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VECTOR-VALUED INEQUALITY OF FRACTIONAL INTEGRAL OPERATOR WITH ROUGH KERNEL ON MORREY–ADAMS SPACES

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Abstract. In 2019, Salim et al proved the vector-valued inequality for maximal operator with rough kernel on Lebesgue spaces and Morrey spaces. This results extend Fefferman-Stein inequality (1971). In 1970's, Adams introduced another variant of Morrey spaces, which called as Morrey-Adams spaces. In this article, we prove vector-valued inequality for maximal operator and fractional integral operator with rough kernel on Morrey–Adams spaces.

Key words and Phrases: Morrey-Adams Space, Fractional Integral Operator, Rough Kernel, Vector-Valued Inequality
Kata kunci: Ruang Morrey-Adams, Operator Fraksional Integral, Rough Kernel, Ketaksamaan bernilai vektor

1. Introduction

Let us first recall the definition of Hardy–Littlewood maximal operator as follows:

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|dy,$$

where $B(x, r)$ is set of $y \in \mathbb{R}^n$ with $|x - y| < r$. The boundedness of M on L^p for $p > 1$ is well known.

In 1971, the boundedness property of M is extended onto vector-valued inequality by Fefferman and Stein [1], in the sense: for $t > 1$, $p > 1$, and sequence of

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functions $\vec{f} = \{f_i\}_{i=1}^{\infty}$ with

$$\|\vec{f}(\cdot)\|_{\ell^t} = \left(\sum_{i=1}^{\infty} |f_i(\cdot)|^t\right)^{\frac{1}{t}} \in L^p,$$

the following is satisfied.

$$\left(\int_{\mathbb{R}^n} \left(\sum_{i=1}^{\infty} [Mf_i(x)]^t\right)^{\frac{p}{t}} dx\right)^{\frac{1}{p}} \leq C \left\| \|\vec{f}(\cdot)\|_{\ell^t} \right\|_{L^p} = \|\vec{f}\|_{L^p(\ell^t))}. \tag{1}$$

We then define $M\vec{f}$ as sequence of maximal function $\{Mf_i\}_{i=1}^{\infty}$, that the left-hand-side of (1) can be written as $\|M\vec{f}\|_{L^p(\ell^t)}$. In 2016, Sawano proved the vector-valued inequality for fractional maximal operator on Lebesgue space (see [9]).

Let Ω be a zero degree homogeneous function on $\mathbb{R}^n$, in the sense: $\Omega(\tau x) = \Omega(x)$ for any $\tau > 0$ and $x \in \mathbb{R}^n$. Hardy–Littlewood maximal operator can be generalized into maximal operator with rough kernel, M_Ω, which is given as follows

$$M_\Omega f(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |\Omega(x-y)|\,|f(y)|dy.$$

For $\Omega \equiv 1$, operator M_Ω is also known as Hardy-Littlewood maximal operator M.

In 2019, Salim et al proved vector-valued inequality for M_Ω on Lebesgue space (see [4]). Let S^{n-1} be the set of $x \in \mathbb{R}^n$ with $|x| = 1$. For $\Omega \in L^1(S^{n-1})$, $t > 1$, and $p > 1$, we have

$$\|M_\Omega \vec{f}\|_{L^p(\ell^t)} \leq C\|\vec{f}\|_{L^p(\ell^t)} \tag{2}$$

where $M_\Omega \vec{f}$ is the sequence of $\{M_\Omega f_i\}_{i=1}^{\infty}$. In this article, the constant C in each row can be differ. The constant C may be depending on Ω, but independent of f.

Based ond Morrey's article in 1930's, Peetre introduced Morrey spaces in 1969 [8]. Nowadays, the research on Morrey spaces is very popular that there are many ways to define the space. In this article, we shall use the following definition of Morrey space. For $0 < \lambda < n$ and $p \geq 1$, Morrey Space $L^{p,\lambda}$ is defined as the set of f which satisfies

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty.$$

For $\lambda = 0$, we have $L^{p,\lambda} \equiv L^p$. For $\lambda > 0$, with Hölder inequality, we have $L^{\frac{pn}{n-\lambda}} \subset L^{p,\lambda}$. This inclusion property is proper since $f(x) = |x|^{-\frac{n-\lambda}{p}}$ is element of $L^{p,\lambda}$, and f is not in L^q for any q. For this reason, Morrey spaces are known as generalization of Lebesgue spaces.

Boundedness of M on Lebesgue spaces was extended onto Morrey spaces by Chiarenza and Frasca in 1987 (see [5]). Therefore, it is challenging to extend inequality (1) onto Morrey space. It was obtained along with the vector-valued inequality for M_Ω on Morrey spaces as follows.

Proposition 1.1. [4, Theorem 5 and Theorem 6] *Let $p > 1$ and one of the followings is satisfied.*

(1) $0 \le \lambda \le \mu < n$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$, $\Omega \in L^s(S^{n-1})$ *with* $s \ge p' = \frac{p}{p-1}$,

(2) $s > 1$, $0 < \lambda < n - \frac{np}{s}$, $\lambda \le \mu < n$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$, *and* $\Omega \in L^s(S^{n-1})$.

Then, for $t > 1$

$$\|M_\Omega \vec{f}\|_{L^{q,\mu}(\ell^t)} = \left\| \|M_\Omega \vec{f}(\cdot)\|_{\ell^t} \right\|_{L^{q,\mu}} \le C \|\vec{f}\|_{L^{p,\lambda}(\ell^t))}.$$

In 1981, Adams introduced another variant of Morrey spaces (see [10]). Suppose that $\theta \ge 1$, $p \ge 1$, and $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$, the functions space $L^{p,\lambda}_\theta$ is set of f with

$$\|f\|_{L^{p,\lambda}_\theta} = \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|f\|^\theta_{L^p(B(x,r))} dr \right)^{\frac{1}{\theta}} < \infty.$$

Note that, we use integral in the Lebesgue norm L^p for $1 < p < \infty$, and we use supremum of the Lebesgue norm L^∞. The same modification is also applied for the case of $\theta = \infty$. Therefore, for $\theta = \infty$, $L^{p,\lambda}_\theta$ is equivalent with $L^{p,\lambda}$. Although $L^{p,\lambda}_\theta$ and $L^{p,\lambda}$ are intersected for $\frac{p}{\theta} < \lambda < n$, there are no inclusion property for both spaces. Since the function space $L^{p,\lambda}_\theta$ is quite similar with Morrey space, we call it as Morrey–Adams space.

In Section 2, we prove the vector-valued inequality of M_Ω on Morrey–Adams space. We then apply our result to investigate the vector-valued inequality for fractional integral operator with rough kernel, which is defined in Section 3.

2. Operator M_Ω on Morrey–Adams Spaces

Let us first state our main result in the following theorem.

Theorem 2.1. *Suppose that* $1 < q \le p$, $\theta > 1$, $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$, $\frac{q}{\theta} < \mu < n + \frac{q}{\theta}$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$, *and one of the following is satisfied.*

(C1) $\Omega \in L^s(S^{n-1})$ *with* $s \ge p' = \frac{p}{p-1}$,

(C2) $s > q$, $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta} - \frac{np}{s}$, *and* $\Omega \in L^s(S^{n-1})$.

Then, for $t > 1$

$$\|M_\Omega \vec{f}\|_{L^{q,\mu}_\theta(\ell^t)} = \left\| \|M_\Omega \vec{f}(\cdot)\|_{\ell^t} \right\|_{L^{q,\mu}} \le C \|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t))}.$$

Remark. The condition $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$ and $q \le p$ in Theorem 2.1 implies one of the followings is satisfied.

- $\lambda \le \mu < n$
- $\lambda = \mu = n$
- $\lambda \ge \mu > n$.

For $q = p$, then there are no λ satisfying (C2). We also note that the condition (C1) are used in the discussion of M_Ω in [2, 4].

Proof of Theorem 2.1. Suppose that $\vec{f}$ satisfies $\|\vec{f}(\cdot)\|_{\ell^t} \in L^{p,\lambda}_\theta$ for $t > 1$. Fix $z \in \mathbb{R}^n$, and decompose

$$\vec{f} = \{f_i\}_{i=1}^\infty = \{g_i\}_{i=1}^\infty + \{h_i\}_{i=1}^\infty = \vec{g} + \vec{h}$$

where $g_i = f_i\chi_{B(z,2r)}$, $h_i = f_i\chi_{B^c(z,2r)}$ and $r > 0$, χ_A is the characteristic function in the set A ($\chi_A(x) = 1$ for $x \in A$, and $\chi_A(x) = 0$ for $x \notin A$). We write $B^c(z, 2r)$ to denote the complement of $B(z, 2r)$ in $\mathbb{R}^n$.
By sublinearity of M_Ω, for each i

$$M_\Omega f_i(x) \le M_\Omega g_i(x) + M_\Omega h_i(x).$$

Hence,

$$\|\chi_{B(z,r)} M_\Omega \vec{f}\|_{L^q(\ell^t)} \le \|\chi_{B(z,r)} M_\Omega \vec{g}\|_{L^q(\ell^t)} + \|\chi_{B(z,r)} M_\Omega \vec{h}\|_{L^q(\ell^t)}. \tag{3}$$

Therefore, it suffices to investigate for both $\vec{g}$ and $\vec{h}$. Let us work for $\vec{g}$ first. By (2),

$$\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,r)} M_\Omega \vec{g}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} \le \left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|M_\Omega \vec{g}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} \tag{4}$$

$$= C\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,2r)} \vec{f}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}}. \tag{5}$$

The right-hand-side of (4) has $\vec{g}$ that are depending on $r > 0$. Although we integrate the r along from 0 to infinite, it is important to note that $\vec{g}$ is in the Lebesgue norm L^q (which is another integral). Therefore, in Lebesgue norm L^q, the r can be treated as a fix number. We then continue our investigation from (5) by applying Hölder's inequality with order p/q, and using the fact $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$, to obtain

$$\begin{aligned}\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,r)} M_\Omega \vec{g}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} &\le C\left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q} - \frac{n\theta}{p}} \|\chi_{B(z,2r)} \vec{f}\|^\theta_{L^p(\ell^t)} dr\right)^{\frac{1}{\theta}}\\ &= C\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|\chi_{B(z,2r)} \vec{f}\|^\theta_{L^p(\ell^t)} dr\right)^{\frac{1}{\theta}}\\ &\le C\left\|\|\vec{f}(\cdot)\|_{\ell^t}\right\|_{L^{p,\lambda}_\theta},\end{aligned} \tag{6}$$

where the last inequality can be obtained by substitution $s = 2r$, and taking the supremum over $z \in \mathbb{R}^n$. The norm in the right-hand-side of (6) is then written as $\|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)}$. Hence, we have

$$\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,r)} M_\Omega \vec{g}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} \le C\|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)}. \tag{7}$$

We now treat $\vec{h}$. For $x \in B(z, r)$, it is easy to confirm $B^c(z, 2r) \subset B^c(x, r)$ (in another word, $B(x, r) \subset B(z, 2r)$). Therefore, for each $i \in \mathbb{N}$

$$\begin{aligned} M_\Omega h_i(x) &\le \int_{B^c(x,r)} \frac{|\Omega(x-y)|}{|x-y|^n} |f_i(y)| dy \\ &= \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \frac{|\Omega(x-y)|}{|x-y|^n} |f_i(y)| dy \\ &\le C \sum_{j=0}^{\infty} (2^j r)^{-n} \int_{B(x,2^{j+1}r)} |\Omega(x-y)|\, |f_i(y)| dy. \end{aligned} \tag{8}$$

Suppose that the condition (C1) is satisfied. By (8), Minkowski's inequality, Hölder's inequality with order p, we can proceed as follows.

$$\begin{aligned} \|M_\Omega \vec{h}(x)\|_{\ell^t} &\le C \left(\sum_{i=1}^{\infty} \left| \sum_{j=0}^{\infty} (2^j r)^{-n} \int_{B(x,2^{j+1}r)} |\Omega(x-y)|\, |f_i(y)| dy \right|^t \right)^{\frac{1}{t}} \\ &\le C \sum_{j=0}^{\infty} (2^j r)^{-n} \left(\sum_{i=1}^{\infty} \left| \int_{B(x,2^{j+1}r)} |\Omega(x-y)|\, |f_i(y)| dy \right|^t \right)^{\frac{1}{t}} \\ &\le C \sum_{j=0}^{\infty} (2^j r)^{-n} \int_{B(x,2^{j+1}r)} |\Omega(x-y)|\, \|\vec{f}(y)\|_{\ell^t} dy \qquad (9) \\ &\le C \sum_{j=0}^{\infty} (2^j r)^{-\frac{n}{p}} \|\chi_{B(z,2^{j+2}r)} \vec{f}\|_{L^p(\ell^t)}. \qquad (10) \end{aligned}$$

Note that from (9) to (10), we use the following inequality

$$\|\Omega(x-\cdot)\|_{L^{p'}(B(x,2^{j+1}r))} = \|\Omega\|_{L^{p'}(B(0,2^{j+1}r))} = (2^{j+1}r)^{\frac{n}{p'}} \|\Omega\|_{L^{p'}(S^{n-1})} \le C(2^j r)^{\frac{n}{p'}}.$$

Since the right-hand-side of (10) is independent of x, by Minkowski's inequality, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$, we have

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,r)} M_\Omega \vec{h}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} \\
&\leq C \left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q}} \left(\sum_{j=0}^\infty (2^j r)^{-\frac{n}{p}} \|\chi_{B(z,2^{j+2}r)} \vec{f}\|_{L^p(\ell^t)}\right)^\theta dr\right)^{\frac{1}{\theta}} \\
&\leq C \sum_{j=0}^\infty 2^{-\frac{jn}{p}} \left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q} - \frac{n}{p}} \|\chi_{B(z,2^{j+2}r)} \vec{f}\|^\theta_{L^p(\ell^t)} dr\right)^{\frac{1}{\theta}} \\
&\leq C \sum_{j=0}^\infty 2^{\frac{j(\lambda-n)}{p} - \frac{j}{\theta}} \left(\int_0^\infty (2^{j+2} r)^{-\frac{\lambda}{p}} \|\chi_{B(z,2^{j+2}r)} \vec{f}\|^\theta_{L^p(\ell^t)} d(2^{j+2}r)\right)^{\frac{1}{\theta}} \\
&\leq C \|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)},
\end{aligned} \tag{11}$$

where the last inequality is obtained by taking the supremum over $z \in \mathbb{R}^n$ and the the convergence of the series $\sum_{j=0}^\infty 2^{\frac{j(\lambda-n)}{p} - \frac{j}{\theta}}$ (due to $\lambda < n + \frac{p}{\theta}$).
On another hand, if the condition (C2) is satisfied, we note that (9) is still valid. From (9), we use Minkowski's inequality, Hölder's inequality with order s/q, and the fact $(B(z,r)) \subset B(y, 2^{j+3}r)$ for $y \in B(z, 2^{j+2}r)$, to obtain

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \|\chi_{B(z,r)} M_\Omega \vec{h}\|^\theta_{L^q(\ell^t)} dr\right)^{\frac{1}{\theta}} \\
&\leq C \left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \left(\int_{B(z,r)} \left|\sum_{j=0}^\infty (2^j r)^{-n} \int_{B(z,2^{j+2}r)} |\Omega(x-y)| \, \|\vec{f}(y)\|_{\ell^t} dy\right|^q dx\right)^{\frac{\theta}{q}} dr\right)^{\frac{1}{\theta}} \\
&\leq C \left(\int_0^\infty r^{-\frac{\mu\theta}{q}} \left(\sum_{j=0}^\infty (2^j r)^{-n} \int_{B(z,2^{j+2}r)} \|\vec{f}(y)\|_{\ell^t} \left(\int_{B(z,r)} |\Omega(x-y)|^q dx\right)^{\frac{1}{q}} dy\right)^\theta dr\right)^{\frac{1}{\theta}} \\
&\leq C \left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q} - \frac{n\theta}{s}} \left(\sum_{j=0}^\infty (2^j r)^{-n} \int_{B(z,2^{j+2}r)} \|\vec{f}(y)\|_{\ell^t} \left(\int_{B(y,2^{j+3}r)} |\Omega(x-y)|^s dx\right)^{\frac{1}{s}} dy\right)^\theta dr\right)^{\frac{1}{\theta}} \\
&\leq C \left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q} - \frac{n\theta}{s}} \left(\sum_{j=0}^\infty (2^j r)^{-n+\frac{n}{s}} \int_{B(z,2^{j+2}r)} \|\vec{f}(y)\|_{\ell^t} dy\right)^\theta dr\right)^{\frac{1}{\theta}}.
\end{aligned} \tag{12}$$

From inequality (12), we proceed by Hölder's inequality with order p, and Minkowski's inequality, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p}$.

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\mu\theta}{q}}\|\chi_{B(z,r)}M_\Omega\vec{h}\|^\theta_{L^q(\ell^t)}dr\right)^{\frac{1}{\theta}}\\
&\leq C\left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q}-\frac{n\theta}{s}}\left(\sum_{j=0}^\infty (2^j r)^{-\frac{n}{p}+\frac{n}{s}}\|\chi_{B(z,2^{j+2}r)}\vec{f}\|_{L^p(\ell^t)}\right)^\theta dr\right)^{\frac{1}{\theta}}\\
&\leq C\sum_{j=0}^\infty (2)^{-\frac{jn}{p}+\frac{jn}{s}}\left(\int_0^\infty r^{\frac{(n-\mu)\theta}{q}-\frac{n\theta}{p}}\|\chi_{B(z,2^{j+2}r)}\vec{f}\|^\theta_{L^p(\ell^t)}dr\right)^{\frac{1}{\theta}}\\
&\leq C\sum_{j=0}^\infty (2)^{\frac{j(\lambda-n)}{p}+\frac{jn}{s}-\frac{j}{\theta}}\left(\int_0^\infty (2^{j+2}r)^{-\frac{\lambda\theta}{p}}\|\chi_{B(z,2^{j+2}r)}\vec{f}\|^\theta_{L^p(\ell^t)}d(2^{j+2}r)\right)^{\frac{1}{\theta}}\\
&\leq C\|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)}, \qquad (13)
\end{aligned}$$

where the last is obtained by taking the supremum over $z \in \mathbb{R}^n$, and the convergence of the series is confirmed by $\lambda < n + \frac{p}{\theta} - \frac{np}{s}$ (see condition (C2)).
By (3), (7), and (11) for (C1), and (13)) for (C2), we can conclude

$$\left(\int_0^\infty r^{-\frac{\mu\theta}{q}}\|\chi_{B(z,r)}M_\Omega\vec{f}\|^\theta_{L^q(\ell^t)}dr\right)^{\frac{1}{\theta}} \leq C\|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)}. \qquad (14)$$

Since the right-hand-side of (14) is independent to $z \in \mathbb{R}^n$,

$$\|M_\Omega\vec{f}\|_{L^{q,\mu}_\theta(\ell^t)} \leq C\|\vec{f}\|_{L^{p,\lambda}_\theta(\ell^t)}.$$

Thus, Theorem 2.1 is confirmed.

3. Fractional Integral Operator with Rough Kernel on Morrey–Adams Space

Suppose that Ω is a zero degree homogeneous function on $\mathbb{R}^n$. For $0 < \alpha < n$, fractional integral operator with rough kernel, $T_{\Omega,\alpha}$, is given as

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy.$$

For $\Omega \equiv 1$, operator $T_{\Omega,\alpha}$ is known as fractional integral operator (or Riesz potential). The vector-valued inequality for fractional integral operator was proven in Lebesgue space by Sawano in 2006 (see [9]). For some study regarding $T_{\Omega,\alpha}$ on Morrey space, the readers can see [6, 7] and the references therein.

If $\Omega \in L^s(S^{n-1})$ with $s \geq p'$, $f \in L^{p,\lambda}_\theta$, and $\alpha < \frac{n-\lambda}{p} + \frac{1}{\theta}$, we have the following pointwise estimation

$$|T_{\Omega,\alpha}f(x)| \leq CM_\Omega f(x)^u\|f\|^{1-u}_{L^{p,\lambda}_\theta} \qquad (15)$$

where $u := 1 - \frac{\alpha p\theta}{(n-\lambda)\theta+p}$ in [3, Lemma 1].

Let $T_{\Omega,\alpha}\vec{f}$ be the sequence of $\{T_{\Omega,\alpha}f_i\}_{i=1}^{\infty}$. Note that, if $\|\vec{f}(\cdot)\|_{\ell^t} \in L_\theta^{p,\lambda}$ with $tu \geq 1$, then for each i, $f_i \in L_\theta^{p,\lambda}$ and

$$\|f_i\|_{L_\theta^{p,\lambda}} \leq \|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}.$$

For any $x \in \mathbb{R}^n$ and $i \in \mathbb{N}$, by (15)

$$|T_{\Omega,\alpha}f_i(x)| \leq CM_\Omega f_i(x)^u \|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}^{1-u}.$$

Thus, we have

$$\|T_{\Omega,\alpha}\vec{f}(x)\|_{\ell^t} \leq C\|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}^{1-u}\|M_\Omega\vec{f}(x)\|_{\ell^{tu}}^u. \tag{16}$$

Let q satisfies $uq = p$ and φ satisfies $u\varphi = \theta$. For $z \in \mathbb{R}^n$, by (16)

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\lambda\varphi}{q}}\|\chi_B(z,r)T_{\Omega,\alpha}\vec{f}\|_{L^q(\ell^t)}^\varphi dr\right)^{\frac{1}{\varphi}} \\
&\leq C\|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}^{1-u}\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}}\|\chi_B(z,r)M_\Omega\vec{f}\|_{L^p(\ell^{tu})}^\theta dr\right)^{\frac{u}{\theta}} \\
&= C\|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}^{1-u}\|M_\Omega\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}^u
\end{aligned} \tag{17}$$

By (17), and Theorem 2.1, we can obtain

$$\|T_{\Omega,\alpha}\vec{f}\|_{L_\varphi^{q,\lambda}(\ell^t)} \leq C\|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^{tu})}. \tag{18}$$

This result is interesting since we approach it without using the boundedness property of $T_{\Omega,\alpha}$ on Morrey–Adams space [3].

However, by using the boundedness property of $T_{\Omega,\alpha}$ in [3], we can actually obtain a better result. For instance, for $t > 1$ and $x \in \mathbb{R}^n$, and by Minkowski's inequality

$$\|T_{\Omega,\alpha}\vec{f}(x)\|_{\ell^t} \leq T_{|\Omega|,\alpha}[\|\vec{f}(\cdot)\|_{\ell^t}](x).$$

Once we apply the boundedness of $T_{\Omega,\alpha}$ on Morrey–Adams space in [3], we have the following theorem.

Theorem 3.1. *Let $p > 1$, $\theta \geq 1$, and $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$. Let $\Omega \in L^s(S^{n-1})$ with $s \geq p'$, and $\alpha < \frac{n-\lambda}{p} + \frac{1}{\theta}$. Let $u = 1 - \frac{\alpha p\theta}{(n-\lambda)\theta + p}$, q and φ satisfy $uq = p$ and $u\varphi = \theta$. Then, for $t > 1$*

$$\|T_{\Omega,\alpha}\vec{f}\|_{L_\varphi^{q,\lambda}(\ell^t)} \leq C\|\vec{f}\|_{L_\theta^{p,\lambda}(\ell^t)}. \tag{19}$$

Inequality (19) is better than (18), since $\ell^{tu} \subset \ell^t$ (due to $tu < t$).

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