J. Indones. Math. Soc.
Vol. 30, No. 01 (2024), pp. 77–88.

GREEN PROBABILISTIC INVENTORY MODEL WITH SHORTAGE BACKORDERING, CARBON EMISSION COST, AND IMPERFECT QUALITY ITEMS: A NEWTON-RAPHSON METHOD APPROACH USING PYTHON

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Abstract. In this paper, a green inventory model has been developed to explain the relationship between a single manufacturer and many retailers in a multiplayer supply chain system. It is assumed that there exist some imperfect items in a quantity lot. In green inventory, one of the important issues is carbon emission with its cost to inventory management. Some troubles in the transportation process yield imperfect quality items. This trouble is the potential to increase the amount of carbon emission and some imperfect quality items. It also affects the cost of the inventory. Therefore, these two aspects will be analyzed under the shortage backorder policy. Due to the complexity of the model, the classical optimization methods cannot be used to determine the optimum values exactly. Therefore, formulation optimization is predominantly conducted using a numerical approach for finding partial derivatives and Newton Raphson's method for finding the optimal solution. These methods are assisted by the Python programming language, operated within the *Google Colab* environment and *Spyder* (Python 3.8) using *Anaconda environment.*

Key words and Phrases: Manufacturer, Carbon Emission, Inventory, Imperfect quality, Python

1. INTRODUCTION

In the production process, products produced by factories managed by manufacturers are not always 100 percent in good condition in terms of quality. This assumption has become the main premise in modern inventory modeling. The

2020 Mathematics Subject Classification: 90B05, 90B50, 65K05, 65Y04.
Received: 30-07-2023, accepted: 24-01-2024.

existence of imperfect products can be stated either deterministically or probabilistically. Numerous studies have been conducted on inventory models that take into account the existence of imperfect products. The inventory models for probabilistically imperfect products with the assumption of controlled lead time have been examined by Lin [1], following the lead-free demand distribution. Similar research involving controllable lead time was continued by Jha and Shanker [2], incorporating the assumption of shortage back-ordering.

Additionally, the inventory model for products with imperfect quality was examined by Mandal and Giri [3] who considered the assumption of increasing quality through lead time reduction. Other related research on imperfect quality aspect has been conducted by Hsu and Hsu [4], Bhowmick and Samanta [5], Konstantaras et al. [6], Alamri et al. [7], and Setiawan et al. [8]. The development of an inventory model depends on predefined assumptions. However, there are still many conditions and situations in supply chain management that have not been included in the inventory model for products with imperfect quality.

Recent research in inventory management has focused on environmental aspects, particularly carbon emissions (greenhouse gases) produced by production machinery and equipment, loading and unloading machinery, and vehicle. There have been several studies that included carbon emission factors in inventory models such as Rahimi et al. [9], Huang et al. [10], Bozorgi et al. [11], Marchi et al. [12], Bazan et al. [13], Beccera et al.[14]. Based on the existing findings, it can be observed that there has been no previous research been previous research in terms of establishing an inventory model for imperfect quality that takes into account greenhouse gas emissions, reorder processes, and shortage back-ordering policies in an objective function formulation. Furthermore, in this research, the analytical analysis will also be complemented by numerical analysis using algorithm-based numerical methods. This is because inventory models, which incorporate various assumptions and parameters, often result in complex forms. The algorithms are implemented in the Python programming language, which has not been previously utilized in inventory model research.

The article is structured into several chapters, namely the introduction, research methodology, formulation of mathematical models, optimum analysis, numerical analysis, and conclusion. The research methodology section explains the theoretical research methodology used in this study. Assumptions, mathematical notations used, inventory model formulations, and optimum analysis are described in the third chapter of the discussion. The explanations in this article are concluded with a summary and suggestions for further research, which are presented in the last section.

2. MATHEMATICAL MODELLING AND OPTIMUM ANALYSIS

In this section, an explanation will be given regarding the formation of an inventory model for products with imperfect quality and considering carbon emissions. Below are the mathematical notations used in the following table:

Table 2.1. Mathematical Notations (1)

Notation	**Descriptions**
q_p	Batch production size at the manufacturing site.
q_i	Lot size or product shipment from the manufacturing site to the retailer i.
	The decision variable (integer).
	The decision variable (integer).
B_i	The maximum quantity of reorder per unit at retailer i.
n	Number of shipments in each batch produced by the manufacturer.
	The decision variable (integer), $q_p = \sum_{i=1}^{n} nq_i$.
D_i	Demand from the retailer i.
D	Cumulative demand.
P	Production rate ($P > D$, with $D = \sum_{i=1}^{n} D_i$).
C_p	Set up the cost per production process for the manufacturer.
C_i^b	Ordering cost per unit for retailer i.
γ_i	Percentage of products with imperfect quality in lot q for retailer i.
$f_i(\gamma_i)$	The probability density function of γ_i for each i.
ω	Compensation cost per unit of imperfect quality products.
s_i	Sorting cost per unit of products for retailer i.
b_i	Reorder cost per unit of products per unit time for retailer i.
h_P	Holding cost per unit product per unit time for the manufacturer.
h_i^b	Holding cost per unit product per unit time for retailer i.
F_i	Transportation cost per shipment from the manufacturer to retailer i.
T	The time length between one shipment and the next.

Table 2.2. Mathematical Notations (2)

Notation	**Descriptions**
T_1	The period during the production process at the manufacturing site.
T_2	The period when the manufacturer fulfills the demands of
	all retailers from the inventory kept at the manufacturing site.
T_t	Cycle Time. $T_t = T_1 + T_2 = nT$.
*	Superscripts for symbols of optimum values.
$E[.]$	Expected value.
J_i	Distance between the manufacturer and retailer i.
EG_1	Carbon gas emissions from a specific type of vehicle per unit distance.
EG_2	Carbon gas emissions from loading equipment per 1 kg product.
EG_3	Carbon gas emissions from unloading equipment per 1 kg product.
C_l	Cost of loading per unit product.
C_{ul}	Cost for unloading per unit product.
C_j	Carbon emission cost per unit distance.
m_{pp}	Weight per unit product.
$\Gamma_i(.)$	Retailer i's objective function.
$\Lambda_i(.)$	Manufacturer's objective function.

The main assumptions used in the formation of the mathematical model are explained.

(1) The supply chain system consists of one manufacturer that produces a single type of product and multiple retailers who place orders for the product with the manufacturer.

(2) The inventory management scheme used is the integration scheme.
(3) The order level is known, constant, and continuous.
(4) The lead time is known and constant.
(5) Imperfect products exist in lot size q. The percentage of imperfect quality products γ_i has a probability density function $f(\gamma_i)$. To ensure that the manufacturer has a sufficient production capacity in the retailer demand fulfillment process, it is assumed that $E[\gamma_i] < 1 - \frac{D}{P}$.
(6) The sorting process for the lot quantity is completed (100%) at the retailer's location before the start of each cycle time T. In this case, the sorting time is counted as part of the delivery lead time. All products with imperfect quality will be returned to the manufacturer through the return process at the time of the next lot shipment.
(7) It is assumed that there are no additional shipping costs for this return process. The manufacturer provides compensation of ω for each product with the imperfect quality found. Furthermore, the manufacturer will resell these products with imperfect quality through the secondary market.
(8) Completely back ordered when a product shortage occurs (shortages back-order).
(9) Carbon emissions are assumed to be generated by the production loading and unloading of equipment and vehicles in the delivery process.
(10) The emission costs caused by loading and unloading equipment are borne by the manufacturer, while the emission costs generated by vehicles are charged to the retailer. The type of equipment and vehicles is determined by the manufacturer. The manufacturer and retailer agree to minimize emission costs by using equipment and vehicles with low emission values.
(11) The manufacturer and all retailers agree to use the principles of synchronization and integration for determining the optimum values of decision variables.

Next, the inventory model for imperfect quality products will be explained, considering shortage back ordering and greenhouse gas emissions. The intended model involves formulating objective functions for each retailer and manufacturer. The objective function for each retailer consists of several components, namely ordering cost, transportation cost, sorting cost, holding cost, reorder cost, and carbon emission cost. The determination of the holding cost components requires an analysis of the inventory level managed by the retailers. The inventory level of retailer i is depicted in the following diagram.

Based on the inventory level depicted in Figure 1, the holding cost component in the cost function for each retailer can be expressed as:

$$h_i^b = \frac{1}{2}\left(\frac{(q_i - \gamma_i - B_i)^2}{D} + \frac{q_i^2\gamma_i(1-\gamma_i)}{D}\right).$$

Simultaneously, the reorder cost component is defined as $\frac{1}{2}b_i n \frac{B_i}{D}$. The emission handling cost is assigned to each retailer i. The specific handling equipment and

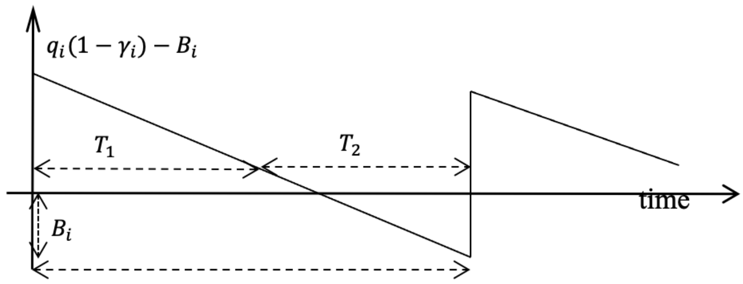

FIGURE 1. Retailer's Inventory Level

vehicles are determined by the manufacturer. The emission handling cost component is calculated as the sum of emission costs from handling equipment and transportation equipment, formulated as $J_iEG_1C_jq_i + C_lEG_2m_{pp}q_i + C_{ul}EG_3m_{pp}q_i$. Consequently, the total cost function for retailer i is denoted as $\Gamma_i(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ with

$$\begin{aligned}\Gamma_i(q_i, n) &= C_i^b + nF_i + s_inq_i + h_i^b\left(\frac{1}{2}\left(\frac{(q_i-\gamma_i-B_i)^2}{D} + \frac{q_i^2\gamma_i(1-\gamma_i)}{D}\right)\right)\\ &+ \frac{1}{2}b_in\frac{B_i}{D} + J_i\left(EG\right)_1C_jq_i + C_l\left(EG\right)_2m_{pp}q_i + C_{ul}\left(EG\right)_3m_{pp}q_i. \quad (1)\end{aligned}$$

Next, the manufacturing objective function, which is the total cost function, will be explained. The manufacturing objective function consists of several components, namely setup cost and compensation cost for imperfect quality products found. Based on Figure 1 and on the work of Lin [1] and Hsu and Hsu [4], the inventory holding cost for the manufacturer is formulated as the product of the holding cost per unit of product multiplied by the cumulative difference between the manufacturing inventory level and the cumulative inventory level of retailer i. Holding cost per cycle is formulated by $h_P\left(\frac{nq^2}{P} - \frac{n^2q^2}{2P} + \frac{n(n-1)q^2(1-\gamma)}{2D}\right)$. All parties in the inventory system are deemed in agreement to use the integration scheme. Thus, the total cost function of the entire supply chain system in each cycle is obtained from the summation of the manufacturing objective function and all retailers $J(q,n,B) = \Lambda_i(q_i,n) + \sum_{i=1}^{k}\Gamma_i(q_i,n)$ as follows.

$$\begin{aligned}
J(q,n,B) &= C_p + \omega nq\gamma + h_P\left(\frac{nq^2}{P} - \frac{n^2q^2}{2P} + \frac{n(n-1)q^2(1-\gamma)}{2D}\right)\\
&+ \sum_{i=1}^{k}\left(C_i^b + nF_i + h_i^b n\left(\frac{1}{2}\left(\frac{(q_i-\gamma_i-B_i)^2}{D}\right)\right)\right)\\
&+ \sum_{i=1}^{k}\left(\frac{\gamma_i(1-\gamma_i)(D_i)^2q^2}{D} + \frac{b_i n B_i}{2D}\right)\\
&+ \sum_{i=1}^{k}\left(J_i\mathrm{EG}_1C_jq_i + C_l\mathrm{EG}_2m_{pp}q_i + C_ul\mathrm{EG}_3m_{pp}q_i\right), \qquad (2)
\end{aligned}$$

where $B = (B_1, ..., B_n)^T$, $q = \sum_{i=1}^{n} q_i$, and $\gamma = (\gamma_1, \ldots, \gamma_n)^T$. Since every party in the inventory system also agrees to use the synchronization principle, the relationship $q_i = \frac{D_i q}{D}$ hold. Thus, Equation (2) is equivalent to the following equation.

$$\begin{aligned}
J(q,n,B) &= C_p + \omega nq\gamma + h_P\left(\frac{nq^2}{P} - \frac{n^2q^2}{2P} + \frac{n(n-1)q^2(1-\gamma)}{2D}\right)\\
&+ \frac{1}{D}\sum_{i=1}^{k}\left(DC_i^b + DnF_i + h_i^b n\left(\frac{1}{2}\left(\frac{D_i^2q^2}{D^2} - \gamma_i^2 - 2\gamma_iB_i^2\right)\right)\right)\\
&+ \frac{1}{D}\sum_{i=1}^{k} h_i^b n\left(\frac{\gamma_i(1-\gamma_i)(D_i)^2q^2}{D^2}\right) + \sum_{i=1}^{k}\left(\frac{b_i n B_i}{2D}\right)\\
&+ \frac{q}{D}\sum_{i=1}^{k}\left((J_i\mathrm{EG}_1C_j + C_l\mathrm{EG}_2m_{pp} + C_ul\mathrm{EG}_3m_{pp})D_i\right). \qquad (3)
\end{aligned}$$

Due to the length of the product replenishment cycle ($T_{tot} = \frac{nq}{(1-\gamma)/D)}$), then we obtain $E[T_{tot}] = nq(1-E[\gamma])/D$. Using the renewal-reward theorem, the expected average total annual cost per unit of time is formulated by $EJ(q,n,B) = \frac{E[J(q,n,B)]}{E[T_tot]}$. Then, we have the expected form of Equation (3) like follows.

$$
\begin{aligned}
J(q,n,B) &= \frac{E[J(q,n,B)]}{E[T_{\text{tot}}]} = \frac{(C_pD)}{nq(1-E[\gamma])} + \frac{\omega E[\gamma]D}{(1-E[\gamma])} \\
&+ \sum_{i=1}^{k}\left(\frac{(DC_i^b)}{nq(1-E[\gamma])} + \frac{(DF_i)}{q(1-E[\gamma])}\right) \\
&+ \sum_{i=1}^{k}\left(\frac{h_i^b}{q(1-E[\gamma])}\left(\frac{1}{2}\left(\frac{(D_i^2q^2)}{D^2} - E[\gamma_i^2] - 2E[\gamma_i]B_i^2\right)\right)\right) \\
&+ \sum_{i=1}^{k}\left(\frac{(E[\gamma_i]-E[\gamma_i^2])(D_i)^2q^2}{D^2}\right) \\
&+ h_P\left(\frac{qD}{P(1-E[\gamma])} - \frac{qD}{2P(1-E[\gamma])} + \frac{n(n-1)q}{2}\right) \\
&+ \frac{1}{2q}\sum_{i=1}^{k}\left(\frac{b_iB_i}{(1-E[\gamma])}\right) \\
&+ \frac{1}{n(1-E[\gamma])}\sum_{i=1}^{k}\left((J_i\mathrm{EG}_1C_j + C_l\mathrm{EG}_2m_{pp} + C_ul\mathrm{EG}_3m_{pp})D_i\right). \quad (4)
\end{aligned}
$$

The determination of the optimum value for each decision variable uses the classical optimization concept, namely the first partial derivative criteria. The first partial derivative process is applied to the combined total cost function $J(q, n, B)$. Due to the intricate nature of the equations involved, manually calculating the partial derivative formulas for the total cost function concerning each decision variable be exceedingly challenging. As a solution, we will utilize Python programming to efficiently derive the partial derivatives of the total cost function. These derivatives are essential for optimizing the decision variables, such as the order quantity (q), number of shipments (n), and the maximum quantity of reorders per unit at each retailer (B_i). By integrating the required integration scheme and synchronization principles, we aim to gain valuable insights into the cost optimization for the entire supply chain cycle, considering the individual maximum reorder quantities (B_i), and the overall number of shipments (n). Thus, we have obtained the partial derivative formulas for each decision variable as follows: First, the partial derivative concerning the order quantity (q) will encompass a multitude of complexities involving the entire manufacturing objective function and the reorder quantities at each retailer (B_i), Second, the partial derivative concerning the number of shipments (n) will involve product-specific parameters such as set-up costs (C_p), annual demand (D), and others. Third, the partial derivative for the total maximum reorder quantity (B) will take into account the individual maximum reorder quantities at each retailer (B_i), and impact the overall supply chain cost. By utilizing Python programming through *Google Colabs*, we can efficiently determine these partial derivative formulas to gain valuable insights into the cost optimization for the entire supply chain cycle. Due to the use of an algorithmic approach to determining the partial derivative formulas, the expected average total annual cost function concerning each decision variable, a specific index value i must be taken. In this case, an attempt is made for $k = 3$, which means simulations are conducted for a total of 3 retailers. It is also assumed that all parties have the same value of γ_i and denoted by γ. Here is the formula for each partial derivative for $k = 3$.

$$\begin{aligned}\frac{\partial J(q,n,B)}{\partial q} &= \frac{2B + 2C_pD}{q^3(1-E[\gamma])} + \frac{(C_1^b + C_2^b + C_3^b)D + F_1 + F_2 + F_3}{nq^2(1-E[\gamma])} \\ &- \frac{h_{1b}/D^2(2D_1^2q(-E[\gamma^2] + E[\gamma] + (D_1^2q)/D^2))}{q(1-E[\gamma])} \\ &+ \frac{h_{1b}/D^2(-B_1^2E[\gamma] - 0.5E[\gamma^2] + D_1^2q^2(-E[\gamma^2] + E[\gamma] + 0.5D_1^2q^2))}{q^2(1-E[\gamma])} \\ &+ \frac{h_{2b}/D^2(2D_2^2q(-E[\gamma]^2 + E[\gamma] + D_2^2q))}{q(1-E[\gamma])} \\ &+ \frac{h_{2b}/D^2(-B_2^2E[\gamma] - 0.5E[\gamma^2] + D_2^2q^2(-E[\gamma^2] + E[\gamma]) + 0.5D_2^2q^2)}{q^2(1-E[\gamma])} \\ &+ \frac{h_{3b}/D^2(2D_3^2q(-E[\gamma^2] + E[\gamma] + D_3^2q))}{q(1-E[\gamma])} \\ &+ \frac{h_{3b}/D^2(-B_3^2E[\gamma] - 0.5E[\gamma^2] + D_3^2q^2(-E[\gamma^2] + E[\gamma] + 0.5D_3^2q^2))}{q^2(1-E[\gamma])} \quad (5)\end{aligned}$$

$$\begin{aligned}\frac{\partial J(q,n,B)}{\partial n} &= \frac{C_1^bD}{nq^2(1-E[\gamma])} + \frac{C_2^bD}{n^2q(1-E[\gamma])} + \frac{C_3^bD}{n^2q(1-E[\gamma])} \\ &+ \frac{C_pD}{n^2q^2(1-E[\gamma])} + n - \frac{1}{2} = 0. \quad (6)\end{aligned}$$

$$\frac{\partial J(q,n,B)}{\partial B} = -\frac{1}{q^2(1-E[\gamma])} = 0. \quad (7)$$

An analytical solution for the complex set of three partial derivative equations (related to q, n, and B) is difficult or even infeasible due to the intricacy of the non-linear cost function and constraints involved. Therefore, a numerical approach is required, which efficiently seeks a numerical solution approximating the optimum. We assume that γ follows uniform distribution in the interval [0,1]. For implementing these numerical methods, Python with the *SciPy* library is an ideal choice, providing flexibility and ease in solving this optimization problem. In the first simulation, we attempted to use the Newton-Raphson method by utilizing the *sympy* package. However, this method alone was not sufficient to obtain the optimum solution. Therefore, we turned to a more robust tool in the *SciPy* library, specifically *optimize.fsolve* function (*import as scipy.optimize*), and combined it with *sympy*.

For the subsequent data analysis, we made modifications to our initial codes. We included a section that prompts the user to input the number of sets of initial guesses (q, n, and B), followed by asking for the initial guesses for each set. The solutions are then stored in a list and presented in a tabular format using *pandas* library. This enhancement allows users to input multiple sets of initial guesses and observe the resulting solutions in a structured manner. The first simulation (single data) is conducted using Python based on *Google Colab*, while the second simulation (multiple data case) is executed using *Spyder* (Python 3.8). We use five packages in Python that are *sympy*, *scipy.optimize*, *matplotlib.pyplot*, and *random*. Package *matplotlib.pyplot* is used to code a graphical representation of the optimal solution. Because we will use some initial guess values in the interval, random packages are needed. Before presenting the numerical simulations and analyzing the numerical results, here is the procedure for determining the solutions as the basis for creating the algorithm based on the Newton-Raphson method.

Procedure for determining the optimum solution:

(1) Input the given parameter values such as C_1^b, C_2^b, C_2^b, B_1, B_2, B_3, D_1, D_2, D_3, F_1, F_2, F_3, C_p, h_1^b, h_2^b, and h_3^b.
(2) Define the variables q, n, and B, and the expected values $E[\gamma]$and $E[\gamma^2]$ according to the desired distribution. In this research, it is assumed that γ follows the uniform distribution.
(3) Set initial guesses for q and n ($initial\_guess - q$ and $initial\_guess - n$)
(4) Set an initial guess for B ($initial\_guess - B$) according to the requirement.
(5) Define the cost function that calculates the vector of partial derivatives with respect to q, n, and B.
(6) Use the numerical method *optimize.fsolve* to find the numerical solution of the formed equations.
(7) Display the numerical solution obtained for q, n, and B.

According to Equations (5), Equation (6), and Equation (7), we take the following value of some cost parameters (in 1000 IDR):

Table 2.3. Parameter values

i	C_i^b	F_i	h_i^b
1	50	200	2
2	55	220	3
3	60	250	2

Then, the other parameters are: $B_1 = 20, B_2 = 15, B_3 = 10, D_1 = 150, D_2 = 160, D_3 = 170$(in unit) and $C_p = 400$ (in 1000 IDR). The Newton–Raphson method is a numerical method that requires guessing the initial value so that iterations in the method can be executed. We simulate twenty pairs of initial guesses (q, n, B) randomly in certain intervals. For the first simulation, initial guess values of q, n, B will be used each of which is taken randomly in the intervals [80,120], [1,10], and [20, 40].

In using the formed algorithm, here is an example of taking one of the optimal values for twenty random initial value selections within a specified interval.

Table 2.4. Simulation Example for Optimum Value

i	q^*	n^*	B^*
1	301.829651	6.949209	6.526886e+07
2	175.433465	8.613923	2.017227e+06
3	173.493334	8.684346	1.151145e+07
4	155.598831	8.851715	6.960295e+06
5	107.736366	10.376168	2.485684e+06
6	486.936392	5.940064	2.803741e+08
7	467.661685	6.092256	2.458636e+08
8	163.987395	8.713638	9.080201e+06
9	989.647099	4.769870	2.419257e+09
10	238.388030	7.636687	2.884494e+07
11	490.153779	5.894341	2.909484e+08
12	1030.297472	4.666678	2.711623e+09
13	169.040810	9.233033	1.027647e+07
14	477.901490	6.098078	2.620549e+08
15	1019.418393	4.709437	2.656991e+09
16	243.466912	7.503973	3.073054e+07
17	347.824387	-5.191177	1.113897e+08
18	191.264496	8.264229	1.509079e+07
19	479.996205	6.018554	2.627387e+08
20	206.213966	8.047127	1.913231e+07

Using *mpl_toolkits.mplot3d* package, the results in Table 2.4 can be presented in the following graphic.

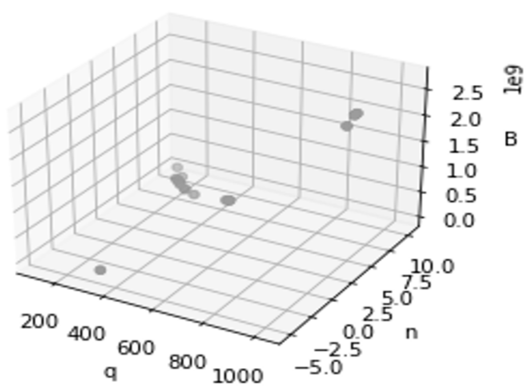

FIGURE 2. Relation between q, n, and B

By adding code to determine the best solution, the Python program selects $q^* = 108$, $n^* = 108$, $B^* = 3$. When we repeat this process multiple times, the program will iterate with any 20 different pairs of initial values. However, in general, it will produce the best approximate optimum solution for q^* within the range of 98 to 110, the best approximate optimum value for n^* within the range of 9 to 11, and the range of B^* values within 1 to 3. Higher values of q will lead to lower values of n; however, collectively, the maximum quantity of reorder points remains relatively the same for several approaches with the same q value. In the optimality analysis, emission costs do not influence the optimality results, as the simulation does not allow for the possibility of q and n values being zero. A supply chain system that meets the criteria in this model can function well, meaning there are products available, and there is a shipping process in place. Consequently, carbon emission costs will always be present. The program's suggested best values do not have to be followed; manufacturers and retailers can choose alternative approach results for optimal values. Additionally, the initial guessed values can also be modified according to the inventory system conditions, which may yield different optimal value approaches.

3. CONCLUDING REMARKS

In this paper we have been able to model an inventory model with a shortage back ordering policy, the existence of products with imperfect quality, and the cost of carbon emissions. The more products need to be shipped to meet retailers' demands, the higher the volume of product shipments will increase. If imperfect quality products become more prevalent, more products will need to be resent to meet retailers' demands. Furthermore, as the number of shipments increases, both inventory costs and carbon emission expenses will rise. Therefore, in this case, quality control is necessary to reduce the presence of imperfect quality products.

Due to the complexity of the model, the optimization analysis uses a numerical method approach, namely the Newton-Raphson method by utilizing packages in Python. For further research, it is possible to improve the approach by using more complex and accurate algorithms such as genetic algorithms.

Acknowledgement. The authors would like to thank the anonymous editors and reviewers, especially the editor-in-chief, for their valuable comments and suggestions, which improved the quality of this article.

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