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PROLONGATIONS OF GOLDEN STRUCTURES TO BUNDLES OF INFINITELY NEAR POINTS

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Abstract. For a Golden-structure ζ on a smooth manifold M and any covariant functor which assigns to M its bundle M^A of infinitely near points of A-king, we define the Golden-structure ζ^A on M^A and prove that ζ is integrable if and only if so is ζ^A. We also investigate the integrability, parallelism, half parallelism and anti-half parallelism of the Golden-structure ζ^A and their associated distributions on M^A.

Key words and Phrases: Prolongations, bundles of infinitely near points, golden structures, integrability, parallelism.

1. INTRODUCTION

The differential geometry of Golden-structure on manifolds has been first initiated by M. Crâşmăreau and C. Hreţcanu in [1]. The concepts of a Golden-Riemannian structure and a Golden-Riemannian manifold has been introduced in [1]-[11] by using a corresponding almost product structure, and some properties of Golden-Riemannian manifold have been studied. And a few years later, some properties of the induced structure on an invariant submanifold in a Golden Reimannian manifolds were investigated by many authors such as C. Hreţcanu and M. Crâşmăreau in [12], M. Gök, S. Keleş and E. Kiliç in [5]-[6]-[7]-[8]. In [3], A. Gezer, N. Cengiz and A. Salimov studied the problem of the integrability for Golden-Riemannian structures. In [16]-[17], M. Ozkan defined the prolongations of Golden-structures to tangent bundles of order $r \geq 1$.

The present paper is mainly focused on a study of prolongations of Golden-structures on manifolds to bundles of infinitely near points. Basicly, this study is inspried from the paper [16]-[17] and [14]. The main goal of this paper is to generalize the results of [16]-[17] to the bundles of infinitely near points of kind A in the sense of A.Weil [19].

The paper has three sections and is organized as follows. In Section 2, we review the notion of bundles of infinitely near points and recall some definitions and properties of the Golden-structure. The Section 3 is devoted to prolongations of the Golden-structure to bundles of near points of kind A and we will investigate some properties of these prolongations. We also discuss of integrability, parallelism,

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half parallelism and anti-half parallelism of a Golden-structure and the associated distributions to bundle of near points. We end this last section by studying the prolongation to bundles of near points of Golden pseudo-Riemannian structure on a smooth manifold M.

2. PRELIMINARIES

2.1. Bundles of Infinitely Near Points

A weil algebra or local algebra (in sense of A. Weil)[19] is a real associative, commutative and unital algebra of finite dimension over $\mathbb{R}$, admitting a unique maximal ideal $\mathcal{M}$ such that $A/\mathcal{M}$ is one-dimensional over $\mathbb{R}$ and that $\mathcal{M}^{h+1} = \{0\}$ for a nonnegative integer h. The smallest h such that $\mathcal{M}^h \neq \{0\}$ and $\mathcal{M}^{h+1} = \{0\}$ is called the height of A. We shall identify the field $\mathbb{R}$ with the subspace of A consisting of all scalar multiples of the unit. Thus $A = \mathbb{R} \oplus \mathcal{M}$. For example,the algebra of dual numbers $D = \mathbb{R}[T]/(T^2)$ is Weil algebra with height 1.

Let us recall this construction of bundles of A-points of M base on [14]. Let's denote by M a smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M and A the weil algebra with the maximal ideal $\mathcal{M}$.
An infinitely near points to x on M of kind A (or A-points of M near x) is a homomorphism of algebras

$$\varphi : C^\infty(M) \to A$$

such that

$$\varphi(f) \equiv f(x)\ mod(\mathcal{M}),$$

for all $f \in C^\infty(M)$.
We denote by M_x^A and $M^A = \bigcup_{x\in M} M_x^A$ respectively the set of all infinitely points on M of kind A and the set of all near points on M of kind A. If M and N are two smooth manifolds and $F \in C^\infty(M,N)$ a differential map, then one defines the differential map

$$F^A : M^A \to N^A,\ \varphi \mapsto F^A(\varphi)$$

such that, for all $g \in C^\infty(N)$,

$$F^A(\varphi)(g) = \varphi(g \circ F).$$

If F is a diffeomorphism, then F^A will be too.

We can identify $(M \times N)^A$ with $M^A \times N^A$ by the following identification

$$\pi_M^A \times \pi_N^A : (M \times N)^A \to M^A \times N^A,\ \varphi \mapsto (\pi_M^A(\varphi), \pi_N^A(\varphi))$$

where $\pi_M^A : (M \times N)^A \to M^A$ (resp. $\pi_N^A : (M \times N)^A \to N^A$) is a projection. If $F_1 : M_1 \to N_1$, $F_2 : M_2 \to N_2$, $F_1' : M_1 \to N_1'$ and $F_2' : N_1 \to N$ are differentiable maps between manifolds, then we have the following equalities

$$\begin{aligned} (F_1, F_1')^A &= (F_1^A, F_1'^A) \qquad & (F_2' \circ F_1)^A = F_2'^A \circ F_1^A \\ (F_1 \times F_2)^A &= F_1^A \times F_2^A \qquad & (1_M)^A = 1_{M^A}. \end{aligned}$$

If (U, u) is a local chart on M with a local coordinate system $(u^1, \cdots, u^n)$, the map

$$u^A : U^A \to A^n,\ \varphi \mapsto (\varphi(u^1), \cdots, \varphi(u^n))$$

is a bijection from U^A to an open subset of A^n and defines a local chart (U^A, u^A) on M^A. Hence the set M^A becomes a differentiable manifold of dimension $\dim(A) \cdot \dim(M)$.
Now, let $\pi_A : M^A \to M,\ M_x^A \ni \varphi \mapsto x$. Therefore, the manifold M^A with the projection $\pi_A : M^A \to M$ is called the bundles of A-points of M (or bundles of infinitely near points of M of kind A).

The notion of bundles of kind $D = \mathbb{R}[T]/(T^2)$ is the same as the tangent bundles. More generally, when $A = \mathbb{R}[T_1, \cdots, T_p]/(T_1, \cdots, T_p)^{r+1}$, M^A is the space $J_0^r(\mathbb{R}^p, M)$ of jet of order r at 0 of differentiable map from $\mathbb{R}^p$ to M and the associated bundle of A-points is the bundles of p^r-velocities.

2.2. Golden-Riemannian Manifolds

Let M be a smooth manifold and $\mathcal{T}_q^p(M)$ the $C^\infty(M)$-module of tensor fields of (p, q)-type on M. An element of $\mathcal{T}_1^1(M)$ is usually called vector 1-form (or affinor) on M. Let $\mathfrak{X}(M) = \Gamma(TM)$ be the $C^\infty(M)$-module of all vector fields on M.

Definition 2.1. *([9]-[10]) An affinor ζ on M is called polynomial structure if it satisfies the following algebraic equation*

$$Q(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0\delta = 0 \tag{2.1}$$

where $\zeta^{n-1}(x)$, $\zeta^{n-2}(x)$, $\cdots$, $\zeta(x)$ and δ are linearly independent for every $x \in M$ and δ is the identity transformation affinor. The monic polynomial $Q(X)$ is named the structure polynomial.

We recall that, a polynomial structure ζ is integrable if the Nijenhuis tensor N_ζ vanishes identically, where

$$N_\zeta(X, Y) = \zeta^2[X, Y] + [\zeta X, \zeta Y] - \zeta[\zeta X, Y] - \zeta[X, \zeta Y], \text{ for all } X, Y \in \mathfrak{X}(M). \tag{2.2}$$

Remark 2.2. *In particular, if $Q(X) = X^2 - \delta$ (resp. $Q(X) = X^2 + \delta$), then we will have an almost product structure ρ (resp. an almost complex structure ν). When $Q(X) = X^2$, we have the notion of almost tangent structure τ.*

Definition 2.3. *Let (M, g) be a Riemannian-manifold. A Golden-structure on (M, g) is a given non-null affinor of class C^∞ ζ on M which verifies the following equation*

$$\zeta^2 - \zeta - \delta_M = 0 \tag{2.3}$$

where δ_M is the identity transformation affinor. In this case, the pair (M, ζ) is called Golden-manifold.

We say that the metric g is ζ-compatible if we have the following equality

$$g(\zeta X, Y) = g(X, \zeta Y) \tag{2.4}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. If we substitute Y into ζY in (2.4), then Equation (2.4) may also be written as

$$g(\zeta X, \zeta Y) = g(\zeta X, Y) + g(X, Y). \tag{2.5}$$

Definition 2.4. *([1]) A Golden-Riemannian manifold is a triple (M, g, ζ), where (M, g) is a Riemannian-manifold, ζ is a Golden-structure on (M, g) and g is ζ-compatible.*

Definition 2.5. *([18]) Let F be a smooth map from a Golden Riemannian manifold (M, g, ζ) to a Golden Riemannian manifold (N, h, ξ). Then F is called a Golden map if the following condition is satisfied*

$$dF \circ \zeta = \xi \circ dF. \tag{2.6}$$

In [1], we have this following proposition which show the connection between the almost product structure and the Golden-structure on M.

Proposition 2.6. *([1])Let M be a smooth manifold.*

(i) Any Golden-structure ζ on M induces two almost product structures on M defined as follows

$$\rho_- = -\frac{1}{\sqrt{5}}(2\zeta - \delta_M) \quad \textit{and} \quad \rho_+ = \frac{1}{\sqrt{5}}(2\zeta - \delta_M). \tag{2.7}$$

(ii) Conversely, any almost product structure ρ on M induces two Golden-structures on M defined as follows

$$\zeta_- = \frac{1}{2}(\delta_M - \sqrt{5}\rho) \quad \textit{and} \quad \zeta_+ = \frac{1}{2}(\delta_M + \sqrt{5}\rho). \tag{2.8}$$

Let (M, ζ) be a Golden-manifold. According to [1], we define these two operators

$$r = \frac{1}{\sqrt{5}}((\sigma - 1)\delta_M + \zeta) \quad \text{and} \quad s = \frac{1}{\sqrt{5}}(\sigma\delta_M - \zeta) \tag{2.9}$$

where the Golden ratio $\sigma = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ is the root of the algebraic equation $t^2 - t - 1 = 0$. We can easily have these following equalities

$$r^2 = r, \;\; s^2 = s, \;\; s \circ r = r \circ s = 0 \;\; \text{and} \;\; s + r = \delta_M. \tag{2.10}$$

This means that, r and s are projection operators splitting the tangent bundle $TM = M^D$ into two complementary parts, and define two globally complementary distributions R and S of M^D (see [1]) as follows

$$R = \bigcup_{x \in M} \{\varphi \in M_x^D : \;\; \zeta(\varphi) = \sigma\varphi\} \;\; \textit{and} \;\; S = \bigcup_{x \in M} \{\varphi \in M_x^D : \;\; \zeta(\varphi) = (1-\sigma)\varphi\}. \tag{2.11}$$

The projection operators r and s verify these following equalities:

$$\zeta = \sigma r + (1 - \sigma)s \tag{2.12}$$

$$\zeta \circ r = r \circ \zeta = \sigma r \quad \text{and} \quad \zeta \circ s = s \circ \zeta = (1 - \sigma)s. \tag{2.13}$$

3. MAIN RESULTS

3.1. A-Lift of Golden-structures to Bundles of Near Points

Let M be a smooth manifold and M^A a manifold of infinitely near points on M of kind A. For a given affinor ζ on a M, Morimoto in [14] gives its A-lift ζ^A and shows that ζ^A is a unique affinor on M^A which verifies

$$\zeta^A(X^A) = (\zeta(X))^A, \tag{3.1}$$

for all $\zeta \in \mathcal{T}_1^1(M)$ and $X \in \mathfrak{X}(M)$. Hence, we can show

$$\zeta^A \circ \xi^A = (\zeta \circ \xi)^A \tag{3.2}$$

for all $\zeta, \; \xi \in \mathcal{T}_1^1(M)$. When $\zeta = \xi$, equation (3.2) becomes

$$(\zeta^2)^A = (\zeta^A)^2. \tag{3.3}$$

Hence, we have these following results.

Proposition 3.1. *Let ζ be an affinor on M. The following assertions are equivalent.*

(i) ζ is a Golden-structure on M.
(ii) ζ^A is a Golden-structure on M^A.
(iii) $\delta_{M^A} - \zeta^A$ is a Golden-structure on M^A.

Proof. It is the use of linearity of the A-lift, equation (3.3) and the fact that $(\delta_M)^A = \delta_{M^A}$. □

Proposition 3.2. *Let (M, ζ) a Golden-manifold.*

(i) The Golden-structure ζ^A on M^A is an isomorphism on $(M^A)^D_\varphi$, for every $\varphi \in M^A$.

(ii) The Golden-structure ζ^A on M^A is invertible and its inverse $(\zeta^A)^{-1} = \zeta^A - \delta_{M^A}$ satisfies the equation

$$\left((\zeta^A)^{-1}\right)^2 + (\zeta^A)^{-1} - \delta_{M^A} = 0.$$

Remark 3.3. *Let $\zeta \in \mathcal{T}^1_1(M)$ be an almost complex (resp. almost product) structure on M. Then ζ^A and $-\zeta^A$ are an almost complex (resp. almost product) structure on M^A. Morever, ζ^A is integrable if an only if so is ζ. (See* [14]*). If τ is an almost tangent structure on M, then τ^A (resp. -τ^A) is also an almost tangent structure on M^A.*

The following proposition shows the connection between Golden and almost product structures on M^A.

Proposition 3.4. *Let M be a smooth manifold.*

(i) If ζ is the Golden-structure on M, then the Golden-structure ζ^A (resp. $\widetilde{\zeta}^A = \delta_{M^A} - \zeta^A$) on M^A induces two almost product structures ρ^A_- and ρ^A_+ on M^A defined as follows

$$\rho^A_- = -\frac{1}{\sqrt{5}}(2\zeta^A - \delta_{M^A}) \quad \textit{and} \quad \rho^A_+ = \frac{1}{\sqrt{5}}(2\zeta^A - \delta_{M^A}).$$

(ii) Conversely, if ρ is an almost product structure on M, then the almost product ρ^A (resp. $\widetilde{\rho}^A = -\rho^A$) on M^A induces two Golden-structures ζ^A_- and ζ^A_+ on M^A defined as follows

$$\zeta^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\rho^A) \quad \textit{and} \quad \zeta^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\rho^A).$$

According to Crâşmăreanu and Hreţcanu in [1], we have the following remark.

Remark 3.5. *(a) If τ is an almost tangent structure on M, then its A-lift τ^A induces two affinor structures on M^A defined as follows*

$$\widetilde{\tau}^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\tau^A) \quad \textit{and} \quad \widetilde{\tau}^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\tau^A)$$

and which are called tangent Golden-structures on M^A. These tangent Golden-structures satisfy the equation

$$(\widetilde{\tau}^A)^2 - \widetilde{\tau}^A + \frac{1}{4}\delta_{M^A} = 0.$$

(b) If ν is the complex structure on M, then its A-lift ν^A induces two affinor structures on M^A defined as follows

$$\widetilde{\nu}^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\nu^A) \quad \textit{and} \quad \widetilde{\nu}^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\nu^A)$$

and which are called complex Golden-structures on M^A. These complex Golden-structures satisfy the equation

$$(\widetilde{\nu}^A)^2 - \widetilde{\nu}^A + \frac{3}{2}\delta_{M^A} = 0.$$

Example 3.6. *(prolongation to M^A of triple structures on M) Let ξ, ρ and ν be three affinors structures on the smooth manifold M such that $\nu = \zeta\circ\rho$. According to* [1] *and* [2]*, the triple (ζ,ρ,ν) is called almost hyperproduct structure (ahps), almost biproduct complex structure (abpcs), almost product bicomplex structure (apbcs) and almost hypercomplex structure (ahcs) on M if ζ, τ and ν verify respectively the following equalities:*
$\xi^2 = \rho^2 = \nu^2 = \xi\circ\rho\circ\nu = \delta_M$, $\xi^2 = \rho^2 = -\nu^2 = \xi\circ\rho\circ\nu = \delta_M$,
$-\xi^2 = \rho^2 = \nu^2 = \xi\circ\rho\circ\nu = -\delta_M$ *and* $\xi^2 = \rho^2 = \nu^2 = \xi\circ\rho\circ\nu = -\delta_M$.
Let

$$\widetilde{\xi}^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\xi^A),\ \ \widetilde{\rho}^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\rho^A)\ \text{ and } \widetilde{\nu}^A_- = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\nu^A)$$

$$(\textit{resp. } \widetilde{\xi}^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\xi^A),\ \ \widetilde{\rho}^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\rho^A)\ \text{ and } \widetilde{\nu}^A_+ = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\nu^A))$$

be the induces structures associated to ξ^A, ρ^A and ν^A respectively (see proposition 3.4). We easily see that, those induces structures verify this following equality

$$\sqrt{5}\widetilde{\nu}^A = 2\widetilde{\xi}^A\circ\widetilde{\rho}^A - \widetilde{\xi}^A - \widetilde{\rho}^A + \sigma\delta_{M^A}$$

and the triple $(\widetilde{\xi}^A_-,\widetilde{\rho}^A_-,\widetilde{\nu}^A_-)$ and $(\widetilde{\xi}^A_+,\widetilde{\rho}^A_+,\widetilde{\nu}^A_+)$ are

1. *(ahps) (resp. (apbcs)) on M^A if and only if (ξ,ρ,ν) is an (ahps) (resp. (apbcs)) on M. In this case, $\widetilde{\nu}^A$ is a Golden-structure on M^A.*
2. *(abpcs) (resp. (ahcs)) on M^A if and only if (ξ,ρ,ν) is an (abpcs) (resp. (ahcs)) on M. In this case, $\widetilde{\nu}^A$ is a complex Golden-structure on M^A.*

3.2. Integrability of Golden-structure to Bundles of Near Point

The purpose of this section is to give some properties of integrability of the Golden-structure ζ^A on M^A and its associated distributions. Let (M,ζ) be a Golden-manifold and A a given Weil algebra.

Definition 3.7. *The Golden-structure ζ^A on M^A is integrable if*

$$N_{\zeta^A}(X^A,Y^A) = 0,$$

for all vector fields X, Y in M.

Proposition 3.8. *ζ is an integrable Golden-structure on M if and only if the Golden-structure ζ^A on (M^A,g^A) is integrable on M^A.*

Proof. It comes from the fact that $N_{\zeta^A}(X^A,Y^A) = (N_\zeta(X,Y))^A$ by using relations (2.2) and (3.3), for all vector fields X, Y in M. □

A. Morimoto in [14] has proved the following proposition.

Proposition 3.9. *(*[14]*) Let M be a smooth manifold.*

(i) The map $\mathfrak{X}(M)\to\mathfrak{X}(M^A)$, $X\mapsto X^A$ is a homomorphism of Lie algebras.
(ii) For all $\zeta\in\mathcal{T}^1_1(M)$ and $X\in\mathfrak{X}(M)$, one has

$$(\zeta(X))^A = \zeta^A(X^A). \tag{3.4}$$

Hence, from the above proposition, we can construct the A-lift of these two projection operators r and s on M as follows

$$r^A = \frac{1}{\sqrt{5}}((\sigma-1)\delta_{M^A} + \zeta^A) \ \text{ and } \ s^A = \frac{1}{\sqrt{5}}(\sigma\delta_{M^A} - \zeta^A). \tag{3.5}$$

These new operators satisfy the following equalities

$$(r^A)^2 = r^A,\ (s^A)^2 = s^A,\ s^A\circ r^A = r^A\circ s^A = 0 \ \text{ and } \ r^A + s^A = \delta_{M^A} \tag{3.6}$$

$$r^A\circ\zeta^A = \zeta^A\circ r^A = \sigma r^A \ \textit{ and } \ s^A\circ\zeta^A = \zeta^A\circ s^A = (1-\sigma)s^A. \tag{3.7}$$

Therefore, r^A and s^A are projection operators splitting the tangent bundle $TM^A = (M^A)^D$ into two complementary parts, and define two globally complementary distributions R^A and S^A of the set of D-point of M^A according to [1].

Let's recall this result from M. Crâşmăreanu and C.E. Hreţcanu.

Proposition 3.10. *([1])Let (M, ζ) be the Golden-manifold. The distribution R (resp. S) is integrable if and only if $[rX, rY] \in \Gamma(R)$ (resp. $[sX, sY] \in \Gamma(S)$) for all vector fields X, Y in M.*

We have this following definition.

Definition 3.11. *The distribution R^A (resp. S^A) is integrable if the vector field $[r^AX^A, r^AY^A]$ (resp. $[s^AX^A, s^AY^A]$) belongs to $\Gamma(R^A)$ (resp. $\Gamma(S^A)$) for all vector fields X and Y in M.*

Hence, we have these following results.

Proposition 3.12. *Let (M, ζ) be the Golden-manifold.*
The distribution R^A (resp. S^A) is integrable if and only if R on (resp. S) is integrable.

Proof. It comes from the fact that

$$s^A[r^AX^A, r^AY^A] = (s[rX, rY])^A \ \ (resp. r^A[s^AX^A, s^AY^A] = (r[sX, sY])^A),$$

for all vector fields X and Y in M. □

Proposition 3.13. *Let (M, ζ) be the Golden-manifold.*
The distribution R^A (resp. S^A) is integrable if and only if

$$N_{\zeta^A}(r^AX^A, r^AY^A) \in \Gamma(R^A) \ \ (resp. \ \ N_{\zeta^A}(s^AX^A, s^AY^A) \in \Gamma(S^A)) \tag{3.8}$$

for all vector fields X, Y in M.

Proof. For all vector fields X and Y in M, one has

$$\begin{aligned} s^AN_{\zeta^A}(r^AX^A, r^AY^A) &= s^A(\zeta^A)^2[r^AX^A, r^AY^A] + s^A[\zeta^A \circ r^AX^A, z^A \circ r^AY^A] \\ &- s^A\zeta^A[\zeta^A \circ r^AX^A, r^AY^A] - s^A\zeta^A[r^AX^A, \zeta \circ r^AY^A] \\ &= (1-\sigma)^2 s^A[r^AX^A, r^AY^A] + \sigma^2[s^AX^A, s^AY^A] \\ &- \sigma(1-\sigma)s^A[r^AX^A, r^AY^A] - \sigma(1-\sigma)s^A[r^AX^A, r^AY^A] \\ &= 5s^A[r^AX^A, r^AY^A]. \end{aligned}$$

With the same manner, $r^AN_{\zeta^A}(s^AX^A, s^AY^A) = 5r^A[r^AX^A, r^AY^A]$. Hence, the proof is finished. □

Proposition 3.14. *Let ζ be a Golden-structure on M and ζ^A its A-lift on M^A. The following assertions are equivalent:*

(i) ζ^A is integrable.
(ii) Both the distribution R^A and S^A are integrable.

Proof. Let X and Y be two vector fields on M. We have

$$\begin{aligned} r^AN_{\zeta^A}(s^AX^A, s^AY^A) + s^AN_{\zeta^A}(r^AX^A, r^AY^A) &= 5r^A[(\delta_{M^A} - r^A)X^A, (\delta_{M^A} - r^A)Y^A] \\ &+ 5(\delta_{M^A} - r^A)[r^AX^A, r^AY^A] \\ &= 5(r^A)^2[X^A, Y^A] + 5[r^AX^A, r^AY^A] \\ &- 5r^A[r^AX^A, Y^A] - 5r^A[X^A, r^AY^A] \\ &= 5N_{r^A}(X^A, Y^A). \end{aligned}$$

Hence, the proof has been finished. □

Corollary 3.15. *Let ζ be a Golden-structure on M and ζ^A its A-lift on M^A. The following assertions are equivalent:*

(a) ζ^A is integrable.
(b) Both R^A and S^A are integrable.
(c) Both R and S are integrable.
(d) ζ is integrable.

Theorem 3.16. *Let ρ be the almost product on a smooth manifold M.*
The almost product ρ^A on M^A is integrable if an only if the associated Golden-structure ζ_+^A (resp. ζ_-) is integrable.

Proof. Let $X,\ Y \in \mathfrak{X}(M)$, ρ^A an almost product structure on M^A and $\zeta_-^A = \frac{1}{2}(\delta_{M^A} - \sqrt{5}\rho^A)$ (resp. $\zeta_+^A = \frac{1}{2}(\delta_{M^A} + \sqrt{5}\rho^A)$) The induced Golden-structure on M^A. One has:

$$\begin{aligned} N_{\zeta_-^A}(X^A,Y^A) &= (\zeta_-^A)^2[X^A,Y^A] + [\zeta_-^A X^A, \zeta_-^A Y^A] - \zeta_-^A[\zeta_-^A X^A, Y^A] - \zeta_-^A[X^A, \zeta_-^A Y^A] \\ &= \frac{1}{4}(\delta_{M^A} - 2\sqrt{5}\rho^A + 5(\rho^A)^2)[X^A,Y^A] \\ &+ \frac{1}{4}[X^A,Y^A] - \frac{\sqrt{5}}{4}[X^A,\rho^A Y^A] - \frac{\sqrt{5}}{4}[\rho^A X^A, Y^A] + \frac{5}{4}[\rho^A X^A, \rho^A Y^A] \\ &- \frac{1}{4}[X^A,Y^A] + \frac{\sqrt{5}}{4}\rho^A[X^A,Y^A] + \frac{\sqrt{5}}{4}\rho^A[\rho^A X^A, Y^A] - \frac{5}{4}\rho^A[\rho^A X^A, Y^A] \\ &- \frac{1}{4}[X^A,Y^A] + \frac{\sqrt{5}}{4}\rho^A[X^A,Y^A] + \frac{\sqrt{5}}{4}[X^A,\rho^A Y^A] - \frac{5}{4}\rho^A[X^A,\rho^A Y^A] \\ &= \frac{5}{4}((\rho^A)^2[X^A,Y^A] + [\rho^A X^A, \rho^A Y^A] - \rho^A[\rho^A X^A, Y^A] - \rho^A[X^A, \rho^A Y^A]) \\ &= \frac{5}{4}N_{\rho^A}(X^A,Y^A) \end{aligned}$$

With the same manner, we have $N_{\zeta_+^A}(X^A,Y^A) = \frac{5}{4}N_{\rho^A}(X^A,Y^A)$. Hence, the proof follows. □

Conversely, we have this following theorem.

Theorem 3.17. *Let (M,ζ) be the Golden-manifold.*
The Golden-structure ζ^A on (M^A, g^A) is integrable if and only if the associated almost product ρ_+^A (resp. ρ_-^A) is integrable.

3.3. Parallelism, Half Parallelism and Anti-half Parallelism of Golden-structure on M^A

In this section, we discuss parallelism, half parallelism and anti half parallelism of the distributions associated with the golden structure on M^A. We recall that, a distribution $\mathcal{D}$ on M is called parallel with respect to the linear connection ∇ if the vector field $\nabla_X Y$ belongs to $\mathcal{D}$ for any vector fields $Y \in \Gamma(\mathcal{D})$ and $X \in \mathfrak{X}(M) = \Gamma(TM)$. Let ζ be a Golden-structure on M. For all vector fields X and Y in M, let's put

$$\triangle\zeta(X,Y) = \zeta(\nabla_X Y) - \zeta(\nabla_Y X) - \nabla_{\zeta X} Y + \nabla_Y \zeta X.$$

We recall this following definition from [4].

Definition 3.18. *([4])Let (M,g,ζ) be a Golden Riemannian manifold.*

(d1) *The distribution R (resp. S) on M is called half-parallel with respect to the linear connection ∇ if*

$$\triangle\zeta(X,Y) \in \Gamma(R) \quad (resp.\ \Gamma(S)), \tag{3.9}$$

for all vector fields $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and $Y \in \mathfrak{X}(M)$.

(d2) *The distribution R (resp. S) on M is called anti-half parallel with respect to the linear connection ∇ if*

$$\triangle\zeta(X,Y) \in \Gamma(S) \quad (resp.\ \Gamma(R)), \tag{3.10}$$

for all vector fields $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and $Y \in \mathfrak{X}(M)$.

Let ∇ be a linear connection on M. Its A-lift ∇^A is a unique linear connection on M^A which satisfies this equality

$$\nabla^A_{X^A} Y^A = (\nabla_X Y)^A, \tag{3.11}$$

where X^A and Y^A mean prolongation to M^A of vector fields X and Y in M (see Theorem 5 of [15]).

From the above consideration, we have these following definitions.

Definition 3.19. *Let ∇ be a linear connection on a Golden-manifold (M, ζ) . The distribution R^A (resp. S^A) is parallel with respect to linear connection ∇^A if*

$$\nabla^A_{X^A} Y^A \in \Gamma(R^A) \ \ (resp.\ \Gamma(S^A))$$

for all vector fields $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and $Y \in \mathfrak{X}(M)$.

Definition 3.20. *Let ∇ be a linear connection on a Golden-manifold (M, ζ) .*

(d1) *The distribution R^A (resp. S^A) on M^A is called half-parallel with respect to the linear connection ∇^A if*

$$\triangle\zeta^A(X^A, Y^A) \in \Gamma(R^A) \quad (resp.\ \Gamma(S^A)), \tag{3.12}$$

for all vector fields $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and $Y \in \mathfrak{X}(M)$.

(d2) *The distribution R^A (resp. S^A) on M^A is called anti-half parallel with respect to the linear connection ∇ if*

$$\triangle\zeta^A(X^A, Y^A) \in \Gamma(S^A) \quad (resp.\ \Gamma(R^A)), \tag{3.13}$$

for all vector fields $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and $Y \in \mathfrak{X}(M)$.

Let ∇ be a linear connection on a Golden-manifold (M, ζ). According to [1]-[13], we can associate to the pair (ζ^A, ∇^A) two other linear connections $\overset{Sc}{\nabla}{}^A$ and $\overset{V}{\nabla}{}^A$ on M^A called respectively Schouten and Vrănceanu connections, and define as follows:

$$\overset{Sc}{\nabla}{}^A_{\overline{X}} \overline{Y} = r^A(\nabla^A_{\overline{X}} r^A \overline{Y}) + s^A(\nabla^A_{\overline{X}} s^A \overline{Y})$$

$$\overset{V}{\nabla}{}^A_{X^A} Y^A = r^A(\nabla^A_{r^A\overline{X}} r^A \overline{Y}) + s^A(\nabla^A_{s^A\overline{X}} s^A \overline{Y}) + r^A[s^A\overline{X}, r^A\overline{Y}] + s^A[r^A\overline{X}, s^A\overline{Y}],$$

for all vector fields $\overline{X}$ and $\overline{Y}$ in M^A.

Hence, we have the following results.

Theorem 3.21. *The Golden-structure ζ^A on M^A is parallel with respect to Schouten and Vrănceanu connections.*

Proof. From the linearity of ∇^A and the relations (3.7)-(3.8), one has

$$\begin{aligned}
(\overset{Sc}{\nabla}{}^A_{\overline{X}} \zeta^A)\overline{Y} &= \overset{Sc}{\nabla}{}^A_{\overline{X}} \zeta^A \overline{Y} - \zeta^A \overset{Sc}{\nabla}{}^A_{\overline{X}} \overline{Y} \\
&= r^A(\nabla^A_{\overline{X}} r^A \circ \zeta^A \overline{Y}) + s^A(\nabla^A_{\overline{X}} s^A \circ \zeta^A \overline{Y}) - \zeta^A \circ r^A(\nabla^A_{\overline{X}} r^A \overline{Y}) \\
&\quad - \zeta^A \circ s^A(\nabla^A_{\overline{X}} s^A \overline{Y}) \\
&= \sigma r^A(\nabla^A_{\overline{X}} r^A \overline{Y}) + (1-\sigma) s^A(\nabla^A_{\overline{X}} s^A \overline{Y}) - \sigma r^A(\nabla^A_{\overline{X}} r^A \overline{Y}) - \\
&\quad (1-\sigma) s^A(\nabla^A_{\overline{X}} s^A \overline{Y}) \\
&= 0.
\end{aligned}$$

With the same manner, $(\overset{V}{\nabla}{}^{A}_{\overline{X}}\zeta^A)\overline{Y} = 0$. □

Theorem 3.22. *The projection operator r^A (resp. s^A) is parallel with respect to Schouten and Vrănceanu connections.*

Proof. It comes from relations (3.7)-(3.8) and the fact that ∇^A and the bracket of vector fields on M^A are linear. □

Proposition 3.23. *Let ∇ be a linear connection on a Golden-manifold (M, ζ). The distribution R (resp. S) is parallel with respect to a fixed linear connection ∇ on M if and only if R^A (resp. S^A) is parallel with respect to linear connection ∇^A on M^A.*

Proof. Let $X \in \Gamma(R)$ (resp. $\Gamma(S)$) and Y be a vector field in M. From the relations (3.11) and (3.5) in this order, on has

$$s^A\nabla^A_{X^A}Y^A = (s\nabla_X Y)^A \quad (resp.\ s^A\nabla^A_{X^A}Y^A = (s\nabla_X Y)^A).$$

Hence,

$$s^A(\nabla^A_{X^A}Y^A) = 0 \Leftrightarrow s(\nabla_X Y) = 0 \quad (resp.\ r^A(\nabla^A_{X^A}Y^A) = 0 \Leftrightarrow r(\nabla_X Y) = 0).$$

□

Theorem 3.24. *The distribution R^A (resp. S^A) is parallel with respect to the Schouten and Vrănceanu connections for every linear connection ∇^A on M^A.*

Proof. Let ∇^A be a linear connection on M^A, $X^A \in \Gamma(R^A)$ and $Y^A \in \mathfrak{X}(M^A)$ be the A-lift of vector fields $Y \in \Gamma(R)$ and $X \in \mathfrak{X}(M)$. From relations (2.10) and (2.11), we easily have $rY = Y$ and $sY = 0$. Hence

$$\begin{aligned} \overset{sc}{\nabla}{}^{A}_{X^A}Y^A &= r^A(\nabla^A_{X^A}r^AY^A) + s^A(\nabla^A_{X^A}s^AY^A) \\ &= r^A(\nabla^A_{X^A}(rY)^A) + s^A(\nabla^A_{X^A}(sY)^A) \\ &= r^A(\nabla^A_{X^A}Y^A) \in \Gamma(R^A) \end{aligned}$$

and

$$\overset{v}{\nabla}{}^{A}_{X^A}Y^A = r^A\left(\nabla^A_{(sX)^A}Y^A + [sX, Y]^A\right) \in \Gamma(R^A).$$

It can be proved analogously that the distribution S^A is parallels with respect to the Schouten and Vrănceanu connections for a linear connection ∇^A. □

Proposition 3.25. *Let ζ be a Golden structure, parallels with respect to a linear connection ∇ on M. Then ζ^A is parallels with respect to linear connection ∇^A on M^A if and only if*

$$(\nabla^A_{X^A}\zeta^A)Y^A = 0$$

for all vector fields X and Y in M.

Proposition 3.26. *Let ∇ be a linear connection on a Golden manifold (M, ζ) and ∇^A its A-lift on (M^A, ζ^A). The distribution R (resp. S) on M is half parallels with respect to ∇ if and only if the distribution R^A (resp. S^A) on M^A is also half parallel with respect to ∇^A.*

Proof. It comes from Equation (3.1) and Equality (3.11). □

Proposition 3.27. *Let ∇ be a linear connection on a Golden manifold (M, ζ) and ∇^A its A-lift on (M^A, ζ^A). The distribution R^A (resp. S^A) on M^A is anti-half parallel with respect to ∇^A.*

Proof. Let $X^A \in \Gamma(R^A)$ and $Y^A \in \mathfrak{X}(M^A)$ be the A-lift of vector fields $X \in \Gamma(R)$ and $Y \in \mathfrak{X}(M)$. From relations (3.8) and (2.11), we have $r^A \circ \zeta^A = \sigma r^A$ and $\zeta X = \sigma X$. Hence

$$r^A\left(\zeta^A(\nabla^A_{X^A}Y^A) - \zeta^A(\nabla^A_{Y^A}X^A) - \nabla^A_{(\zeta X)^A}Y^A + \nabla^A_{Y^A}(\zeta X)^A\right) = 0$$

since ∇^A is linear. Therefore,

$$\triangle\zeta^A(X^A, Y^A) \in \Gamma(S^A)$$

and R^A is anti-half parallel with respect to ∇^A. S^A is anti-half parallel with respect to ∇^A by using the same method. □

Proposition 3.28. *Let ∇ be a fixed linear connection on Golden-manifold (M, ζ). The the distribution R (resp. S) is half parallels with respect to Schouten and Vrănceanu connections if and only if so is R^A (resp. S^A).*

3.4. Prolongation to M^A of Golden Pseudo-Riemannian Structure on M

Let g be a pseudo-Riemannian metric on M. Its A-lift is a unique pseudo-Riemannian metric on M^A which satisfies

$$g^A(X^A, Y^A) = (g(X, Y))^A, \tag{3.14}$$

where X^A and Y^A mean prolongation to M^A of vector fields X and Y in M (see proposition 12 of [15]. Hence, the pair (M^A, g^A) becomes a pseudo-Riemannian manifold. Then, we easily have the following results.

Proposition 3.29. *If the triple (M, g, ζ) is a Golden pseudo-Riemannian manifold, then so is the triple (M^A, g^A, ζ^A).*

Corollary 3.30. *Let (M, g, ζ) be a pseudo-Riemannian manifold. For all vector fields in M, we have*

- *(a)* $g^A(r^AX^A, Y^A) = g^A(X^A, r^AY^A)$ *(resp.* $g^A(s^AX^A, Y^A) = g^A(X^A, s^AY^A)$*): This means that the projection operators r^A and s^A are g^A-symmetric*
- *(b)* $g^A(r^AX^A, s^AY^A) = 0$*: This means that the distribution R^A and S^A are g^A-orthogonal.*
- *(c)* $N_{\zeta^A}(\zeta^AX^A, Y^A) = N_{\zeta^A}(X^A, \zeta^AY^A)$*. This means that the Golden structure ζ^A is N_{ζ^A}-symmetric.*

Remark 3.31. *If (g, ρ) is a pseudo-Riemannian almost product on M (that is, ρ is a g-symmetric almost product structure on pseudo-Riemannian manifold (M, g), then the pair (g^A, ρ^A) is also a pseudo-Riemannian almost product on M^A and the triple (M^A, g^A, ζ^A) is a Golden pseudo-Riemannian structure on M^A where ζ^A is the Golden-structure on M^A induced by ρ^A (see Proposition 3.4).*

Proposition 3.32. *If $F : M \to N$ is a Golden map between Golden pseudo-Riemannian manifolds (M, g, ζ) and (N, h, ξ), then $F^A : M^A \to N^A$ is a Golden map between Golden pseudo-Riemannian manifolds (M^A, g^A, ζ^A) and (N^A, h^A, ξ^A).*

Proof. Since F is a Golden map, then we have:

$$dF \circ \zeta = \xi \circ dF.$$

Taking the A-lift on the both sides of the above equality and from the relation (3.2), we obtain

$$dF^A \circ \zeta^A = \xi^A \circ dF^A.$$

□

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