $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ be an intuitionistic r-fuzzy almost continuous function. Then the following statements holds:*

(1) $\text{Cl}(f^{-1}(\mu), r) \leq f^{-1}(\text{Cl}(\mu, r))$ *for each intuitionistic r-fuzzy open set μ of Y.*
(2) $f^{-1}(\text{Int}(\mu, r)) \leq \text{Int}(f^{-1}(\mu), r)$ *for each intuitionistic r-fuzzy closed set μ of Y.*

Theorem 4.7. *For a function $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$, the following statements are equivalent:*

(1) f *is an intuitionistic* r*-fuzzy almost continuous function.*
(2) for each intuitionistic fuzzy point $x(a,b)$ *of* X *and intuitionistic* r*-fuzzy open set* μ *contain* $f(x(a,b))$*, there exists an intuitionistic* r*-fuzzy open set* λ *of* X *containing* $x(a,b)$ *such that* $f(\lambda) \leq \mathrm{Int}(\mathrm{Cl}(\mu,r),r)$*.*
(3) for each intuitionistic fuzzy point $x(a,b)$ *of* X *and an intuitionistic* r*-fuzzy regular open set* μ *containing* $f(x(a,b))$*, there exists an intuitionistic* r*-fuzzy open set* λ *of* X *containing* $x(a,b)$ *such that* $f(\lambda) \leq \mu$*.*

Proof. (1) $\Rightarrow$ (3): Let f be intuitionistic r-fuzzy almost continuous, $x(a,b)$ be an intuitionistic fuzzy point of X and let μ be an intuitionistic r-fuzzy open set of Y such that $f(x(a,b)) \leq \mu$. Then $x(a,b) \leq f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu,r),r)),r)$. Let $\lambda = \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu,r),r)),r)$. Then λ is an intuitionistic r-fuzzy open set and $f(\lambda) = f(\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu,r),r)),r)) \leq f(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mu,r),r))) \leq \mathrm{Int}(\mathrm{Cl}(\mu,r),r)$.
(2) $\Rightarrow$ (3): Let $x(a,b)$ be an intuitionistic fuzzy point of X and let μ be an intuitionistic r-fuzzy regular open set of Y containing $f(x(a,b))$. Then μ is an intuitionistic r-fuzzy open set. By assumption, there exists an intuitionistic r-fuzzy open set λ of X containing $x(a,b)$ such that $f(\lambda) \leq \mathrm{Int}(\mathrm{Cl}(\mu,r),r) = \mu$.
(3) $\Rightarrow$ (1): Let μ be an intuitionistic r-fuzzy regularly open set of Y and let $x(a,b)$ be an intuitionistic fuzzy singleton of X such that $x(a,b) \leq f^{-1}(\mu)$. By assumption, there exists an intuitionistic fuzzy open set λ of X such that $x(a,b) \leq \lambda$ and $f(\lambda) \leq \mu$. Hence $x(a,b) \leq \lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\mu)$ and $x(a,b) \leq \lambda = \mathrm{Int}(\lambda,r) \leq \mathrm{Int}(f^{-1}(\mu),r)$. Since $x(a,b)$ is arbitrary and $f^{-1}(\mu)$ is the union of all intuitionistic fuzzy points of $f^{-1}(\mu)$, $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mu),r)$. Thus f is an intuitionistic r-fuzzy almost continuous function. □

Definition 4.8. *An intuitionistic fuzzy subset* λ *of an intuitionistic smooth fuzzy topological space* $(X,\tau,\tau^\star)$ *is said to be an intuitionistic* r*-fuzzy* δ*-open set if it is the union of a family of intuitionistic* r*-fuzzy regular open sets. The complement of an intuitionistic* r*-fuzzy* δ*-open set is called an intuitionistic* r*-fuzzy* δ*-closed.*

Definition 4.9. *For an intuitionistic fuzzy set* λ *in an intuitionistic fuzzy topological space* (X,τ)*, we define the following*

(1) $\mathrm{Cl}_\delta(\lambda,r) = \bigwedge\{\mu \in I^X : \mu$ *is intuitionistic* r*-fuzzy* δ*-closed and* $\lambda \leq \mu\}$,
(2) $\mathrm{Int}_\delta(\lambda,r) = \bigvee\{\mu \in I^X : \mu$ *is intuitionistic* r*-fuzzy* δ*-open and* $\mu \leq \lambda\}$

Theorem 4.10. *For a function* $f : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$*, the following statements are equivalent:*

(1) f *is an intuitionistic* r*-fuzzy almost continuous function.*
(2) $f^{-1}(\mu)$ *is an intuitionistic* r*-fuzzy open set of* X *for each intuitionistic* r*-fuzzy* δ*-open set* μ *of* Y*.*
(3) $f^{-1}(\mu)$ *is an intuitionistic* r*-fuzzy closed set of* X *for each intuitionistic* r*-fuzzy* δ*-closed set* μ *of* Y*.*
(4) $f(\mathrm{Cl}(\mu,r)) \leq \mathrm{Cl}_\delta(f(\lambda),r)$ *for each* $\lambda \in I^X$*.*
(5) $\mathrm{Cl}(f^{-1}(\mu),r) \leq f^{-1}(\mathrm{Cl}_\delta(\mu,r))$ *for each* $\mu \in I^Y$*.*

(6) $f^{-1}(\mathrm{Int}_\delta(\mu, r)) \leq \mathrm{Int}(f^{-1}(\mu), r)$ *for each* $\mu \in I^Y$.

Proof. (1) $\Rightarrow$ (2): Let μ be any intuitionistic r-fuzzy δ-open set of Y. Then $\mu = \bigvee_{\alpha\in\Delta} \mu_\alpha$, where μ_α is an intuitionistic r-fuzzy regular open set of Y for each $\alpha \in \Delta$. From $f^{-1}(\mu) = f^{-1}(\bigvee_{\alpha\in\Delta} \mu_\alpha) = \bigvee_{\alpha\in\Delta} f^{-1}(\mu_\alpha)$ it follows that $f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set as a union of intuitionistic r-fuzzy open sets.
(2) $\Rightarrow$ (3): Can be proved by using the complement.
(3) $\Rightarrow$ (4): Let $\lambda \in I^X$. Then $\mathrm{Cl}_\delta(f(\lambda), r)$ is an intuitionistic r-fuzzy δ-closed set of Y. By assumption, $f^{-1}(\mathrm{Cl}_\delta(f(\lambda), r))$ is an intuitionistic r-fuzzy closed set of X. Hence $\mathrm{Cl}(\lambda, r) \leq \mathrm{Cl}(f^{-1}(f(\lambda)), r) \leq \mathrm{Cl}(f^{-1}(\mathrm{Cl}_\delta(f(\lambda), r)), r) = f^{-1}(\mathrm{Cl}_\delta(f(\lambda), r))$ so $f(\mathrm{Cl}(\lambda, r)) \leq \mathrm{Cl}_\delta(f(\lambda), r)$.
(4) $\Rightarrow$ (5): Let μ be any intuitionistic fuzzy set of Y. From the assumption it follows that $\mathrm{Cl}(f^{-1}(\mu), r) \leq f^{-1}(f(\mathrm{Cl}(f^{-1}(\mu), r))) \leq f^{-1}(\mathrm{Cl}_\delta(\mu, r))$.
(5) $\Rightarrow$ (6): Can be proved by using the complement.
(6) $\Rightarrow$ (1): Let μ be any intuitionistic r-fuzzy regular open set of Y. Then $\mu = \mathrm{Int}_\delta(\mu, r)$. By assumption $f^{-1}(\mu, r) = f^{-1}(\mathrm{Int}_\delta(\mu, r)) \leq \mathrm{Int}(f^{-1}(\mu), r) \leq f^{-1}(\mu)$. Hence $f^{-1}(\mu) = \mathrm{Int}(f^{-1}(\mu), r)$, so $f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set. Thus f is an intuitionistic r-fuzzy almost continuous function.

□

Theorem 4.11. *A bijective function* $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ *is intuitionistic* r*-fuzzy almost continuous if and only if* $\mathrm{Int}_\delta(f(\lambda), r) \leq f(\mathrm{Int}(\mu, r))$ *for each* $\mu \in I^X$.

Proof. Suppose that f is intuitionistic r-fuzzy almost continuous. Then we have $f^{-1}(\mathrm{Int}_\delta(f(\lambda), r))$ is an intuitionistic r-fuzzy open set of X for any $\lambda \in I^X$. Since f is injective, $f^{-1}(\mathrm{Int}(f(\lambda), r)) = \mathrm{Int}(f^{-1}(\mathrm{Int}_\delta(f(\lambda), r)), r) \leq \mathrm{Int}(f^{-1}(f(\lambda)), r) = \mathrm{Int}(\lambda, r)$. Again, since f is surjective, $\mathrm{Int}_\delta(f(\lambda), r) = f(f^{-1}(\mathrm{Int}_\delta(f(\lambda), r))) \leq f(\mathrm{Int}(\lambda, r))$. Conversely, let μ be any intuitionistic r-fuzzy δ-open set of Y. Then $\mathrm{Int}_\delta(\mu, r) = \mu$. By assumption, $f(\mathrm{Int}(f^{-1}(\mu), r)) \leq \mathrm{Int}_\delta(f(f^{-1}(\mu)), r) = \mathrm{Int}_\delta(\mu, r) = \mu$. Thus implies that $f^{-1}(f(\mathrm{Int}(f^{-1}(\mu), r))) \geq f^{-1}(\mu, r)$. Since f is injective we obtain $\mathrm{Int}(f^{-1}(\mu), r) = f^{-1}(f(\mathrm{Int}(f^{-1}(\mu), r))) \geq f^{-1}(\mu)$. Hence $\mathrm{Int}(f^{-1}(\mu), r) = f^{-1}(\mu)$, so $f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set. Thus f is an intuitionistic r-fuzzy almost continuous function.

□

Theorem 4.12. *For a function* $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$*, the following statements are equivalent:*

(1) f *is intuitionistic* r*-fuzzy almost continuous;*
(2) $f^{-1}(\lambda, r) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r)$ *for every intuitionistic* r*-fuzzy open set* λ *in* Y*;*
(3) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r) \leq f^{-1}(\mu)$ *for every intuitionistic* r*-fuzzy closed set* μ *in* Y*;*

(4) $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mu, r), r), r)), r) \leq f^{-1}(\mathrm{Cl}(\mu, r))$ *for every* $\mu \in I^Y$*;*
(5) $f^{-1}(\mathrm{Int}(\mu, r)) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r)), r)$ *for every* $\mu \in I^Y$*.*

Proof. (1) $\Rightarrow$ (2): Let λ be any intuitionistic r-fuzzy open set in Y and $x(a,b) \in f^{-1}(\lambda)$. Then there exists an intuitionistic r-fuzzy open set μ containing $x(a,b)$ such that $f(\mu) \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$. Then $x(a,b) \in \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r)$. Hence $f^{-1}(\lambda) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r)$.
(2) $\Rightarrow$ (3): Let μ be any intuitionistic r-fuzzy closed set in Y. Then $f^{-1}(\overline{1} - \mu) \leq \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\overline{1} - \mu, r), r)), r) = \overline{1} - \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r)$. It is clear that $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r)), r) \leq f^{-1}(\mu)$.
(3) $\Rightarrow$ (4) and (4)$\Rightarrow$(5): It is obvious.
(5) $\Rightarrow$ (1): Let μ be any intuitionistic r-fuzzy regular open set in Y.
Since $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mu, r), r), r) = \mu$, from (5), $f^{-1}(\mu) \leq \mathrm{Int}(f^{-1}(\mu), r)$ and so $f^{-1}(\mu) = \mathrm{Int}(f^{-1}(\mu), r)$. Then $f^{-1}(\mu)$ is intuitionistic r-fuzzy open in X. Hence f is intuitionistic r-fuzzy almost continuous. □

Theorem 4.13. *For a function* $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$*, the following statements are equivalent:*

(1) f *is intuitionistic* r*-fuzzy almost continuous;*
(2) $\mathrm{Cl}(f^{-1}(\lambda), r) \leq f^{-1}(\mathrm{Cl}(\lambda, r))$ *for every intuitionistic* r*-fuzzy open set* λ *of* Y*;*
(3) $\mathrm{Cl}(f^{-1}(\lambda), r) \leq f^{-1}(\mathrm{Cl}(\lambda, r))$ *for every intuitionistic* r*-fuzzy semiopen set* λ *of* Y*;*
(4) $\mathrm{Cl}(f^{-1}(\lambda), r) \leq f^{-1}(\mathrm{Cl}(\lambda, r))$ *for every intuitionistic* r*-fuzzy preopen set* λ *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let λ be any intuitionistic r-fuzzy open set of Y. Since $\mathrm{Cl}(\lambda, r)$ is intuitionistic r-fuzzy regular closed, $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(\lambda, r)), r) = f^{-1}(\mathrm{Cl}(\lambda, r))$. Thus, $\mathrm{Cl}(f^{-1}(\lambda), r) \leq \mathrm{Cl}(f^{-1}(\mathrm{Cl}(\lambda, r)), r) = f^{-1}(\mathrm{Cl}(\lambda, r))$.
(2) $\Rightarrow$ (3): It is obvious since every intuitionistic r-fuzzy semiopen set is intuitionistic r-fuzzy open.
(3) $\Rightarrow$ (1): Let λ be any intuitionistic r-fuzzy regular closed set of Y. Since λ is intuitionistic r-fuzzy semiopen, $\mathrm{Cl}(f^{-1}(\lambda), r) \leq f^{-1}(\mathrm{Cl}(\lambda, r)) = f^{-1}(\lambda)$. Thus, f is intuitionistic r-fuzzy almost continuous.
(1) $\Rightarrow$ (4): Let λ be any intuitionistic r-fuzzy preopen set of Y. Then $\lambda \leq \mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ and $\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)$ is intuitionistic r-fuzzy regular open. By Theorem 4.4 (2), $f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)) = \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r)$. It follows that

$$f^{-1}(\lambda) \leq f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)) = \mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r).$$

(4) $\Rightarrow$ (1): Let λ be any intuitionistic r-fuzzy regular open set of Y. Then since λ is intuitionistic r-fuzzy preopen and $f^{-1}(\lambda) \leq \mathrm{Cl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\lambda, r), r)), r) = \mathrm{Int}(f^{-1}(\lambda), r)$. Hence by Theorem 4.4 (2), f is intuitionistic r-fuzzy almost continuous. □

5. CONCLUDING REMARKS

In this paper, we have introduced the notions of intuitionistic fuzzy regular open sets in intuitionistic fuzzy topological spaces. Several characterizations are also studied. Relevant examples are given. In our future study, we will apply this concept/results to intuitionistic fuzzy bitopological spaces Also, we will study this concept to intuitionistic fuzzy ideal topological spaces.

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