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# SCREEN PSEUDO-SLANT LIGHTLIKE SUBMERSIONS

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**Abstract.** In this article, we introduce the notion of screen pseudo-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold which include complex (invariant), screen real (anti-invariant), screen slant and SCR lightlike submersions. We study some properties of proper screen pseudo-slant lightlike submersions with non-trivial examples and gave a characterization theorem. We also obtain integrability conditions of distributions involved in the definition of such submersions.
*Key words and Phrases*: Submersion, Slant manifold, Lightlike manifold, Lightlike submersion, Kaehler manifold.

## 1. INTRODUCTION

A smooth map $f : (M, g) \to (B, g')$ between Riemannian manifolds $M$ and $B$ is called a Riemannian submersion if the derivative map $f_*$ is surjective and $g(X, Y) = g'(f_*X, f_*Y)$, where X and Y are vector fields tangent to the horizontal space $(Ker\ f_*)^\perp$. Riemannian submersions between Riemannian manifolds were studied by ONeill [9] and Gray [8]. In [10], O' Neill studied Semi-Riemannian submersions between semi-Riemannian manifolds. In [14], Sahin and G*ü*nd*ü*zalp defined lightlike submersions from semi-Riemannian manifolds onto lightlike manifolds. In [5], Duggal and Sahin gave the definition of SCR-lightlike submanifolds of an indefinite Kaehler manifold. Sahin [12, 13] introduced the notion of a slant and screen-slant lightlike submanifold of an indefinite Hermitian manifold. In [16], Shukla and Yadav gave the notion of screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold. In the present paper, we study screen pseudo slant lightlike submersions as a natural generalization of screen slant and SCR lightlike submersions.

The present article is organized as follows. In Section 2, we give some basic definitions and formulas related to this paper. In Section 3, we define screen pseudo-slant lightlike submersions with non- trivial examples. In this section, we also obtain

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a characterization theorem and investigate integrability conditions of distributions involved in the definition of such submersions.

## 2. PRELIMINARIES

Let $(M, J)$ be a 2m-dimensional almost complex manifold, where J is an almost complex structure and $g$ is a semi-Riemannian metric with index $0 < r \leq 2m$. Then $M$ is called an indefinite almost Hermitian manifold, if

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{1}$$

Also, if $J$ is a complex structure on $M$, then $M$ is said to be an indefinite Hermitian manifold. Now, let $(M, J, g)$ is an indefinite almost Hermitian manifold with Levi-Civita connection $\nabla$. Then, $M$ is called an indefinite Kaehler manifold if

$$(\nabla_X J)Y = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2}$$

Let $(M, g)$ be a real m-dimensional $C^\infty$ manifold. The Radical (or null) space $Rad\ T_pM$ of $T_pM$ is defined as $Rad\ T_pM = \{\xi \in T_pM : g(\xi, X) = 0,\ \forall X \in T_pM\}$. If $Rad\ TM : p \in M \to Rad\ T_pM$ defines a smooth distribution of rank $r > 0$ of $M$ such that $0 < r \leq m$, then $Rad\ TM$ is called a radical or null distribution of $M$ and the manifold $M$ is called an r-lightlike manifold.

Let $f : (M, g) \to (B, g')$ be a smooth submersion from a semi-Riemannian manifold $M$ onto an r-lightlike manifold $B$. Then, $Ker\ f_{*p} = \{X \in T_pM : f_{*p}X = 0\}$ and $(Ker\ f_{*p})^\perp = \{Y \in T_pM : g(Y, X) = 0,\ \forall X \in Ker\ f_{*p}\}$. As $T_pM$ is a semi-Riemannian vector space $(Ker\ f_{*p})^\perp$ may not be a complementary space to $Ker f_{*p}$. Assume that $Ker\ f_{*p} \cap (Ker\ f_{*p})^\perp = \Delta_p \neq \{0\}$. In this case $\Delta : p \to \Delta_p$ is said to be a radical distribution of $M$. As $\Delta$ is a lightlike distribution, we have $Ker\ f_* = \Delta \perp S(Ker\ f_*)$. Similarly $(Ker\ f_*)^\perp = \Delta \perp S(Ker\ f_*)^\perp$. Here $S(Ker\ f_*)^\perp$ is the complementary distribution to $\Delta$ in $(Ker\ f_*)^\perp$. Now, let $dim(\Delta) = r > 0$. Since $\Delta \subset (S(ker\ f_*)^\perp)^\perp$ and $(S(ker f_*)^\perp)^\perp$ is non-degenerate, then there exists null vectors $N_1, N_2..., N_r$, such that $g(N_i, N_j) = 0,\ g(\xi_i, N_j) = \delta_{ij}$, where $\{N_i\}$ and $\{\xi_i\}$ are smooth null vector fields in $S(Ker\ f_*)^\perp$ and lightlike basis of $\Delta$, respectively. Assume that $ltr(ker\ f_*)$ denotes the distribution spanned by null vector fields $N_1, N_2..., N_r$. Then $tr(ker f_*) = ltr(ker f_*) \perp S(ker f_*)^\perp$. Moreover, we have

$$TM = (\Delta \oplus ltr(Ker\ f_*)) \perp S(Ker\ f_*) \perp S(Ker\ f_*)^\perp. \tag{3}$$

A Riemannian submersion $f : (M, g) \to (B, g')$ is said to be r-lightlike submersion if

$$dim\,\Delta = dim\{(Ker\ f_*) \cap (Ker\ f_*)^\perp\} = r,\ 0 < r < min\{dim(ker\ f_*), dim(ker\ f_*)^\perp\};$$

isotropic submersion if $dim\ \Delta = dim(Ker\ f_*) < dim(Ker\ f_*)^\perp$; co-isotropic submersion if $dim\ \Delta = dim(Ker\ f_*)^\perp < dim(Ker\ f_*)$ and totally lightlike submersion if $dim\ \Delta = dim(Ker\ f_*)^\perp = dim(Ker\ f_*)$. A lightlike submersion $f : (M, g) \to (B, g')$ determines two (1,2) type tensors fields $T$ and $A$ on $M$, given as

$$T_X Y = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY, \tag{4}$$

$$A_X Y = \nu\nabla_{hX} hY + h\nabla_{hX}\nu Y. \tag{5}$$

Here $T$ and $A$ are vertical and horizontal tensors, respectively. For vertical tensor $T$, we have

$$T_X Y = T_Y X, \quad \forall\ X, Y \in \Gamma(Ker\ f_*). \tag{6}$$

Now, we suppose that $f$ is a lightlike submersion from a real (m + n)-dimensional semi-Riemannian manifold $(M, g)$ onto a lightlike manifold $(B, g')$, with $m, n > 1$. Further, let $Ker\ f_*$ be an m-dimensional lightlike distribution of $M$ and $tr(Ker\ f_*)$ is the complementary distribution of $Ker\ f_*$ in $M$ with respect to the pair $\{S(Ker\ f_*),\ S(Ker\ f_*)^\perp\}$. Let us denote by $\hat{g}$ the induced metric on $Ker\ f_*$ of $g$ and by $\nabla$ the Levi-Civita connection on $M$. Then, in view of (4), we have

$$\nabla_U V = \hat{\nabla}_U V + T_U V, \tag{7}$$

$$\nabla_U X = T_U X + \nabla^\perp_U X, \tag{8}$$

$\forall\ U, V \in \Gamma(Ker\ f_*), X \in \Gamma(Ker\ f_*)^\perp$, where $\hat{\nabla}_U V = \nu\nabla_U V$ and $\nabla^\perp_U X = h\nabla_U X$. Here $\{\hat{\nabla}_U V, T_U X\}$ and $\{T_U V, \nabla^\perp_U X\}$ belong to $\Gamma(Ker\ f_*)$ and $\Gamma(tr(Ker\ f_*))$, respectively. Let $S(Ker\ f_*)^\perp \neq \{0\}$. Now, we denote by L and S the projections of $tr(Ker\ f_*)$ on $ltr(Ker\ f_*)$ and $S(Ker\ f_*)^\perp$, respectively. Then, from (7) and (8), we have

$$\nabla_U V = \hat{\nabla}_U V + T^l_U V + T^s_U V, \tag{9}$$

$$\nabla_U N = T_U N + \nabla^{\perp l}_U N + D^{\perp s}(U, N), \tag{10}$$

$$\nabla_U W = T_U W + D^{\perp l}(U, W) + \nabla^{\perp s}_U W, \tag{11}$$

$\forall\ U, V \in \Gamma(Ker\ f_*), N \in \Gamma(ltr(Ker\ f_*))$ and $W \in \Gamma(S(Ker\ f_*)^\perp)$. From equations (9)-(11) and the fact that $\nabla$ is a metric connection, we obtain

$$g(T^s_U V, W) + g(V, D^{\perp l}(U, W)) = -\hat{g}(T_U W, V), \tag{12}$$

$$g(D^{\perp s}(U, N), W) = -g(N, T_U W) \tag{13}$$

If $f$ is either r-lightlike or co-isotropic submersion, then we write

$$\hat{\nabla}_U \xi = T^*_U \xi + \nabla^{*\perp}_U \xi, \tag{14}$$

$\forall\ U \in \Gamma(Ker\ f_*),\ \xi \in \Gamma(\Delta)$. Here $T^*_U \xi \in \Gamma(S(Ker\ f_*))$ and $\nabla^{*\perp}_U \xi \in \Gamma(\Delta)$.

## 3. Screen Pseudo-Slant Lightlike Submersions

In this section, we introduce the notion of screen pseudo-slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold. First, we gave the following lemma, which is useful to define screen-pseudo slant lightlike submersions.

**Lemma 3.1.** *Let $f : (M, g) \to (B, g')$ be a $2r$-lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$ and $Ker\ f_*$ is a lightlike distribution on $M$. Then the screen distribution $S(Ker\ f_*)$ is Riemannian.*

*Proof.* Let $M$ be a real $(m+n)$-dimensional indefinite Kaehler manifold and $Ker\ f_*$ be a lightlike distribution of dimension $m$. Then there exists a local quasi orthonormal field of frames on $M$ along $Ker\ f_*$

$$\{\xi_i, N_i, U_\alpha, Z_a\}, i \in \{1, ..., 2r\}, \alpha \in \{2r+1, ..., m\}, a \in \{2r+1, ..., n\},$$

where $\{\xi_i\}$, $\{N_i\}$ are lightlike basis of $\Delta$, $ltr(Ker\ f_*)$ and $U_\alpha$, $Z_a$ are orthonormal basis of $S(Ker\ f_*)$, $S(Ker\ f_*)^\perp$, respectively. With the help of null basis $\{\xi_1, ..., \xi_{2r}, N_1, ..., N_{2r}\}$ of $\Delta \oplus ltr(Ker\ _*)$, we construct following orthonormal basis $\{X_1, ..., X_{4r}\}$

$$\begin{aligned}
X_1 &= \frac{1}{\sqrt{2}}(\xi_1 + N_1), & X_2 &= \frac{1}{\sqrt{2}}(\xi_1 - N_1),\\
X_3 &= \frac{1}{\sqrt{2}}(\xi_2 + N_2), & X_4 &= \frac{1}{\sqrt{2}}(\xi_2 - N_2),\\
&... & &...\\
&... & &...\\
X_{4r-1} &= \frac{1}{\sqrt{2}}(\xi_{2r} + N_{2r}), & X_{4r} &= \frac{1}{\sqrt{2}}(\xi_{2r} - N_{2r}).
\end{aligned}$$

Thus, Span $\{\xi_i, N_i\}$ is a non-degenerate space of index $2r$, which enables us to conclude that $\Delta \oplus ltr(Ker\ _*)$ is non-degenerate with constant index $2r$ on $M$. Moreover,

$$ind(TM) = ind(\Delta \oplus ltr(Ker\ f_*)) + ind(S(Ker\ f_*) \perp (S(Ker\ f_*))^\perp),$$

implies that $S(Ker\ f_*) \perp S(Ker\ f_*)^\perp$ has a constant index zero. Hence, $S(Ker\ f_*)$ and $S(Ker\ f_*)^\perp$ are Riemannian distributions. □

**Definition 3.2.** *Let $f : (M, g, J) \to (B, g')$ be a 2r-lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$, such that $2r < dim(Ker\ f_*)$. Then we say that f is a screen pseudo-slant lightlike submersion if*

*(a) the lightlike distribution $\Delta$ is invariant with respect to $J$,*
*(b) there exists two non-null distributions $D_1$ and $D_2$, such that $S(Ker\ f_*) = D_1 \oplus D_2$,*
*(c) $D_1$ is anti-invariant, i.e., $JD_1 \subseteq S(Ker\ f_*)^\perp$,*
*(d) $D_2$ is slant with slant angle $\theta\Big(\neq \frac{\pi}{2}\Big)$, that is, for every $p \in M$ and for every non-zero vector $U \in (D_2)_p$, the angle $\theta(U)$ between the vector subspace $(D_2)_p$ and $JU$ is a constant $\Big(\neq \frac{\pi}{2}\Big)$.*

From the definition, it is clear that

(a) if $D_1 = 0$, then f is a screen slant lightlike submersion.
(b) if $D_2 = 0$, then f is a screen real lightlike submersion.
(c) if $D_1 = 0$ and $\theta = 0$, then f is a complex lightlike submersion.
(d) if $D_1 \neq 0$ and $\theta = 0$, then f is a SCR-lightlike submersion.

Thus, the above class of lightlike submersions is a natural generalization of screen slant, screen real, complex and SCR-lightlike submersions. If $D_1 \neq 0$, $D_2 \neq 0$ and

$\theta \neq 0$, then f is called a proper screen pseudo-slant lightlike submersion. Now, we give some non-trivial examples of screen pseudo-slant lightlike submersions.

Denote by $\mathbb{R}^n_{r,q,p}$ the space $\mathbb{R}^n$ equipped with the semi-Riemannian metric $g$, such that $g(e_i, e_j)_{r,q,p} = (G_{r,q,p})_{ij}$, $i \in \{1, ..., n\}$, where $e_i$ is the standard basis of $\mathbb{R}^n$ and $G_{r,q,p}$ is the diagonal matrix determined by $g$, i.e., $G_{ij} =$ diagonal$(\underbrace{0, ..., 0}_{\text{r-times}}, \underbrace{-1, ..., -1}_{\text{q-times}}, \underbrace{1, ..., 1}_{\text{p-times}})$.

**Example 3.3.** *Let $\mathbb{R}^{12}_{0,2,10}$ and $\mathbb{R}^6_{2,0,4}$ endowed with the semi-Riemannian metric*

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2 + (dx_9)^2 + (dx_{10})^2 + (dx_{11})^2 + (dx_{12})^2,$$

*and degenerate metric $g' = (dy_3)^2 + (dy_4)^2 + (dy_5)^2 + (dy_6)^2$, where $x_1, \dots, x_{12}$ and $y_1, \dots, y_6$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^6$, respectively. Define the mapping $f : (\mathbb{R}^{12}, g) \to (\mathbb{R}^6, g')$ as*

$$(x_1, ..., x_{12}) \longmapsto \Big(x_1 + x_5,\ x_2 + x_6,\ x_3,\ x_7,\ \frac{x_9 + x_{12}}{\sqrt{2}},\ x_{11}\Big).$$

*Then, we can see easily that $f$ is a 2-lightlike submersion with*

$$\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\Big\{\xi_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_5}, \xi_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_6}\Big\}.$$

*Since $J\xi_1 = \xi_2$, $\Delta$ is invariant with respect to $J$. By easy calculation we can see that*

$$D_1 = Span\Big\{\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_8}\Big\}$$

*is anti-invariant distribution. Further, we see that*

$$D_2 = Span\Big\{\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_9} - \frac{\partial}{\partial x_{12}}\Big), \frac{\partial}{\partial x_{10}}\Big\}$$

*is slant distribution with slant angle $\theta = \frac{\pi}{4}$. Thus, $f$ is a proper screen pseudo-slant lightlike submersion.*

**Example 3.4.** *Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be endowed with the semi-Riemannian metric*

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2,$$

*and degenerate metric $g' = (dy_3)^2 + (dy_4)^2$, where $x_1, \dots, x_8$ and $y_1, \dots, y_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(x_1, ..., x_8) \longmapsto \Big(x_1 + x_7,\ x_2 + x_8,\ \frac{x_4 + x_6}{\sqrt{2}}, x_3\Big)$. Then*

$$Ker\ f_* = Span\Big\{U_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_7},\ U_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_8},\ U_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_6}\Big),\ U_4 = \frac{\partial}{\partial x_5}\Big\}$$

*and*

$$(Ker\ f_*)^\perp = Span\Big\{U_1, U_2,\ X = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}\Big),\ Y = \frac{\partial}{\partial x_3}\Big\}$$

*Thus $f$ is a 2-lightlike submersion with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{U_1, U_2\}$, which is invariant with respect to $J$. Also, $D_2 = S(Ker\ f_*) = Span\{U_3, U_4\}$ is slant with slant angle $\theta = \frac{\pi}{4}$. Hence, $f$ is a screen slant lightlike submersion.*

**Example 3.5.** *Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be endowed with the semi-Riemannian metric $g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2$, and degenerate metric $g' = (dy_3)^2 + (dy_4)^2$, where $x_1, \ldots, x_8$ and $y_1, \ldots, y_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as $(x_1, ..., x_8) \longmapsto \Big(x_1 + x_5,\ x_2 + x_6,\ \frac{x_3 - x_7}{\sqrt{2}}, \frac{x_4 - x_8}{\sqrt{2}}\Big)$. Then, we obtain*

$$Kerf_* = Span\Big\{U_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_5},\ U_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_6},$$
$$U_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}\Big),\ U_4 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8}\Big)\Big\},$$

*and*

$$(Kerf_*)^\perp = Span\Big\{U_1,\ U_2,\ X = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_7}\Big),\ Y = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_8}\Big).$$

*Then, $f$ is a 2-lightlike submersion with $\Delta = Span\{U_1, U_2\}$. Since $JU_1 = U_2$, $\Delta$ is invariant with respect to $J$. Further, since $JU_3 = U_4$, $S(Ker\ f_*) = D_2 = Span\{U_3, U_4\}$ is slant distribution with slant angle $\theta = 0$, that is, $D_2$ is invariant. Thus, $D_1 = 0$. Hence $f$ is a complex lightlike submersion.*

**Example 3.6.** *Let $\mathbb{R}^{12}_{0,4,8}$ and $\mathbb{R}^6_{4,0,2}$ be endowed with the semi-Riemannian metric*

$$g = -(dx_1)^2 - (dx_2)^2 - (dx_3)^2 - (dx_4)^2 + (dx_5)^2 + (dx_6)^2$$
$$+ (dx_7)^2 + (dx_8)^2 + (dx_9)^2 + (dx_{10})^2 + (dx_{11})^2 + (dx_{12})^2,$$

*and degenerate metric $g' = (dy_5)^2 + (dy_6)^2$, where $x_1, \ldots, x_{12}$ and $y_1, \ldots, y_6$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^6$, respectively. Let us define the map*

$$f : (\mathbb{R}^{12}, g) \to (\mathbb{R}^6, g'), \quad (x_1, ..., x_{12}) \longmapsto \Big(\frac{x_1 - x_7}{\sqrt{2}}, \frac{x_2 - x_8}{\sqrt{2}}, \frac{x_3 + x_9}{2}, \frac{x_4 + x_{10}}{2}, x_6, x_{12}\Big).$$

*Then, we obtain*

$$Kerf_* = Span\Big\{U_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\Big),\ U_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_8}\Big),\ U_3 = \frac{1}{2}\Big(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_9}\Big),$$
$$U_4 = \frac{1}{2}\Big(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_{10}}\Big),\ U_5 = \frac{\partial}{\partial x_5},\ U_6 = \frac{\partial}{\partial x_{11}}\Big\},$$

*and*

$$(Kerf_*)^\perp = Span\Big\{U_1,\ U_2,\ U_3,\ U_4,\ X = \frac{\partial}{\partial x_6},\ Y = \frac{\partial}{\partial x_{12}}\Big\}.$$

*Thus, $f$ is a 4-lightlike submersion with $\Delta = Span\{U_1, U_2, U_3, U_4\}$. As $JU_1 = U_2$ and $JU_3 = U_4$, $\Delta$ is invariant with respect to $J$. Also $JU_5 = X$ and $JU_6 = Y$, implies that $S(Ker\ f_*) = D_1 = Span\{U_5, U_6\}$ is anti-invariant. Also $D_2 = 0$. Hence $f$ is a screen real lightlike submersion.*

**Example 3.7.** *Let $\mathbb{R}^{16}_{0,2,14}$ and $\mathbb{R}^{8}_{2,0,6}$ be endowed with the semi-Riemannian metric*

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2 + (dx_9)^2 + (dx_{10})^2 + (dx_{11})^2 + (dx_{12})^2 + (dx_{13})^2 + (dx_{14})^2 + (dx_{15})^2 + (dx_{16})^2,$$

*and degenerate metric $g' = (dy_3)^2 + (dy_4)^2 + (dy_5)^2 + (dy_6)^2 + (dy_7)^2 + (dy_8)^2$, where $x_1, \ldots, x_{16}$ and $y_1, \ldots, y_8$ are the canonical coordinates on $\mathbb{R}^{16}$ and $\mathbb{R}^8$, respectively. Let us define the map $f : (\mathbb{R}^{16}, g) \to (\mathbb{R}^8, g')$ as*

$$(x_1, ..., x_{16}) \longmapsto \Big(x_1 + x_3,\ x_2 + x_4,\ \frac{x_5 - x_{11}}{\sqrt{2}},\ \frac{x_6 - x_{12}}{\sqrt{2}},\ \frac{x_7 + x_9}{\sqrt{2}},\ \frac{x_8 + x_{10}}{\sqrt{2}},\ x_{14},\ x_{16}\Big).$$

*Then, we obtain*

$$Ker f_* = Span\Big\{U_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},\ U_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4},\ U_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_{11}}\Big), U_4 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_{12}}\Big),\ U_5 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}\Big), U_6 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_{10}}\Big),\ U_7 = \frac{\partial}{\partial x_{13}},\ U_8 = \frac{\partial}{\partial x_{15}}\Big\},$$

*and*

$$(Ker f_*)^{\perp} = Span\Big\{U_1,\ U_2,\ V_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_{11}}\Big),\ V_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_{12}}\Big), V_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}\Big),\ V_4 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}}\Big),\ V_5 = \frac{\partial}{\partial x_{14}},\ V_6 = \frac{\partial}{\partial x_{16}}\Big\},$$

*Since $JU_1 = U_2$. So $\Delta = Span\{U_1, U_2\}$ is invariant with respect to J. It follows that $f$ is a 2-lightlike submersion. Also, $JU_7 = V_5$ and $JU_8 = V_6$ implies that $D_1 = Span\{U_7, U_8\}$ is anti-invariant. Finally, since $JU_3 = U_4$ and $JU_5 = U_6$, $D_2 = Span\{U_3, U_4, U_5, U_6\}$ is slant with slant angle zero, i.e., $D_2$ is invariant. Hence, $f$ is a proper SCR lightlike submersion.*

For any $U \in \Gamma(Ker\ f_*)$, we assume that

$$JU = \phi U + FU. \tag{15}$$

Here $\phi U$ and $FU$ are tangential and normal components of $JU$ respectively. Now, let $\phi_1, \phi_2$ and $\phi_3$ denotes the projections of $Ker\ f_*$ on $\Delta$, $D_1$ and $D_2$, respectively. Also, denote the projections of $tr(Ker\ f_*)$ on $ltr(Ker\ f_*)$ , $JD_1$ and $D'$ by $Q_1$, $Q_2$ and $Q_3$, respectively. Here $D'$ is non-null orthogonal complementary distribution of $JD_1$ in $S(Ker\ f_*)^{\perp}$. Then, for any vector field $U$ tangent to $Ker\ f_*$, we have

$$U = \phi_1 U + \phi_2 U + \phi_3 U. \tag{16}$$

Above equation gives $JU = J\phi_1 U + J\phi_2 U + J\phi_3 U$, which implies

$$JU = J\phi_1 U + J\phi_2 U + \psi\phi_3 U + F\phi_3 U, \tag{17}$$

where $\psi\phi_3 U$ and $F\phi_3 U$ denotes the tangential and normal components of $J\phi_3 U$, respectively. Therefore, $J\phi_1 U \in \Gamma(\Delta)$, $J\phi_2 U \in \Gamma(D_1)$, $\psi\phi_3 U \in \Gamma(D_2)$ and $F\phi_3 U \in \Gamma(S(Ker\ f_*)^\perp)$. Further, for any vector field $W$ tangent to $tr(Ker\ f_*)$, we put

$$W = Q_1 W + Q_2 W + Q_3 W, \tag{18}$$

which gives $JW = JQ_1 W + JQ_2 W + JQ_3 W$. Then, we have

$$JW = JQ_1 W + JQ_2 W + BQ_3 W + CQ_3 W, \tag{19}$$

where $BQ_3 W$ and $CQ_3 W$ denotes the tangential and normal components of $JQ_2 W$, respectively. Here $JQ_1 W \in \Gamma(ltr(Ker\ f_*))$, $JQ_2 W\ \Gamma(D_1)$, $BQ_3 W \in \Gamma(D_2)$ and $CQ_3 W \in \Gamma(D')$. Now, from (2), (9), (11) and (16)-(19) and identifying the components of $\Delta$, $D_1$, $D_2$, $ltr(Ker\ f_*)$, $JD_1$ and $D'$, we have

$$\nabla^{*\perp}_U J\phi_1 V + \phi_1(T_U J\phi_2 V) + \phi_1(\hat{\nabla}_U \psi\phi_3 V) + \phi_1(T_U F\phi_3 V) = J\phi_1(\hat{\nabla}_U V), \tag{20}$$

$$\phi_2(T^*_U J\phi_1 V) + \phi_2(T_U J\phi_2 V) + \phi_2(T_U F\phi_3 V) + \phi_2(\hat{\nabla}_U \psi\phi_3 V) = JQ_2 T^s_U V, \tag{21}$$

$$\phi_3(T^*_U J\phi_1 V) + \phi_3(T_U J\phi_2 V) + \phi_3(T_U F\phi_3 V) + \phi_3(\hat{\nabla}_U \psi\phi_3 V) = \psi\phi_3(\hat{\nabla}_U V) + BQ_3 T^s_U V, \tag{22}$$

$$T^l_U J\phi_1 V + D^{\perp l}(U, J\phi_2 V) + T^l_U \psi\phi_3 V + D^{\perp l}(U, F\phi_3 V) = JT^l_U V \tag{23}$$

$$Q_2(\nabla^{\perp s}_U J\phi_2 V) + Q_2(\nabla^{\perp s}_U F\phi_3 V) + Q_2(T^s_U J\phi_1 V) + Q_2(T^s_U \psi\phi_3 V) = J\phi_2(\hat{\nabla}_U V) \tag{24}$$

$$Q_3(\nabla^{\perp s}_U J\phi_2 V) + Q_3(\nabla^{\perp s}_U F\phi_3 V) + Q_3(T^s_U \psi\phi_3 V) + Q_3(T^s_U J\phi_1 V) = F\phi_3(\hat{\nabla}_U V) + CQ_3 T^s_U V \tag{25}$$

Next, we give a characterization of screen pseudo-slant lightlike submersions:

**Theorem 3.8.** *Let $f$ be a 2r-lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then, $f$ is a screen pseudo-slant lightlike submersion if and only if*

*(i) $ltr(Ker\ f_*)$ is invariant with respect to $J$,*
*(ii) $D_1$ is anti-invariant with respect to $J$,*
*(iii) there exists a constant $\lambda \in (0, 1]$ such that $\phi^2 U = -\lambda U$, $\forall U \in \Gamma(D_2)$, where $D_1$ and $D_2$ are non-null orthogonal distributions, such that $S(Ker\ f_*) = D_1 \oplus D_2$ and $\lambda = cos^2\theta$, $\theta$ is a slant angle of $D_2$.*

*Moreover, there also exists a constant $\kappa \in [0, 1)$, such that $BFU = -\kappa U$, $\forall U \in \Gamma(D_2)$.*

*Proof.* Let $f$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then the distribution $D_1$ is anti-invariant with respect to $J$. Using (1) and (17), we have

$$g(JN, U) = -g(N, JU) = -g(N, J\phi_1 U + J\phi_2 U + \psi\phi_3 U + F\phi_3 U) = 0,$$

for any $U \in \Gamma(S(Ker\ f_*))$, $N \in \Gamma(ltr(Ker\ f_*))$. So $JN$ does not belong to $S(Ker\ f_*)$. Now, for any $W \in \Gamma(S(Ker\ f*)^{\perp})$, using (2.1) and (3.5) we derive

$$g(JN, W) = -g(N, JW) = -g(N, JQ_1W + JQ_2W + BQ_3W + CQ_3W) = 0,$$

which implies that $JN$ does not belongs to $\Gamma(S(Ker\ f_*)^{\perp})$. Now, if $JN \in \Gamma(\Delta)$, then $J(JN) = J^2N = -N \in \Gamma(ltr(Ker\ f_*))$. But, it is absurd as $\Delta$ is invariant with respect to $J$. Thus $ltr(Ker\ f_*)$ is invariant with respect to $J$. Now, let $U \in \Gamma(D_2)$, then we have

$$cos(\theta)(U) = \frac{g(JU, \phi U)}{|J(U)||\phi U|} = -\frac{g(U, \phi^2 U)}{|JU||\phi U|}.$$

Also, we have

$$cos(\theta)(U) = \frac{|\phi U|}{|JU|}.$$

Thus, we obtain

$$cos^2\theta(U) = -\frac{\hat{g}(U, \phi^2 U)}{|U|^2}.$$

Since $\theta(U)$ is constant, we have $\phi^2 U = -\lambda U, \lambda \in (0, 1]$, where $\lambda = cos^2\theta$.

Now, applying J to (15) and comparing the tangential parts, we get $-U = \phi^2 U + BFU,\ \forall\ U \in \Gamma(D_2)$. It gives $BFU = -\mu U$, where $1 - \lambda = \mu \in [0, 1)$. The reverse implication can be proved in a similar way. □

As an immediate consequence of the above theorem, we have following lemma:

**Corollary 3.1.** *Let $f$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$, with slant angle $\theta$. Then, for any $U, V \in \Gamma(D_2)$, we have*

$$\hat{g}(\phi U, \phi V) = cos^2\theta \hat{g}(U, V), \tag{26}$$

*and*

$$\hat{g}(FU, FV) = sin^2\theta \hat{g}(U, V). \tag{27}$$

**Theorem 3.9.** *Let $f : M \to B$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then, the radical distribution $\Delta$ is integrable if and only if $\forall\ U, V \in \Gamma(\Delta)$, we have*

*(i)* $Q_2(T^s_U J\phi_1 V) = Q_2(T^s_V J\phi_1 U)$,
*(ii)* $Q_3(T^s_U J\phi_1 V) = Q_3(T^s_V J\phi_1 U)$,
*(iii)* $\phi_3(T^*_U J\phi_1 V) = \phi_3(T^*_V J\phi_1 U)$.

*Proof.* If $U, V \in \Gamma(\Delta)$, then using (24), we get $Q_2(T^s_U J\phi_1 V) = J\phi_2(\hat{\nabla}_U V)$, which implies

$$Q_2(T^s_U J\phi_1 V) - Q_2(T^s_V J\phi_1 U) = J\phi_2[U, V]. \tag{28}$$

From (25), we have $Q_3(T^s_U J\phi_1 V) = F\phi_3(\hat{\nabla}_U V) + CQ_3(T^s_U V)$, which implies

$$Q_3(T^s_U J\phi_1 V) - Q_3(T^s_V J\phi_1 U) = F\phi_3[U, V]. \tag{29}$$

Finally, using (22), we obtain $\phi_3(T^*_U J\phi_1 V) = \psi\phi_3(\hat{\nabla}_U V) + BQ_3 T^s_U V$, which implies

$$\phi_3(T^*_U J\phi_1 V) - \phi_3(T^*_V J\phi_1 U) = \psi\phi_3[U, V], \tag{30}$$

Our assertion follows from (28), (29) and (30). □

**Theorem 3.10.** *Let $f : M \to B$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then, the anti-invariant distribution $D_1$ is integrable if and only if*

*(i)* $\phi_1(T_U J\phi_2 V) = \phi_1(T_U J\phi_2 V)$,
*(ii)* $\phi_3(T_U J\phi_2 V) = \phi_3(T_U J\phi_2 V)$,
*(iii)* $Q_3(\hat{\nabla}^{\perp s}_U J\phi_2 V) = Q_3(\hat{\nabla}^{\perp s}_V J\phi_2 U)$,

*for any $U, V \in \Gamma(D_1)$.*

*Proof.* Let $U, V \in \Gamma(D_1)$. Using (20), we have $\phi_1(T_U J\phi_2 V) = J\phi_1(\hat{\nabla}_U V)$, which implies

$$\phi_1(T_U J\phi_2 V) - \phi_1(T_U J\phi_2 V) = J\phi_1[U, V]. \tag{31}$$

From (22), we obtain $\phi_3(T_U J\phi_2 V) = \psi\phi_3(\hat{\nabla}_U V) + BQ_3 T^s_U V$, which gives

$$\phi_3(T_U J\phi_2 V) - \phi_3(T_U J\phi_2 V) = \psi\phi_3[U, V]. \tag{32}$$

Finally, from (25), we get $Q_3(\hat{\nabla}^{\perp s}_U J\phi_2 V) = F\phi_3(\hat{\nabla}_U V) + CQ_3 T^s_U V$, which implies

$$Q_3(\hat{\nabla}^{\perp s}_U J\phi_2 V) - Q_3(\hat{\nabla}^{\perp s}_V J\phi_2 U) = F\phi_3[U, V]. \tag{33}$$

The proof follows from (31), (32) and (33). □

**Theorem 3.11.** *Let $f : M \to B$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then, the slant distribution $D_2$ is integrable if and only if $\forall\ U, V \in \Gamma(D_2)$, we have*

*(i)* $\phi_1(\hat{\nabla}_U \psi\phi_3 V - \hat{\nabla}_V \psi\phi_3 U) = \phi_1(T_V F\phi_3 U - T_U F\phi_3 V)$,
*(ii)* $Q_2(\hat{\nabla}^{\perp s}_U F\phi_3 V - \hat{\nabla}^{\perp s}_V F\phi_3 U) = Q_2(T^s_V \psi\phi_3 U - T^s_U \psi\phi_3 V)$.

*Proof.* Assume that $U, V \in \Gamma(D_2)$. Using (20), we have $\phi_1(\hat{\nabla}_U \psi\phi_3 V) + \phi_1(T_U F\phi_3 V) = J\phi_1 \hat{\nabla}_U V$, which gives

$$\phi_1(\hat{\nabla}_U \psi\phi_3 V - \hat{\nabla}_V \psi\phi_3 U) + \phi_1(T_U F\phi_3 V - T_V F\phi_3 U) = J\phi_1[U, V]. \tag{34}$$

In view of (24), we get $Q_2(\hat{\nabla}^{\perp s}_U F\phi_3 V) + Q_2(T^s_U \psi\phi_3 V) = J\phi_2(\hat{\nabla}_U V)$, which implies

$$Q_2(\hat{\nabla}^{\perp s}_U F\phi_3 V - \hat{\nabla}^{\perp s}_V F\phi_3 U) + Q_2(T^s_U \psi\phi_3 V - T^s_V \psi\phi_3 U) = J\phi_2[U, V]. \tag{35}$$

Using (34) and (35), we have the required proof. □

**Theorem 3.12.** *Let $f : M \to B$ be a screen pseudo-slant lightlike submersion from an indefinite Kaehler manifold $M$ onto a lightlike manifold $B$. Then, the induced connection $\hat{\nabla}$ on $S(Ker\ f_*)$ is a metric connection if and only if $\forall\ U \in \Gamma(S(Ker\ f_*))$ and $\xi \in \Gamma(\Delta)$, we have*

*(i)* $JQ_2 T^s_U \xi = 0$,
*(ii)* $BQ_3 T^s_U \xi = 0$,
*(iii)* $JT^*_U \xi = 0$ *on* $\Gamma(Ker\ f_*)$.

*Proof.* The induced connection $\hat{\nabla}$ on $S(Ker\ f_*)$ is a metric connection if and only if $\Delta$ is a parallel distribution with respect to $\hat{\nabla}$. In view of (2), (9) and (14), we derive $\nabla_U J\xi = JT_U^*\xi + J\nabla_U^{*\perp}\xi + JT_U^l\xi + JQ_2T_U^s\xi + BQ_3T_U^s\xi + CQ_3T_U^s\xi$, for any $U \in \Gamma(S(Ker\ f_*))$ and $\xi \in \Gamma(\Delta)$. Comparing the tangential components of above equation, we get $\hat{\nabla}_U J\xi = JT_U^*\xi + J\nabla_U^{*\perp}\xi + JQ_2T_U^s\xi + BQ_3T_U^s\xi$. Thus the proof is completed. □

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